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Expansivité et distorsion bornée des systèmes dynamiques C^r

Expansiveness and bounded distorsion of C^r dynamical systems

préparée au LPSM

soutenue le

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Contents

List of main presented works

- **Symbolic extensions and uniform generators for topological regular flows**. J. Differential Equations, 267(7):4320–4372, 2019.
- **Uniform generators, symbolic extensions with an embedding, and structure of periodic orbits.** (with Tomasz Downarowicz) J. Dynam. Differential Equations, 31(2):815–852, 2019.
- **Asymptotic h-expansiveness rate of C^∞ maps.** (with Gang Liao, and Jiagang Yang) Proc. Lond. Math. Soc. (3), 111(2):381–419, 2015.
- **Periodic expansiveness of smooth surface diffeomorphisms and applications.** J. Eur. Math. Soc. (JEMS), 22(2):413–454, 2020.
- **Symbolic extensions in intermediate smoothness on surfaces.** Ann. Sci. Éc. Norm. Supér. (4), 45(2):337–362, 2012.
- **Entropy of physical measures for C^∞ dynamical systems.** Comm. Math. Phys., 375(2):1201–1222, 2020.
- **SRB measures for C^∞ surface diffeomorphisms,** arxiv 2021.

Introduction

Dans ce mémoire je présente une partie de mes travaux de recherche reliée à l'étude des systèmes dynamiques de classe C^r et de leurs modèles symboliques, du point de vue de l'entropie.

L'entropie d'un système estime sa complexité en comptant le nombre d'orbites vue par la topologie (entropie topologique) ou par une mesure invariante (entropie de Kolmogorov). Le principe variationnel énonce que l'entropie topologique est le supremum des entropies de Kolmogorov des mesures invariantes. Un système topologique (X, T) est expansif lorsqu'il existe une échelle (uniforme) à laquelle on peut distinguer deux orbites quelconques différentes. L'expansivité et certaines de ses formes faibles entraîne l'existence de mesures d'entropie maximale. Dans les années 2000, Boyle et Downarowicz ont cherché à caractériser les systèmes dynamiques topologiques (X, T) pouvant être codés par des sous-décalages à alphabet fini (pas nécessairement de type fini). Ils ont développé pour cela une nouvelle théorie de l'entropie, qui permet de relier l'existence (et l'entropie) de tels codages à des propriétés d'expansivité entropique de (X, T).

Lorsque l'espace des phases X est une variété compacte lisse, la mesure de Lebesgue est une mesure de référence naturelle pour laquelle on cherche à décrire le comportement statistique du système : quelle est la limite des mesures empiriques $\mu_n^x = \frac{1}{n}\sum_{0\leq k<n}\delta_{T^k x}$ pour Lebesgue presque tout $x \in X$? Pour une mesure invariante ν, l'ensemble des points x pour lesquels μ_n^x converge vers ν est appelé le bassin de ν. Une mesure physique est une mesure dont le bassin est de mesure de Lebesgue positive. Les mesures ergodiques de Sinai-Ruelle-Bowen (SRB) hyperboliques sont des exemples importants de mesures physiques. Une propriété essentielle dans la construction de mesures de Sinai-Ruelle-Bowen pour un système différentiable (M, f) de classe C^2 est la propriété de distorsion bornée : si D_n est un disque "instable" tels que D_n, $f(D_n)$, ... , $f^n(D_n)$ restent bornées alors pour une constante C indépendante de n on a

$$\forall x, y \in D_n,\ \frac{\mathrm{Jac}_x(f^n|_{D_n})}{\mathrm{Jac}_y(f^n|_{D_n})} < C.$$

En théorie ergodique des systèmes différentiables, deux classes de mesures invariantes sont ainsi particulièrement étudiées : les mesures d'entropie maximale et les mesures de Sinai-Ruelle-Bowen. Les systèmes dynamiques uniformément hyperboliques, qui sont désormais bien compris, satisfont les propriétés suivantes :

- ils peuvent être codés par un sous-décalage de type fini à l'aide de partitions de Markov,
- ils admettent un nombre fini de mesures ergodiques d'entropie maximale,
- les points périodiques s'équidistribuent le long des mesures d'entropie maximales,
- ils admettent un nombre fini de mesures ergodiques de Sinai-Ruelle-Bowen hyperboliques dont les bassins forment un ensemble de mesure de Lebesgue totale.

Dans les années 80, Yomdin a résolu la conjecture de Shub pour les systèmes C^∞ en montrant que la croissance locale des volumes est nulle pour ces systèmes. La preuve de Yomdin s'appuie sur des outils de géométrie semi-algébrique. Il montre que les systèmes C^r satisfont certaines propriétés faibles d'expansivité, qui se renforcent avec la régularité r. J'ai établi différents résultats dans ce sens, en particulier l'existence d'extensions symboliques en régularité intermédiaire (Théorème 21) et l'équidistribution des mesures périodiques le long des mesures d'entropie maximale en régularité C^∞ (Théorème 18), pour les difféomorphismes de surface. En appliquant la théorie de Yomdin à la différentielle et non pas au système lui-même j'ai obtenu des propriétés de distorsion bornée qui permettent de comparer l'entropie et les exposants de Lyapunov d'un point de vue physique pour les systèmes C^∞ en toute dimension (Théorème 25). Plus récemment en utilisant ces propriétés de distorsion bornée j'ai montré pour les difféomorphismes de surface (M, f) de classe C^r, que Lebesgue presque tout point avec un exposant supérieur à $\frac{\log \|df\|_\infty}{r}$ est dans le bassin d'une mesure de Sinai-Ruelle-Bowen (Théorème 27).

Ce manuscrit est divisé en trois parties (une par rapporteur!). Dans la première, nous rappelons quelques notions d'expansivité puis nous évoquons la théorie des extensions symboliques pour les systèmes discrets. Nous présentons enfin quelques contributions au sujet : tout d'abord une extension de cette théorie aux flots, puis une notion de générateurs qui permet de voir la théorie des extensions symboliques comme un problème de générateurs.

Dans la seconde partie nous rappelons l'approche de Yomdin pour montrer la conjecture de Shub. Un raffinement de cette théorie nous a permis d'obtenir des estimées quantitatives de l'expansivité entropique dans les classes ultradifférentiables (Théorème 12). Nous présentons enfin une approche alternative à la théorie de Yomdin en dimension 1 qui permet un contrôle plus fin de la géométrie des courbes sous l'action d'un difféomorphisme C^r.

La dernière partie est consacrée aux applications à la théorie ergodique des systèmes de classe C^r avec $r > 1$ déjà évoquées :

- mesures d'entropie maximale en dimension 1 et 2,
- extensions symboliques en régularité intermédiaire,
- mesures physiques pour les difféomorphismes de surface.

Part I

Generators in topological dynamics

Chapter 1

Entropy and Expansiveness

Related personal works : [28],[31]

Contents

1.1 Entropy

The entropy of a dynamical system quantifies the dynamical complexity by counting the number of orbits that can be differentiated at arbitrarily small scales. Depending on the structure of the system (topological or measured), we distinguish two notions of entropy: the topological entropy introduced by Adler, Conrad and Weiss [1] and the measured entropy due to Kolmogorov [64].

When the system is topological, the scale is defined by an open cover (topological approach of Adler, Konheim and McAndrew) or by a positive real number when the phase space is endowed with a metric (Bowen metric approach [13]). When the scale goes to zero, the Bowen entropy coincides with the topological definition (for the topology induced by the metric).

We recall here Bowen's definition. For a compact metric space (X, d) and a continuous application $T : X \circlearrowleft$, we define for any $n \in \mathbb{N}^*$ the dynamical distance d_n as follows:

$$\forall x, y \in X,\ d_n(x, y) = \max_{0 \leq k < n} d(T^k x, T^k y).$$

We will also write $B_n(x, \varepsilon)$ to denote the ball for the metric d_n of radius $\varepsilon > 0$ centered at $x \in X$. Such dynamical balls are called (n, ε)-balls. Finally for $Y \subset X$, we let $r_n(Y, \varepsilon)$ be the minimal number of (n, ε)-balls needed to cover Y.

The **topological entropy** $h_{top}(T)$ of (X, T) is the exponential growth rate in n of the minimal cardinality of a cover by d_n-balls of arbitrarily small radius:

$$h_{top}(T) = \lim_{\varepsilon \to 0} h_{top}(T, \varepsilon)$$

with

$$h_{top}(T, \varepsilon) = \limsup_{n \to +\infty} \frac{1}{n} \log r_n(X, \varepsilon).$$

When the dynamical system is invertible, we can define the bilateral dynamical distances $\overline{d_n}$:

$$\forall x, y \in X, \overline{d_n}(x, y) = \max_{|k|<n} d(T^k x, T^k y)$$

and the associated dynamical balls $\overline{B}_n(x, \varepsilon)$.

For measure preserving system systems $(X, \mathcal{B}, f, \mu)$, the entropy may be defined with finite measurable partitions following Kolmogorov [64] (see also [84]). For such a partition P, let $P^n = \bigvee_{0 \leq k < n} f^{-k} P$ be the n-iterated partition and $P^n(x)$ be the atom of P^n containing a point x of X. We then define the Kolmogorov entropy $h(\mu)$ as follows:

$$h(\mu) = h_f(\mu) = \sup_P h(\mu, P)$$

with

$$h(\mu, P) = \lim_n \frac{1}{n} \int -\log \mu(P^n(x))\, d\mu(x).$$

The Kolmogorov entropy and the topological entropy are related by a variational principle [51, 57, 56]:

$$h_{top}(T) = \sup_{\mu \in \mathcal{M}(X,T)} h(\mu),$$

where $\mathcal{M}(X, T)$ is the set of T-invariant Borel probability measures. An invariant measure realizing the supremum is called a maximal measure or a measure of maximal entropy. Such measures do not always exist. An abstract example of a topological system without maximal measure may be given as follows. Consider a sequence $(X_n, T_n)_{n \in \mathbb{N}}$ of topological systems, such that the topological entropy $h_{top}(T_n)$ goes (strictly) increasingly in n to some $0 < a < +\infty$. Let $X = \{*\} \cup \coprod_n X_n$ be the one point compactification of the X_n's. Then the system $T : X \circlearrowleft$ coinciding with T_n on each X_n and fixing $*$ has no measure of maximal entropy.

1.2 Strong expansiveness

A topological system (X, T) is said to be **positively expansive**, when there exists a scale $\varepsilon > 0$ such that for any $x \in X$, the intersection $\bigcap_{n \in \mathbb{N}^*} B_n(x, \varepsilon)$, is reduced to the singleton x. In other words, all future orbits can be distinguished on the scale ε. In particular $h_{top}(T) = h_{top}(T, \varepsilon) < \infty$. Equivalently a system is positively expansive when there exists a finite open cover $\mathcal{U}$ such that for all $x \in X$ we have $\bigcap_{k \in \mathbb{N}} \mathcal{U}(T^k x) = \{x\}$ for some $\mathcal{U}(T^k x) \in \mathcal{U}$ containing $T^k x$, $k \in \mathbb{N}$. Such a cover is then called a positive **topological generator**.

An invertible topological dynamical system is said to be **expansive** when it satisfies the previous property where we consider bilateral dynamical balls insteal of unilateral ones, i.e. for any $x \in X$ the intersection $\bigcap_{n \in \mathbb{N}^*} \overline{B_n}(x, \varepsilon)$ is reduced to $\{x\}$. One may also define topological generators as in the *positive* case. Symbolic dynamical systems, which are given by the shift on sequences with values in a finite alphabet, are expansive. Among differentiable systems uniformly hyperbolic diffeomorphisms are (robustly) expansive and they are almost the unique one's [69].

The expansiveness generates constraints on the topology of X. Fathi [55] has shown that all expansive dynamics are hyperbolic in some sense. In particular, he deduced that in any expansive system the phase space has finite topological dimension. In the case where the dynamical system is minimal, Mané [70] had previously shown that the system is zero-dimensional.

A zero-dimensional expansive system is topologically conjugate to a subshift. Moreover Krieger [66] showed that given a zero-dimensional expansive system (Z, R), any subshift of finite type, with larger entropy and with as many n-periodic points for any $n \in \mathbb{N}^*$, contained a subshift topologically conjugate to (Z, R).

1.3 Entropy expansiveness

Bowen and Misiurewicz [13, 73] have introduced finer notions of expansiveness. For $\varepsilon > 0$, we let

$$h^*(T, \varepsilon) := \lim_{\delta \to 0} \limsup_n \frac{1}{n} \sup_{x \in X} \log r_n(B_n(x, \varepsilon), \delta).$$

The quantity $h^*(T, \varepsilon)$ estimates the complexity of the system which remains in dynamical balls of size ε. The system is said to be h**-expansive** when there exists $\varepsilon > 0$ with $h^*(T, \varepsilon) = 0$. By an easy compactness argument, one checks that expansive systems are h-expansive. An example of a non expansive but h-expansive system is the time-1 map of the geodesic flow of a surface with negative curvature. More generally, C^1 diffeomorphisms far from homoclinic tangencies are h-expansive [67] (see also [49, 50]).

Finally, a topological system (X, T) is said to be **asymptotically** h**-expansive** when

$$h^*(T, \varepsilon) \xrightarrow{\varepsilon \to 0} 0.$$

We will see in Chapter 5 that C^∞ systems are asymptotically h-expansive but not h-expansive in general. An important consequence of this last property is the upper semicontinuity of the Kolmogorov entropy $h : \mathcal{M}(X,T) \to \mathbb{R}$ on the simplex of invariant probability measures $\mathcal{M}(X,T)$ equipped with the weak-$*$-topology. In particular an asymptotically h-expansive system always admits maximal measures.

The limit $h^*(T) = \lim_{\varepsilon\to 0} h^*(T,\varepsilon)$ is a topological invariant, called the **tail entropy** after Downarowicz [52]. In [28] we define a notion of robust tail entropy. If $\mathcal{C}$ is a family of continuous functions of X in itself, endowed with a topology stronger than the topology of uniform convergence (for example for a smooth manifold X, we can consider the C^r topology on the C^r diffeomorphisms of X), then for $T \in \mathcal{C}$ we define

$$h^*_{\mathcal{C}}(T) = \lim_{\varepsilon\to 0} \limsup_{\mathcal{C}\ni S\to T} h^*(S,\varepsilon).$$

In this framework if $h^*_{\mathcal{C}} = \sup_{T\in\mathcal{C}} h^*_{\mathcal{C}}(T) = 0$ then the topological entropy $h_{top}(T)$ is upper semicontinuous in $T \in \mathcal{C}$ and the simplex $\mathcal{M}_{max}(X,T) \subset \mathcal{M}(X)$ of maximal measures also varies upper semicontinuously in $T \in \mathcal{C}$ for the Hausdorff topology on the set $\mathcal{M}(X)$ of probability measures of X endowed with the weak-$*$ topology.

We show for example in [28] that the set M^1_k of functions f of class $\mathcal{C}^1$ of the interval admitting a partition in at most k-intervals on which f is monotone verifies $h^*_{M^1_k} = 0$. Moreover it is well known that the topological entropy of continuous applications of the interval is lower semicontinuous [72]. It follows that the topological entropy is continuous on M^1_k. This approach gives thus an elementary proof of this result due to Misiurewicz [74].

1.4 Periodic expansiveness

For a topological system, we introduce the notion of periodic expansiveness. Let $\mathrm{Per}(X)$ be the set of periodic points of (X,T). For a subset $\mathcal{P}$ of $\mathrm{Per}(X)$, let $\mathcal{P}_n$ be the subset of n-periodic points of $\mathcal{P}$ for any $n \in \mathbb{N}^*$.

When (X,T) is expansive, one shows easily following Bowen that if the exponential growth $g_{\mathcal{P}} := \lim_n \frac{1}{n}\log \sharp\mathcal{P}_n$ is well defined and is equal to the topological entropy then any weak limit of $\frac{1}{\sharp\mathcal{P}_n}\sum_{x\in\mathcal{P}_n}\delta_x$, when n goes to infinity, is a maximal measure.

As asymptically h-expansiveness is the good generalization of expansiveness for the entropy, we introduced in [31] the property of periodic expansiveness which is adapted to the growth of periodic points. For all $\varepsilon > 0$, we let

$$g^*_{\mathcal{P}}(\varepsilon) := \limsup_n \frac{1}{n} \log \sup_{x\in X} \sharp\mathcal{P}_n \cap B_n(x,\varepsilon).$$

The system (X,T) is said to be **asymptotically $\mathcal{P}$-expansive**, when

$$g^*_{\mathcal{P}}(\varepsilon) \xrightarrow{\varepsilon\to 0} 0.$$

It is shown in [31] that if $g_{\mathcal{P}} := \lim_k \frac{1}{n_k} \log \sharp\mathcal{P}_{n_k}$ is additionally equal to the topological entropy, then the periodic measures equidistribute along the maximum entropy measure, in other words any weak limit of $\frac{1}{\sharp\mathcal{P}_{n_k}} \sum_{x\in\mathcal{P}_{n_k}} \delta_x$, when k goes to infinity, is a maximal measure.

When $\mathcal{P}$ is the set Per of all periodic points, then g^*_{Per} is a topological invariant playing a key role in the theory of perfect generators developed in Chapter 3.

Chapter 2

Symbolic extensions

Related personal works : [23],[20],[29],[38]

Contents

2.1 Problematic

In the previous chapter we mentioned that an expansive zero-dimensional topological dynamical system is topologically conjugate to a subshift over a finite alphabet.

A zero-dimensional system (X, T) always admits a factor given by a subshift of entropy arbitrarily close to $h_{top}(T)$. But does it admit a **symbolic extension**, i.e. a topological extension by a subshift? What can be the entropy of this subshift? A necessary condition is the finiteness of the topological entropy, as the entropy of an extension is always larger than or equal to the entropy of the factor, but this is not sufficient. Boyle built the first examples of finite entropy (zero-dimensional) topological systems without symbolic extension. With Downarowicz, they relate the existence of a symbolic extension with new entropic expansiveness properties of the system.

2.2 Entropic characterization of symbolic extensions

Let us first assume that (X, T) is a zero-dimensional system. We consider a sequence $\mathcal{P} = (P_k)_{k\in\mathbb{N}}$ of clopen partitions such that P_{k+1} is finer than P_k for all k and $\text{diam}(P_k) \xrightarrow{k\to+\infty} 0$. Then we let

$$h_k : \mathcal{M}(X, T) \to \mathbb{R}^+,$$
$$\mu \mapsto h(\mu, P_k).$$

The sequence of functions $(h_k)_k$ is nondecreasing and is converging pointwisely to the measure theoretical entropy function $h : \mathcal{M}(X, T) \to \mathbb{R}^+$. For a bounded function $f : \mathcal{M}(X,T) \to \mathbb{R}^+$ we let $\tilde{f}$ be the smallest upper semicontinuous function larger than or equal to f, i.e. $\tilde{f}(\mu) = \limsup_{\nu\to\mu} f(\nu)$ for any $\mu \in \mathcal{M}(X,T)$. A function $E : \mathcal{M}(X,T) \to \mathbb{R}^+$ is called an **affine superenvelope** when E is affine and satisfies

$$\lim_k (E + h - h_k)\tilde{} = E.$$

When $\pi : (Y,S) \to (X,T)$ is a symbolic extension of (X,T) then one checks easily that the fiber entropy function $h^\pi : \mathcal{M}(X,T) \to \mathbb{R}^+$, defined as $h^\pi(\mu) = \sup_{\pi^*\nu=\mu} h_S(\nu) - h_T(\mu)$ for any $\mu \in \mathcal{M}(X,T)$, is an affine superenvelope.

Boyle and Downarowicz have shown that this property completely characterizes the fiber entropy function of symbolic extensions, in particular if there is an affine superenvelope then (X,T) admits a symbolic extension.

Theorem 1. *[16] The fiber entropy functions h^π of symbolic extensions π are exactly the affine superenvelopes.*

For a general topological system there are many ways to compute the measure theoretical entropy function h as a nondecreasing sequence of functions $(h_k)_k$. For example we may use the formulas due to Brin-Katok, Katok or Newhouse [18, 62, 76]. In these three cases, these sequences $(h_k)_k$ are equivalent in some sense and Downarowicz called them **entropy structures**. The above theorem still holds true for a general system when one replaces the entropy with respect to clopen partitions by such an entropy structure. Moreover for entropy structures, we have the following variational principle for the tail entropy :

Theorem 2. *[20, 52] For any entropy structure $(h_k)_k$ of a topological system (X,T) we have*

$$h^*(T) = \lim_k \sup_{\mu\in\mathcal{M}(X,T)} (h - h_k)(\mu).$$

The symbolic extension π is said **principal** when it preserves the entropy of measures, i.e. $h^\pi = 0$. By the above variational principle, the system is asymptotically h-expansive if and only if the sequence h_k are converging uniformly to the entropy function h when k goes to infinity. Equivalently, the zero function is an affine superenvelope.

Corollary 1. *[17] A topological system is asymptotically h-expansive if and only if it admits a principal symbolic extension.*

The **symbolic extension entropy** $h_{sex}(T)$ is defined as the infimum of the topological entropy of symbolic extensions of (X,T).

2.3 Symbolic extensions for flows

In [29] I have developed a similar theory for flows. A topological flow without fixed points is said to be **regular**. A **symbolic flow** (Y, Ψ) is a (regular) suspension flow over a subshift. A symbolic extension of a topological flow (X, Φ) is a topological extension $\pi : (Y, \Psi) \to (X, \Phi)$ by a symbolic flow.

Theorem 3. *[29, 38] Let (X, Φ) be a continuous regular flow. Then Φ admits a (principal) symbolic extension if and only if so does its time t-map for any $t \neq 0$.*

Ideas of the proof. We first build a principal extension of the flow by a suspension flow over a zero-dimensional system. Then, by using Abramov's like formulas we relate the affine superenvelopes of the time t-map of the flow with the affine superenvelopes of this zero-dimensional system. Finally one easily build a symbolic extension of the suspension flow from a symbolic extension of the discrete zero-dimensional system on the basis. □

In the previous statement we can in fact always choose the roof function of the symbolic flow constant equal to one. A topological extension $\pi : (Y, \Psi) \to (X, \Phi)$ is said **isomorphic** when π is an isomorphism between the measure preserving systems (Y, Ψ, ν) and $(X, \Phi, \pi^*\nu)$ for any $\nu \in \mathcal{M}(X, \Phi)$. A discrete system (X, T) is said to have the **small boundary property** when there is a basis of neighborhoods U with small boundary, i.e. $\mu(\partial U) = 0$ for any $\mu \in \mathcal{M}(X, T)$. For flows, we have introduced a similar concept by working with sections building on works of Bowen and Walters [15]. By using standard smooth transversality argument, we proved this small flow boundary property is satisfied for regular C^2 flows with countably many periodic orbits. Recently, building on works of Lindenstrauss [68] for discrete systems, Gutman and Shi extended this result to C^0 flows. A flow with the small flow boundary property admits an isomorphic extension by a suspension flow over a zero-dimensional system. In particular it follows from the asymptotic h-expansiveness of C^∞ systems (see Corollary 4), that :

Corollary 2. *[29] Any C^∞ regular flow with countably many periodic orbits admits an isomorphic symbolic extension.*

Note that in this last statement we can not in general take the roof function of the symbolic flow to be constant. In the aperiodic case we do not know if one can choose this roof function to be a two step function as in famous Rudolph's work for ergodic flows [80].

Two topological flows (X, ϕ) and (Y, Ψ) are **orbit equivalent** when there is a homeomorphism Λ from X onto Y mapping Φ-orbits to Ψ-orbits by preserving their orientation. For regular flows, it is well known that finiteness/nullity of the topological entropy [77] and expansiveness [15], are invariant properties under orbit equivalence. This is also the case for the existence of symbolic extensions:

Theorem 4. *[29, 38] Existence of symbolic extensions, but also asymptotic h-expansiveness, are preserved under orbit equivalence for regular flows.*

The above theorem is no more true for singular flows, i.e. flows with fixed points. In [38], we consider suspension flows over the two full shift with one fixed point at the zero sequence 0^∞. Then the existence of (principal) symbolic extensions depends on the "flatness" of the roof function at the singularity.

Chapter 3

Perfect generators

Related personal works : [32],[27],[35]

Contents

3.1 Ergodic generators

Let $(X, \mathcal{B}, f, \mu)$ be an invertible ergodic measure preserving system. A finite measurable partition P is called an **ergodic generator** when the iterated partitions $P^{[-n,n]} = \bigvee_{|k|\leq n} f^{-k}P$, $n \in \mathbb{N}$, generate $\mathcal{B}$. In other terms, an ergodic measure preserving system admits a generator with cardinality K if and only if this ergodic measure preserving system is isomorphic to an invariant ergodic measure of the full shift with K letters.

Theorem 5. *[65] Any invertible ergodic non atomic measure preserving system $(X, \mathcal{B}, f, \mu)$ with $h(\mu) < +\infty$ admits a generator P with $\sharp P \leq [e^{h(\mu)}] + 1$.*

The idea of Krieger's proof is to encode with markers the dynamics at a certain scale P_0 while leaving free codes at identifiable places, which we then use to describe the dynamics at smaller scales P_k, $k \in \mathbb{N}$. Here $(P_k)_k$ denotes a sequence of finer and finer partitions with $\bigvee_k P_k = \mathcal{B}$. The number of "bits" needed for the coding at the k^{th}-step is given by the conditional entropy $h(\mu, P_{k+1}|P_k)$. As $h(\mu) = \sum_k h(\mu, P_{k+1}|P_k) < +\infty$, we may iterate the process and the limit defines the desired embedding of the ergodic system into the full shift with $[e^{h(\mu)}] + 1$ letters.

Another way of stating Theorem 5 is to say that the full K-shift σ, $K \in \mathbb{N}^*$, is **universal** in the sense that any aperiodic ergodic invertible measure preserving system of entropy less than $\log K = h_{top}(\sigma)$ is isomorphic to an invariant measure of σ.

A topological system satisfies a **specification property** when we can glue finitely many bounded pieces of orbits. More precisely for any $\varepsilon > 0$ and any finite collection of finite orbits

$$T^{a_1}x_1, T^{a_1+1}x_1, ..., T^{b_1}x_1$$
$$\cdots$$
$$T^{a_p}x_p, T^{a_p+1}x_p, ..., T^{b_p}x_p$$

with $a_1 < b_1 < a_2 < ... < b_p$ there exists $x \in Y$ with $d(T^k x, T^k x_i) < \varepsilon$ for $k \in \bigcup_i [a_i, b_i]$ provided that $a_{i+1} - b_i \geq L(b_{i+1} - a_{i+1})$ for some function $L = L_\varepsilon$.

When L is constant and independent of ε, we speak of strong specification. If L_ε verifies $\lim_{n\to+\infty} L_\varepsilon(n)/n = 0$ for all ε, we say that the system has the weak specification property. Subshifts of finite type satisfy the strong specification property.

Generalizing a result of Quas and Soo [79] (see also [47]), we have obtained the following result:

Theorem 6. *[32] Any invertible topological system satisfying the weak specification property is universal.*

We also showed a stronger topological universality property for subshifts satisfying the strong specification property.

3.2 Symbolic extension with an embedding

Let (X, T) be a topological invertible system. A symbolic extension $\pi : (Y, S) \to (X, T)$ is said to be **with an embedding** if there exists a Borel section, i.e. a Borel map $\psi : X \to Y$ satisfying $\psi \circ T = S \circ \psi$ and $\pi \circ \psi = \mathrm{Id}_X$. When a system admits a principal symbolic extension, we can always choose this extension to be aperiodic. For symbolic extensions with an embedding, entropy is not the only constraint, one must also take into account the periodic points.

A finite Borel measurable partition P of X is called a **perfect generator** if the diameter of the iterated partitions $P^{[-n,n]} = \bigvee_{|k|\leq n} T^{-k}P$ goes to 0 when n goes to infinity. Remark that if P is moreover a clopen partition then P is a topological generator and the system is expansive and conjugated to a subshift. Perfect generators and symbolic extensions with an embedding are related as follows :

Proposition 1. *[35] An invertible topological system (X, T) admits a perfect generator if and only if it admits a symbolic extension with an embedding.*

3.3 Asymptotic expansive systems

A system is said to be **asymptotically expansive** when it is asymptotically h-expansive and asymptotically Per-expansive. For a topological system (X, T) we consider the following invariant:

$$g_{\mathrm{Per}}^{sup}(T) = \sup_n \frac{\log \sharp \mathrm{Per}_n}{n}.$$

Observe that $g_{\mathrm{Per}}^{sup}(T)$ is finite when (X, T) is asymptotically Per-expansive. By adapting Krieger's proof of ergodic generators to the topological setting, we obtained :

Theorem 7. *[27] Let (X, T) be an invertible topological system. The following properties are equivalent:*

- *(X, T) admits a perfect generator P with null boundaries,*
- *(X, T) has the small boundary property and (X, T) is asymptotically expansive.*

Moreover we can chose P with $\sharp P \leq e^{\max\left(h_{top}(T), g_{\mathrm{Per}}^{sup}(T)\right)} + 1$.

In this case, the symbolic extension associated to the perfect generator is an isomorphic extension.

3.4 The general case

With Downarowicz we generalized the previous result to any topological systems with the small boundary property as follows.

Theorem 8. *Let (X, T) be an invertible topological system with the small boundary property. The following assertions are equivalent:*

- *(X, T) admits a perfect generator P,*
- *(X, T) admits a symbolic extension and $g_{\mathrm{Per}}^{sup}(T) < +\infty$.*

Moreover, in this case we can choose P with

$$\sharp P \leq e^{max\left(h_{sex}(T) + g_{\mathrm{Per}}^{*}(T), g_{\mathrm{Per}}^{sup}(T)\right)} + 1.$$

Ideas of the proof : We first prove that for an aperiodic system any affine superenvelope is a the fiber entropy of a symbolic extension with an embedding. Then we deal with the periodic case by building for any topological system an aperiodic zero-dimensional extension of (X, T), which is isomorphic on aperiodic measures, while each periodic orbit of (X, T) lifts to a collection of odometers. □

Part II

Reparametrization lemmas

Chapter 4

Yomdin's theory

Related personal works : [19],[37]

Contents

4.1 The Algebraic Lemma

A semi-algebraic set is a subset of some Euclidean space $\mathbb{R}^d$, which can be defined by a finite union of polynomial equalities and inequalities. To estimate its algebraic complexity one can define the degree of such a set as the sum of the total degrees of these polynomials. By a well-known result of semi-algebraic geometry, any semi-algebraic A set may be decomposed into *cells* (Theorem 2.3.6 in [11]), in particular $A = \bigcup_{\phi\in\Phi} \mathrm{Ima}(\phi)$ with Φ being a finite family of real-analytic maps $\phi : (0,1)^d \to A$, where $\mathrm{Ima}(\phi)$ denotes the image of ϕ. For a C^l map $f = (f_1, \cdots, f_d) : U \to \mathbb{R}^d$ defined on an open set U of $\mathbb{R}^k$, we let $\|d^l f\| = \max_{1\leq i\leq d} \max_{\alpha\in\mathbb{N}^k,\ |\alpha|=l} \sup_{x\in U} |\partial^\alpha f_i(x)|$. A C^r map $\phi : (0,1)^k \to \mathbb{R}^d$ is called **a** C^r **unit** when

$$\|\phi\|_r := \max_{1\leq l\leq r} \|d^l\phi\| \leq 1$$

Given an order of smoothness $r \in \mathbb{N}$, one can define the r-smooth complexity of A as the minimal cardinality of a family Φ of C^r units satisfying $A = \bigcup_{\phi\in\Phi} \phi(0,1)^d$.

Roughly speaking the Algebraic Lemma states that the r-smooth complexity of a semi-algebraic set depends only on the smoothness r, its algebraic complexity and its diameter (not on the coefficient of the polynomials involved in its definition). We state below the version of the Algebraic Lemma used in the dynamical applications. Fix the order of smoothness $r \in \mathbb{N}$, the dimension d of the ambient space and some intermediate dimension $0 \leq k \leq d$. We call

reparametrization of $(0,1)^k$ any real-analytic map from $(0,1)^k$ to $(0,1)^k$. We also denote by B the closed unit Euclidean ball in $\mathbb{R}^d$.

Lemma 1 (Algebraic Lemma [58, 87]). *For any $r \in \mathbb{N}$ and any $P : (0,1)^k \to \mathbb{R}^d$ with $P = (P_1, \cdots, P_d) \in \mathbb{R}^d[X_1, \cdots, X_k]$ and $\max_i \deg P_i \leq r$, there exists a family $\Theta = \{\theta\}$ of reparametrizations of $(0,1)^k$ such that :*

1. $\bigcup_{\theta \in \Theta} \mathrm{Ima}(\theta) = P^{-1}(B)$,
2. $\forall \theta \in \Theta$, θ *and* $P \circ \theta$ *are* C^r *units,*
3. $\sharp\Theta \leq \mathfrak{C}$ *with* $\mathfrak{C} = \mathfrak{C}(k, d, r)$.

With Yang and Liao [37], we obtained a polynomial estimate of $\mathfrak{C}$ for $k = d = 1$ which was generalized by Binyamini and Novikov in higher dimensions [10].

Lemma 2. *[37, 10] There exists $R_{k,d} \in \mathbb{R}[X]$, s.t.*

$$\mathfrak{C}(k, d, r) = R_{k,d}(r).$$

In the next chapter we give new dynamical applications of this improved version of the Algebraic Lemma. This polynomial estimate of $\mathfrak{C}$ has also been useful in diophantine geometry [48, 10].

4.2 Local complexity of C^r maps

Following Yomdin, we show now how the Algebraic Lemma allows to estimate the local complexity of smooth maps.

We consider a C^r, $2 \leq r \in \mathbb{N}$, smooth map $f : \mathbb{R}^d \supset B \to \mathbb{R}^d$ with $\max_{l=2,\cdots,r} \|d^l f\| \leq \|df\|$. The next lemma estimates the local complexity of the image by f of a C^r unit.

Lemma 3. *Let $\mathfrak{s} : (0,1)^k \to \mathbb{R}^d$ be a C^r unit with $r \in \mathbb{N}^*$. There exists a family $\Theta = \Theta(\mathfrak{s}) = \{\theta\}$ of reparametrizations of $(0,1)^k$ such that*

- $\bigcup_{\theta \in \Theta} \mathrm{Ima}(\theta) \supset (f \circ \mathfrak{s})^{-1}(B)$,
- $\forall \theta \in \Theta$, $\mathfrak{s} \circ \theta$ *and* $f \circ \mathfrak{s} \circ \theta$ *are* C^r *units,*
- $\sharp\Theta \leq \mathfrak{D} \max(\|df\|, 1)^{\frac{k}{r}}$ *with* $\mathfrak{D} = \mathfrak{D}(k, d, r)$.

Proof. We may assume $\|d^r(f \circ \mathfrak{s})\| \leq 1$. Indeed we can consider subcubes C of $(0,1)^k$ of size $c = \max(\|d^r(f \circ \mathfrak{s})\|, 1)^{-1/r}$ covering $(0,1)^k$ and $\sharp\{\mathsf{C}\} \leq (c^{-1} + 1)^k$. Let $\psi_{\mathsf{C}} : (0,1)^k \to \mathsf{C}$ be an affine parametrization of C. Then $\|d^r(f \circ \mathfrak{s} \circ \psi_{\mathsf{C}})\| = c^r \|d^r(f \circ \mathfrak{s})\| \leq 1$ and we can take $\Theta(\mathfrak{s}) = \bigcup_{\mathsf{C}} \Theta(\mathfrak{s} \circ \psi_{\mathsf{C}})$. By Faa-di Bruno formula, $\|d^r(f \circ \mathfrak{s})\|$ is bounded by a universal polynomial in

$\|d^k\mathfrak{s}\|$ and $\|d^k f\|$, $1 \leq k \leq r$. As $\mathfrak{s}$ is a C^r unit and $\max_{k=2,\cdots,r}\|d^k f\| \leq \|df\|$ by assumption, we get $\|d^r(f\circ\mathfrak{s})\| \leq \mathfrak{C}\|df\|$ with $\mathfrak{C} = \mathfrak{C}(k,d,r)$.

If $\|d^r(f\circ\mathfrak{s})\| \leq 1$, we consider the $(r-1)$-Lagrange polynomial P of $f\circ\mathfrak{s}$ at some $x_0 \in (0,1)^k$. Let $\Theta = \{\theta\}$ be the family of reparametrizations as in the Algebraic Lemma for $\frac{P}{2} : (0,1)^k \to \mathbb{R}^d$. Then we have

- $(f\circ\mathfrak{s})^{-1}(B) \subset P^{-1}(2B) = \bigcup_{\theta\in\Theta}\mathrm{Ima}(\theta)$,
- $\|f\circ\mathfrak{s}\circ\theta\|_r \leq \|P\circ\theta\|_r + \|(f\circ\mathfrak{s}-P)\circ\theta\|_r \leq \mathfrak{E} = \mathfrak{E}(k,d,r)$.

We get the required family by composing each θ with affine contractions of rate $\mathfrak{E}^{-1}$. □

4.3 Dynamical reparametrization lemma

We may iterate Lemma 3 to reparametrize dynamical balls of a C^r smooth system (M,f). As an intermediate step we consider non autonomous dynamical systems and apply it later to the non autonomous system given by the local dynamics along the future f-orbit of a given point $x \in M$.

4.3.1 Non autonomous C^r dynamical version

Let $2 \leq r \in \mathbb{N}$ and $0 < \alpha \leq 1$. We consider a family $\mathcal{F} = (\mathsf{f}_m)_{m\in\mathbb{N}^*}$ of C^r maps from B to $\mathbb{R}^d$ with $\mathsf{f}_0 = \mathrm{Id}_B$ and $\|d^l\mathsf{f}_m\| \leq \|d\mathsf{f}_m\| < +\infty$ for $l = 2,\cdots,r$, for all m. We consider the associated non autonomous system by letting :

$$\forall m \in \mathbb{N},\ \mathsf{f}^m = \mathsf{f}_m \circ \cdots \circ \mathsf{f}_0 : B_m \to \mathbb{R}^d$$

with $B_m = B_m(\mathcal{F})$ being the dynamical ball $B_m := \bigcap_{0\leq l<m}(\mathsf{f}^l)^{-1}B$. For a α-Hölder map $\phi : B \to \mathbb{R}$ we let $|\phi|_\alpha = \sup_{x\neq y}\frac{|\phi(x)-\phi(y)|}{\|x-y\|^\alpha}$ be its α-Hölder semi-norm. We consider a family $\Phi = (\phi_m)_{m\in\mathbb{N}}$ of α-Hölder maps from B to $\mathbb{R}$ with $\sup_m|\phi_m|_\alpha \leq 1$. We denote the associated Birkhoff sums by $S_m\Phi = \sum_{l=0}^{m-1}\phi_l\circ\mathsf{f}^l$, $m \in \mathbb{N}$.

Lemma 4. *Let $\mathfrak{s} : (0,1)^k \to \mathbb{R}^d$ be a C^r map. For any $m \in \mathbb{N}^*$ there exists a family $\Theta_m = \{\theta_m\}$ of reparametrizations of $(0,1)^k$ satisfying :*

- $\bigcup_{\theta_m\in\Theta_m}\mathrm{Ima}(\theta_m) \supset \mathfrak{s}^{-1}(B_m)$,
- $\forall\theta_m \in \Theta_m\ \forall 0 \leq l < m$, $\mathsf{f}^l\circ\mathfrak{s}\circ\theta_m$ *is a C^r unit,*
- $\forall\theta_m \in \Theta_m,\ |S_m\Phi\circ\mathfrak{s}\circ\theta_m|_\alpha \leq 1$,
- $\sharp\Theta_m \leq \mathfrak{D}^m\max\left(\|d^r\mathfrak{s}\|,1\right)^{\frac{k}{r}}\prod_{l=0}^{m-1}\max\left(\|d\mathsf{f}_l\|,1\right)^{\frac{k}{r}}$ *with $\mathfrak{D} = \mathfrak{D}(k,d,r)$.*

Proof. We argue by induction on m. For each $\theta_m \in \Theta_m$, we apply Lemma 3 to f_m and to the C^r unit $\mathsf{f}^{m-1} \circ \mathfrak{s} \circ \theta_m$. Take $\Theta'_{m+1} = \{\theta_m \circ \theta \mid \theta_m \in \Theta_m,\ \theta \in \Theta(\mathsf{f}^{m-1} \circ \mathfrak{s} \circ \theta_m)\}$. Again by Faa-di Bruno formula, for any $\theta'_{m+1} \in \Theta'_{m+1}$ and for any $0 \leq l \leq m$ we have $\|\mathsf{f}^l \circ \mathfrak{s} \circ \theta'_{m+1}\|_r \leq \mathfrak{C}$ for some constant $\mathfrak{C} = \mathfrak{C}(k, d, r)(\geq 1)$. Moreover $|\phi_m \circ \mathsf{f}^l \circ \mathfrak{s} \circ \theta'_{m+1}|_\alpha \leq |\phi_m|_\alpha \|d(\mathsf{f}^l \circ \mathfrak{s} \circ \theta'_{m+1})\|^\alpha \leq \mathfrak{C}^\alpha$, thus $|S_{m+1}\Phi \circ \mathfrak{s} \circ \theta'_m|_\alpha \leq \mathfrak{C}^\alpha + 1 \leq 2\mathfrak{C}$. We get the required reparametrizations θ_m by composing each θ'_m with affine contractions of rate $c = (2\mathfrak{C})^{-1}$. □

4.3.2 Dynamical reparametrization lemma for C^∞ smooth systems

We consider now a usual C^∞ system, i.e. a C^∞ map $f : M \circlearrowleft$ on a compact smooth Riemannian manifold, with a α-Hölder potential $\phi : M \to \mathbb{R}$ with $0 < \alpha \leq 1$. We let $S_n\phi = \sum_{l=0}^{n-1} \phi \circ f^l$, $n \in \mathbb{N}$, be the associated Birkhoff sums. Let $\sigma : (0,1)^k \to M$ be a C^∞ map with $\|d^r\sigma\| < +\infty$ for all $r \in \mathbb{N}$.

Lemma 5. *For all $\gamma > 0$, there exists $\varepsilon = \varepsilon(f, \phi, \gamma)$ and $C = C(f, \phi, \sigma, \gamma) > 0$ such that for all $x \in M$ the following properties hold.*

For any $n \in \mathbb{N}^*$ there exists a family $\Theta_n = \{\theta_n\}$ of reparametrizations of $(0,1)^k$ satisfying :

- $\bigcup_{\theta_n \in \Theta_n} \mathrm{Ima}(\theta_n) \supset \sigma^{-1}(B_n(x, \varepsilon))$,
- $\forall \theta_n \in \Theta_n\ \forall 0 \leq l < n,\ \|d(f^l \circ \sigma \circ \theta_n)\| \leq 1$,
- $\forall \theta_n \in \Theta_n,\ \forall t, s \in \mathrm{Ima}(\theta_n),\ |S_n\phi \circ \sigma(t) - S_n\phi \circ \sigma(s)| \leq 1$,
- $\sharp\Theta_n \leq Ce^{\gamma n}$.

The third item may be seen as a weak Bowen property for ϕ on σ. Recall the Bowen property for the potential ϕ is defined as follows [14] :

$$\exists \varepsilon > 0\, \exists C > 0 \text{ such that}$$

$$\forall n \in \mathbb{N}\, \forall y \in B_n(x, \varepsilon),\ |S_n\phi(x) - S_n\phi(y)| < C.$$

Lemma 5 is deduced from its non autonomous version Lemma 4 as follows. To avoid some technical points we assume M is the torus $M = \mathbb{R}^d/\mathbb{Z}^d$. Fix $x \in \mathbb{R}^d$ and denote by $\overline{x} \in \mathbb{R}^d/\mathbb{Z}^d$ its projection on the torus. For $\varepsilon > 0$ we let $\psi^\varepsilon_{\overline{x}} = \overline{x + \varepsilon \cdot} : \mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$. Without loss of generality we may assume $\varepsilon < 1/2$ and $\mathrm{Ima}(\sigma) \subset B(x, 2\varepsilon)$. Then we let

$$\mathfrak{s} = (\psi^\varepsilon_{\overline{x}})^{-1} \circ \sigma : (0,1)^k \to \mathbb{R}^d.$$

We consider the non-autonomous system associated to the local dynamics of f^p, $p \in \mathbb{N}^*$, at x given by $\mathcal{F} = (\mathsf{f}_m)_m$ with

$$\mathsf{f}_m = \left(\psi^{\varepsilon}_{f^{p(m+1)}\overline{x}}\right)^{-1} \circ f^p \circ \psi^{\varepsilon}_{f^{pm}\overline{x}}.$$

Finally the corresponding potentials $\Phi = (\phi_m)_m$ are defined as

$$\phi_m = \sum_{l=0}^{p-1} \phi \circ f^l \circ \psi^{\varepsilon}_{f^{pm}\overline{x}}.$$

Observe that

$$\forall s \geq 1,\ \sup_m \|d^s \mathsf{f}_m\| = O(\varepsilon^{s-1})$$
$$\text{and } \sup_m |\phi_m|_\alpha = O(\varepsilon^\alpha) \text{ uniformly in } x \in M.$$

Proof of Lemma 5. Choose r, p, then ε with respect to a small fixed error term $\gamma > 0$ such that

- $r \in \mathbb{N}^*$ with $\|df\|^{k/r} < e^{\gamma/2}$,
- $p \in \mathbb{N}^*$ with $\mathfrak{D}^{1/p} < e^{\gamma/2}$, where $\mathfrak{D}$ is the constant in Lemma 4,
- $\varepsilon > 0$ with $2\varepsilon \max(\|df^p\|, 1) < 1$, $\|\mathsf{f}_m\|_r = \|d\mathsf{f}_m\| \leq \|df^p\|$ and $|\phi_m|_\alpha \leq 1$ for all $m \in \mathbb{N}$ with $\mathcal{F} = (\mathsf{f}_m)_m$ and $(\phi_m)_m$ as above.

For $n = pm$, we have

$$B^f_n(\overline{x}, \varepsilon) \subset \psi^{\varepsilon}_{\overline{x}}(B_m(\mathcal{F})) \text{ and}$$
$$S_n\phi = S_m\Phi \circ (\psi^{\varepsilon}_{\overline{x}})^{-1}.$$

Let $(\Theta_m(\mathcal{F}))_m$ be the families of reparametrizations given by Lemma 4 applied to $\mathcal{F}$. The family $\Theta'_n = \{\theta'_n = \theta_m,\ \theta_m \in \Theta_m(\mathcal{F})\}$ satisfies the conlusion of Lemma 5 for $l, n \in p\mathbb{N}$, as we have for some C independent of $n = pm$:

$$\begin{aligned}\sharp\Theta'_n &\leq \mathfrak{D}^m \prod_{l=0}^{m-1} \max\left(\|\mathsf{f}_l\|_r, 1\right)^{\frac{k}{r}},\\ &\leq \mathfrak{D}^m \max\left(\|df\|, 1\right)^{n\frac{k}{r}},\\ &\leq Ce^{\gamma n}.\end{aligned}$$

One easily concludes the proof in the general case (for any l and n) by composing the reparametrizations of Θ'_n with affine contractions of rate depending only on p, r, d.

□

Chapter 5

h-expansiveness of C^∞ smooth systems

Related personal works : [37]

Contents

5.1 Shub's entropy conjecture

Let M be a compact manifold and let $f : M \circlearrowleft$ be a continuous map. We may consider the map f_* induced on its homology groups, $f_* : H_*(M) = \oplus_i H_i(M, \mathbb{R}) \circlearrowleft$. We let $\rho(f_*)$ be the spectral radius of f_*. Shub has conjectured that:

Conjecture 1 (Shub). *For any C^1 smooth system,*

$$h_{top}(f) \geq \log \rho(f_*).$$

The conjecture is known to hold true under some general expansive properties, e.g. :

- for diffeomorphisms C^1 far from homoclinic tangencies (Liao-Viana-Yang [67]).
- for C^∞ systems (Yomdin [87], see below).

Manning [70] has proved that $h_{top}(f) \geq \rho(f_*^1)$ for general continuous map f, where $\rho(f_*^1)$ is the spectral radius of the action induced on the first homology group $H_1(M)$. Finally note there are Lipschitz counter-examples [78]. The conjecture is still open in the general case (even for C^2 systems).

5.2 Volume growth

For a smooth Riemannian manifold $(M, \|\cdot\|)$ we let $\Lambda^k TM$ be the k^{th} exterior power tangent bundle endowed with the Riemannian structure inherited from $(M, \|\cdot\|)$. If $f : M \to N$ is a smooth map between smooth manifolds M and N we let $\Lambda^k df : \Lambda^k TM \to \Lambda^k TN$ be the induced map.

Fix a C^∞ smooth Riemannian manifold $(M, \|\cdot\|)$. We let $\mathcal{D}(M)$ be the set of C^∞ maps $\sigma : (0,1)^k \to M$, $k \leq \dim(M)$, with $\|d^r\sigma\| < +\infty$ for all $r \in \mathbb{N}$. Given a C^1 map $f : M \circlearrowleft$ and a map $\sigma : (0,1)^k \to M$ in $\mathcal{D}(M)$ we may define the volume growth $v(\sigma)$ of σ as the exponential growth rate in n of the k-volume of $f^n \circ \sigma$:

$$\begin{aligned} v(\sigma, f) &= \limsup_n \frac{1}{n} \log \mathrm{vol}_k(f^{n-1} \circ \sigma), \\ &= \limsup_n \frac{1}{n} \log \int_{(0,1)^k} \|\Lambda^k d_t(f^n \circ \sigma)\| \, dt, \end{aligned}$$

We can also consider local quantities as follows

$$\begin{aligned} \forall \varepsilon > 0, \ v^*(\sigma, f, \varepsilon) &= \limsup_n \frac{1}{n} \log \sup_{x \in M} \mathrm{vol}_k(f^n \circ \sigma|_{\sigma^{-1}B_n(x,\varepsilon)}), \\ &= \limsup_n \frac{1}{n} \log \sup_{x \in M} \int_{\sigma^{-1}B_n(x,\varepsilon)} \|\Lambda^k d_t(f^n \circ \sigma)\| \, dt. \end{aligned}$$

Then we let

$$v(f) = \sup_{\sigma \in \mathcal{D}(M)} v(\sigma, f),$$

$$v^*(f) = \overset{\searrow}{\lim_{\varepsilon \to 0}} \sup_{\sigma \in \mathcal{D}(M)} v^*(\sigma, f, \varepsilon).$$

The volume growth may be compared with the topological entropy. Firstly its is easily seen that

$$v(\sigma, f) \leq h_{top}(f, \varepsilon) + v^*(\sigma, f, \varepsilon),$$

thus

$$v(f) \leq h_{top}(f) + v^*(f). \tag{5.1}$$

Moreover building on Pesin's theory, Newhouse has proved the following theorem:

Theorem 9. *[75, 76] For any C^{1+} map $f : M \circlearrowleft$ we have :*

- $h_{top}(f) \leq v(f)$,
- $\forall \mu \in \mathcal{M}(M, f), \ \limsup_{\nu \to \mu} h(\nu) \leq h(\mu) + v^*(f)$.

Finally by using De Rham cohomology, one checks easily that

$$\rho(f_*) \leq v(f). \tag{5.2}$$

5.3 Local volume growth and tail entropy of smooth systems

Theorem 10. *[87] For any C^∞ map $f : M \circlearrowleft$, we have*

$$v^*(f) = 0.$$

The entropy conjecture for C^∞ systems then follows from (5.2) and (5.1).

Proof. Theorem 10 is a direct consequence of Lemma 5 in the previous chapter. Indeed, with the notations of this lemma, we have $\|d(f^{n-1} \circ \sigma \circ \theta_n)\| \leq 1$ for any $\theta_n \in \Theta_n$, therefore

$$\begin{aligned} \mathrm{vol}\left(f^{n-1} \circ \sigma|_{\sigma^{-1}B_n(x,\varepsilon)}\right) &\leq \sum_{\theta_n \in \Theta_n} \mathrm{vol}\left(f^{n-1} \circ \sigma \circ \theta_n\right), \\ &\leq \sharp\Theta_n, \\ &\leq Ce^{\gamma n}. \end{aligned}$$

□

As a consequence of Theorem 9, Newhouse obtained:

Corollary 3. *[75] For any C^∞ map $f : M \circlearrowleft$, we have*

- *$h_{top}(f) = v(f)$,*
- *the Kolmogorov entropy $h : \mathcal{M}(M, f) \to \mathbb{R}$ is upper semi-continuous. In particular there exists a measure of maximal entropy.*

Buzzi observed that the conditions $\|d(f^k \circ \sigma \circ \theta_n)\| \leq 1$ for $k = 0, \cdots, n-1$ implies that the (n, δ)-balls centered at $\sigma \circ \theta_n(x)$ with $x \in (0, 1)^k \cap \delta\mathbb{Z}^k$ are covering the image of $\sigma \circ \theta_n$. Therefore with the notations of Lemma 5, we get

$$\forall n \in \mathbb{N}, \forall x \in M, \forall \delta > 0,\ r_n\left(B_n(x, \varepsilon), \delta\right) \leq \delta^{-k}\sharp\Theta_n,$$

then

$$h^*(f, \varepsilon) \leq \limsup_n \frac{1}{n} \log \sharp\Theta_n \leq \gamma.$$

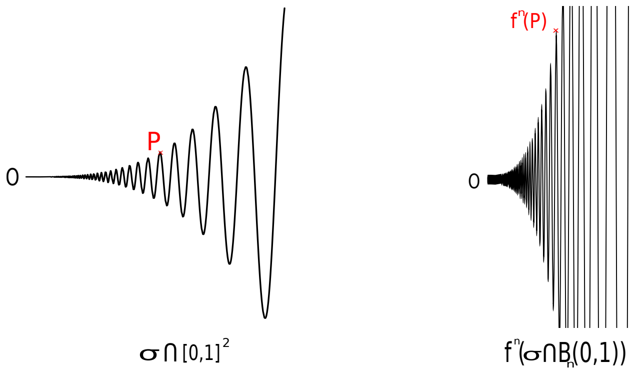
Corollary 4. *[39] For any C^∞ system $f : M \circlearrowleft$, we have*

$$h^*(f) := \lim_{\varepsilon \to 0} h^*(f, \varepsilon) = 0.$$

In fact, for the class $\mathcal{C} = C^\infty(M)$ of C^∞ maps on M endowed with the C^∞ topology, the robust tail entropy $h^*_{\mathcal{C}}$ is zero. For the sake of simplicity we only focused here on C^∞ systems, but there are C^r versions, $1 \leq r < +\infty$, of all these results, e.g. with $R(f) = \lim_n \frac{\log^+ \|df^n\|}{n}$ we have for $f : M \circlearrowleft C^r$:

$$\max\left(v^*(f), h^*(f)\right) \leq \frac{\dim(M)}{r} R(f).$$

These upperbounds in intermediate smoothness are essentially sharp. Moreover there are C^r examples without maximal measures [71, 39]. We recall below an easy example of a C^∞ map $f : \mathbb{R}^2 \circlearrowleft$ with a C^r, but not C^{r+1} curve, $\sigma : (0,1) \to \mathbb{R}^2$ with $\lim_{\varepsilon\to 0} v^*(\sigma, f, \varepsilon) > 0$. Let $f : \mathbb{R}^2 \circlearrowleft$ be the linear map given by $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ with $\lambda > 1$ and let $\sigma : [-1,1] \ni t \mapsto (t, t^{2r+1}\sin(1/t))$. Fix $\varepsilon > 0$ arbitrarily small and let P be the last top of a branch of σ such



that $f^n P$ lie in the square $[-\varepsilon, \varepsilon]^2$. Then we have $y_{f^n(P)} = x_P^{2r+1} \times \lambda^n \simeq 1$ and there are $\simeq 1/x_P$ disconnected branches in $f^n\left(\sigma \cap B_n(0,\varepsilon)\right)$ each of size larger than ε. Therefore

$$v^*(\sigma, f, \varepsilon) \geq \lim_n \frac{\log(1/x_P)}{n} = \frac{\log \lambda}{2r+1}.$$

5.4 Rate of convergence of the tail entropy

In [37] we study the rate of convergence of $h^*(f,\varepsilon)$ to zero for C^∞ systems (M, f). We show in particular that the convergence may be arbitrarily slow :

Theorem 11. *[37] For any function $l : (0,+\infty) \to (0,+\infty)$ with $l(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$, there is a C^∞ smooth diffeomorphism f with $h^*(f,\varepsilon) \geq l(\varepsilon)$ for ε small enough.*

However if we fix some bounds on the supremum norms of the k-derivatives, i.e. in a given ultradifferentiable class $C_{\mathcal{A}}(M) = \{f \in C^\infty(M),\ \forall k\ \|d^k f\| \leq A_k\}$ with a given sequence $\mathcal{A} = (A_k)_k \in (\mathbb{R}^+)^{\mathbb{N}}$, we manage to give explicit rates of convergence for $h^*(f,\varepsilon)$ for $f \in C_{\mathcal{A}}(M)$ in terms of $\mathcal{A}$. In particular for analytic maps we obtain :

Theorem 12. *[88, 37, 10] Let $f : M \circlearrowleft$ be a real-analytic map. Then there is $C = C(\|df\|)$ such that for $\varepsilon > 0$ small enough :*

$$h^*(f, \varepsilon) \leq C \frac{\log(|\log \varepsilon|)}{|\log \varepsilon|}.$$

The above theorem was first proved by Yomdin [88] for surface diffeomorphisms by using Bernstein inequalities. With Liao and Yang [37], we gave another complete proof in this setting. The higher dimensional case follows straightforwardly from our proof and the polynomial estimate of the constant $\mathfrak{C}$ in Lemma 2 for $d > 1$, which was proved later by Binyamini-Novikov [10].

The main idea of Theorem 12 in [37] consists in observing that in the proof of Lemma 5 for C^∞ systems, for any scale $\varepsilon > 0$ we may choose (thanks to an explicit estimate in r of the constant $\mathfrak{D}$) an optimal order of smoothness $r = r(\varepsilon)$ for which we can apply the C^r reparametrization Lemma 4.

In [37] we manage to build optimal examples in most ultradifferentiable classes, but we only manage to get real analytic examples f with $h^*(f, \varepsilon) \geq \frac{1}{|\log \varepsilon|}$. These examples are just given by analytic maps with a quadratic homoclinic tangency. We expect this last bound to be optimal, that is we can replace $\frac{\log(|\log \varepsilon|)}{|\log \varepsilon|}$ by $\frac{1}{|\log \varepsilon|}$ in Theorem 12, but our proof does not allow to reach such an estimate.

Chapter 6

Another approach to control the geometry of curves

Related personal works : [24],[33]

Contents

6.1 Bounded curves

Following [24, 33], a C^r, $2 \le r \in \mathbb{N}$, smooth curve $\sigma : (-1,1) \to \mathbb{R}^2$ is said :

- **r-bounded** when
$$\max_{k=2,\cdots,r} \|d^k\sigma\| \le \frac{1}{10}\|d\sigma\|,$$
- **r-strongly bounded** when σ is r-bounded and $\|d\sigma\| \le 1$,
- **to have r-bounded geometry** when σ is r-strongly bounded and satisfies $\|\sigma'(0)\| \ge \frac{1}{10}$.

We fix now the order of smoothness r, so that we will just speak of bounded curves. We list below some elementary properties of such curves.

Proposition 2. *[33]*

1. *A bounded curve σ has bounded distorsion:*
$$\forall t, s \in (-1,1),\ \frac{\|\sigma'(t)\|}{\|\sigma'(s)\|} \le 3/2,$$
and such a curve lies in a cone:
$$\forall t, s \in (-1,1),\ \angle \sigma'(t), \sigma'(s) \le \frac{\pi}{6},$$

2. *a strongly bounded curve* σ *is a* C^r *unit,* $\|\sigma\|_r \leq 1$,

3. *the intersection of the image of a bounded curve with a ball of radius* $1/4$ *is the image of a strongly bounded curve,*

4. *a curve with bounded geometry has bounded curvature. More precisely, the image of* σ *is the graph of a* C^r *map* $\psi : \mathbb{R}\sigma'(0) \supset I \to \sigma'(0)^\perp$ *with* I *being a segment of length larger than* $1/100$ *and with* $\|\psi\|_r$ *bounded by a constant depending only on* r.

6.2 Image of a bounded curve

We consider a C^r smooth map $f : B \to \mathbb{R}^2$ with $\max_{k=2,\cdots,r} \|d^k f\| \leq \|df\|$. For $\hat{x} = (x, v) \in PTB = B \times P\mathbb{R}^2$ we let $\phi_f(\hat{x}) = \log\left(\frac{\|d_x f(v)\|}{\|v\|}\right)$. For a C^1 embedded curve $\mathfrak{s} : (-1, 1) \to B$ we let $\hat{\mathfrak{s}} : (-1, 1) \to PTB$, $t \mapsto \left(\mathfrak{s}(t), \frac{\mathfrak{s}'(t)}{\|\mathfrak{s}'(t)\|}\right)$, be the induced map on the projective space PTB.

Lemma 6. *If* $\mathfrak{s} : (-1, 1) \to B$ *is a strongly bounded curve, then for any* $k \in \mathbb{Z}$, *there is a finite family* $\Theta = \{\theta_i\}_{i \in I}$, $I = \underline{I} \coprod \overline{I}$, *of affine reparametrizations of* $(-1, 1)$, *such that we have for some constant* $\mathfrak{C} = \mathfrak{C}(r)$:

- $\bigcup_{i \in I} \mathrm{Ima}(\mathfrak{s} \circ \theta_i) \supset \{x = \mathfrak{s}(t),\ \phi_f(\hat{\mathfrak{s}}(t)) \in [k, k+1[\}$,
- $\forall i \in \overline{I}$ *the curve* $f \circ \mathfrak{s} \circ \theta_i$ *is strongly bounded and* $\forall i \in \underline{I}$ *the curve* $f \circ \mathfrak{s} \circ \theta_i$ *has bounded geometry,*
- $\sharp \underline{I} \leq 100 e^k$ *and* $\sharp \overline{I} \leq \mathfrak{C}\left(\frac{\|df\|}{e^k}\right)^{\frac{1}{r-1}}$.

The above lemma is similar to Lemma 3. Beyond the distinct ways in estimating the C^r complexity (C^r unit versus r-bounded curve), the main difference in the two lemmas comes from the reparametrized set : in Lemma 3 we reparametrize $\mathfrak{s}$ on $f^{-1}B$, rather here on $\phi_f^{-1}([k, k+1[)$ for some k. In others terms we previously fix the value of f and we fix now the size of the derivative on the curve $\mathfrak{s}$.

Sketch of proof. For any $k < l \in \mathbb{Z}$ we let

$$A_{k,l} := \{t \in (-1, 1),\ \phi_f(\hat{\mathfrak{s}}(t)) \in [k, l[\}.$$

<u>First step : control of the r-derivative.</u> Let $\theta_b : (-1, 1) \circlearrowleft$ be an affine contraction of rate b with $\mathrm{Ima}(\theta_b) \cap A_{k,k+1} \neq \emptyset$. By using Faa-di Bruno formula one gets easily with $b = \mathfrak{C}\left(\frac{\|df\|}{e^k}\right)^{-\frac{1}{r-1}}$ for some constant $\mathfrak{C}$ depending only on r :

$$\|d^{r-1}(d(f \circ \mathfrak{s} \circ \theta_b))\| \leq e^{-4+k}.$$

Second step : Polynomial interpolation. Let $P \in \mathbb{R}^2[X]$ be the Taylor polynomial of degree $r-2$ of $d(f \circ \mathfrak{s} \circ \theta_b)$ at 0 and let $A = \left\{t \in (-1,1),\ \|P(t)\| \in [e^{-3+k}\|d(\mathfrak{s} \circ \theta_b)\|, e^{3+k}\|d(\mathfrak{s} \circ \theta_b)\|]\right\}$. By interpolation, one checks easily with the previous bound on the r-derivative that

$$\theta_b^{-1}(A_{k,k+1}) \subset A \subset \theta_b^{-1}(A_{k-3,k+4}).$$

Third step : Landau-Kolmogorov inequality. Landau-Kolmogorov inequality claims there is some constant $\mathfrak{C}(s)$ such that for a C^s map $g : (-1,1) \to \mathbb{R}$:

$$\forall 0 \leq k \leq s,\ \|d^k g\| \leq \mathfrak{C}(s)\left(\|g\| + \|d^s g\|\right). \tag{6.1}$$

Let J be a connected component of A with $\theta_b(J) \cap A_{k,k+1} \neq \emptyset$ and let $\psi_J : (-1,1) \to J$ be the affine reparametrization of J. Then $\tilde{\theta} = \theta_b \circ \psi_J$ satisfies by applying (6.1) to $g = d(f \circ \mathfrak{s} \circ \tilde{\theta})$ with $s = r-1$ (the constants $\mathfrak{C}(r)$ below may change at each step):

$$\begin{aligned}
\forall 1 \leq k \leq r,\ \|d^k(f \circ \mathfrak{s} \circ \tilde{\theta})\| &\leq \mathfrak{C}(r)\left(\|d(f \circ \mathfrak{s} \circ \tilde{\theta})\| + \|d^r(f \circ \mathfrak{s} \circ \tilde{\theta})\|\right),\\
&\leq \mathfrak{C}(r) e^k |J| \|d(\mathfrak{s} \circ \theta)\|,\\
&\leq \mathfrak{C}(r) \min_{t \in (-1,1)} \|d_t(f \circ \mathfrak{s} \circ \tilde{\theta})\|.
\end{aligned}$$

Last step : conclusion. Let finally $\tilde{\tilde{\theta}} = \tilde{\theta} \circ \theta_c$ with $c = (10\mathfrak{C}(r))^{-1}$. We have for all $2 \leq k \leq r$:

$$\begin{aligned}
\|d^k(f \circ \mathfrak{s} \circ \tilde{\tilde{\theta}})\| &\leq c^2 \mathfrak{C}(r) \min_{t \in (-1,1)} \|d_t(f \circ \mathfrak{s} \circ \tilde{\theta})\|,\\
&\leq \frac{1}{10}\|d(f \circ \mathfrak{s} \circ \tilde{\tilde{\theta}})\|,
\end{aligned}$$

i.e. $\tilde{\tilde{\theta}}$ is bounded. Observe also that:

$$\sharp\left\{\tilde{\tilde{\theta}}\right\} \leq \mathfrak{C}(r) b^{-1} \leq \mathfrak{C}(r)\left(\frac{\|df\|}{e^k}\right)^{\frac{1}{r-1}}.$$

For the reparametrizations $f \circ \mathfrak{s} \circ \tilde{\tilde{\theta}}$, which are strongly bounded, we just let $\tilde{\tilde{\theta}} \in \Theta$. Then we may cover $f \circ \mathfrak{s}\left(\{t,\ \phi_f\left(\hat{\mathfrak{s}}(t)\right) \in [k, k+1[\}\right)$ by at most $100e^k$ balls of radius $1/4$ and the intersection of each ball with the image of $f \circ \mathfrak{s} \circ \tilde{\tilde{\theta}}$ for the others $\tilde{\tilde{\theta}}$ allow us to define another element of $\theta \in \Theta$ by using the third item of Proposition 2.

□

6.3 Dynamical reparametrization lemma

We consider a C^r, $r \in \mathbb{N}^*$, diffeomorphism $g : M \circlearrowleft$ on a compact Riemannian surface. We denote by $PTM = \coprod_x PT_xM$ the projective tangent bundle of M. We let $G : PTM \circlearrowleft$, $(x, v) \mapsto (x, d_xg(v))$ be the map induced by g on the projective tangent bundle. Let exp be the exponential map of M.

We may transpose the notion of bounded curve from the Euclidean space $\mathbb{R}^2$ to the Riemannian surface via the exponential map as follows. We say that a C^r curve $\sigma : (-1, 1) \to M$ is ε-bounded (resp. ε-strongly bounded, ε-bounded geometry) when $\varepsilon^{-1} \exp_x^{-1} \circ \sigma : (-1, 1) \to \mathbb{R}^2$ with $x = \sigma(0)$ is well defined and is bounded (resp. strongly bounded, has bounded geometry). We may also consider dynamically bounded curves : for $n \in \mathbb{N}^*$ the curve σ is said (n, ε)-bounded (resp. (n, ε)-strongly bounded) when $g^k \circ \sigma$ is ε-bounded (resp. ε-strongly bounded) for all $0 \leq k < n$. For $x \in \text{Ima}(\sigma)$ we let $\hat{x} = (x, v_x)$ be the projective vector tangent to the curve σ at x. A (n, ε)-bounded curve σ satisfies the following bounded distorsion property :

$$\forall x, y \in \text{Ima}(\sigma), \ |\phi_{g^{n-1}}(\hat{x}) - \phi_{g^{n-1}}(\hat{y})| \leq 10.$$

We choose now $\varepsilon > 0$ so small that $\|d^s g_{2\varepsilon}^x\|_\infty \leq 3\varepsilon \|d_xg\|$ for all $s = 1, \cdots, r$ and all $x \in M$, where $g_{2\varepsilon}^x$ is $g \circ \exp_x(2\varepsilon \cdot) : \{w_x \in T_xM, \ \|w_x\| \leq 1\} \to M$. We fix some embedded ε-strongly bounded C^r curve σ with ε as above.

For $x \in \text{Ima}(\sigma)$ we let $K(x) \geq k(\hat{x})$ be the following integers:

$$K(x) := [\log \|d_xg\|],$$

$$k(\hat{x}) := [\log \|d_xg(v_x)\|].$$

We present below two reparametrization lemmas on σ under g (with $\varepsilon > 0$ fixed as above) which are proved by induction from Lemma 6.

6.3.1 Local version

We first state a *local* reparametrization lemma, meaning that we reparametrize the curve σ restricted to a dynamical ball as in Lemma 5.

For all $n \in \mathbb{N}^*$ and $x \in \text{Ima}(\sigma)$ we let let $k^n(x)$ be the n-uple of integers

$$k^n(x) = (k(\hat{x}), \cdots, k(G^l\hat{x}), \cdots, k(G^{n-1}\hat{x})) \in \mathbb{Z}^n.$$

Given $\mathbf{k}^n = (k_0, \cdots, k_{n-1}) \in \mathbb{Z}^n$ we consider the subset

$$\mathcal{H}(\mathbf{k}^n) := \{x \in \text{Ima}(\sigma), \ k^n(x) = \mathbf{k}^n\}.$$

Lemma 7. *For any $x \in M$ and any $\mathbf{k}^n = (k_0, \cdots, k_{n-1}) \in \mathbb{Z}^n$, there is a family $\Theta_n = \{\theta_n : (-1,1) \circlearrowleft\}$ of affine maps satisfying for some constant $\mathfrak{C} = \mathfrak{C}(r)$:*

- $\bigcup_{\theta_n \in \Theta_n} \mathrm{Ima}(\theta_n) \supset \sigma^{-1}\left(B_n(x,\varepsilon) \cap \mathcal{H}(\mathbf{k}^n)\right)$,
- $\forall \theta_n \in \Theta_n$, *$\sigma \circ \theta_n$ is (n,ε)-strongly bounded,*
- $\sharp\Theta_n \leq \mathfrak{C}^n \prod_{0 \leq l < n} e^{\frac{K(g^l x) - k_l}{r-1}}$.

We do not detail the proof, which goes by induction on n by using Lemma 6.

6.3.2 Global version

We state now a *global* reparametrization lemma to describe the dynamics on the whole set $\mathrm{Ima}(\sigma)$. We will encode the dynamics of g on $\mathrm{Ima}(\sigma)$ with a tree. A weighted directed rooted tree $\mathcal{T}$ is a directed rooted tree whose edges are labelled. Here the weights on the edges are pairs of integers. Moreover the nodes of our tree will be coloured, either in blue or in red.

We let $\mathcal{T}_n$ (resp. $\underline{\mathcal{T}_n}$, $\overline{\mathcal{T}_n}$) be the set of nodes (resp. blue, red nodes) of level n. For all $k \leq n-1$ and for all $\mathbf{i}^n \in \mathcal{T}_n$, we also let $\mathbf{i}^n_k \in \mathcal{T}_k$ be the *parent* node of level k leading to $\mathbf{i}^n$. For $\mathbf{i}^n \in \mathcal{T}_n$, we let $K(\mathbf{i}^n) = (K_1(\mathbf{i}^n), k_1(\mathbf{i}^n), K_2(\mathbf{i}^n), \cdots, k_n(\mathbf{i}^n))$ be the $2n$-uple of integers given by the sequence of labels along the path from the root $\mathbf{i}^0$ to $\mathbf{i}^n$, where $(K_l(\mathbf{i}^n), k_l(\mathbf{i}^n))$ denotes the label of the edge joining $\mathbf{i}^n_{l-1}$ and $\mathbf{i}^n_l$.

For all n and $x \in \mathrm{Ima}(\sigma)$ we let $K_n(x)$ be the $2n$-uple of integers

$$K^n(x) = (K(x), k(\hat{x}) \cdots, K(g^{n-2}x), k(G^{n-2}\hat{x}), K(g^{n-1}x), k(G^{n-1}\hat{x})).$$

Given $\mathbf{K}^n = (K_1, k_1, \cdots K_n, k_n) \in \mathbb{Z}^{2n}$ we consider then

$$\mathcal{K}(\mathbf{K}^n) := \{x \in \mathrm{Ima}(\sigma),\ K^n(x) = \mathbf{K}^n\}.$$

Lemma 8. *There is a bicoloured weighted directed rooted tree $\mathcal{T}$ and $(\theta_{\mathbf{i}^n})_{\mathbf{i}^n \in \mathcal{T}_n}$, $n \in \mathbb{N}$, families of affine reparametrizations of $(-1,1)$, such that for some constant $\mathfrak{C}(r)$ depending only on r:*

1. *$\forall \mathbf{K}^n \in (\mathbb{Z} \times \mathbb{Z})^n$, we have* $\bigcup_{\substack{\mathbf{i}^n \in \mathcal{T}_n \\ K(\mathbf{i}^n) = \mathbf{K}^n}} \mathrm{Ima}(\theta_n) \supset \sigma^{-1}\mathcal{H}(\mathbf{K}^n)$,
2. *$\forall \mathbf{i}^n \in \underline{\mathcal{T}_n}$, the curve $\sigma \circ \theta_{\mathbf{i}^n}$ is strongly (n,ε)-bounded and $\forall \mathbf{i}^n \in \overline{\mathcal{T}_n}$, the curve $f^n \circ \sigma \circ \theta_{\mathbf{i}^n}$ has ε-bounded geometry,*

3. $\forall \mathbf{i}^{n-1} \in \mathcal{T}_{n-1}$ *and* $(K_n, k_n) \in \mathbb{Z} \times \mathbb{Z}$ *we have :*

$$\sharp \left\{ \mathbf{i}^n \in \overline{\mathcal{T}_n},\ \mathbf{i}^n_{n-1} = \mathbf{i}^{n-1} \text{ and } k_n(\mathbf{i}^n) = k_n \right\} \leq 100 e^{k_n},$$
$$\sharp \left\{ \mathbf{i}^n \in \underline{\mathcal{T}_n},\ \mathbf{i}^n_{n-1} = \mathbf{i}^{n-1} \text{ and } (K_n(\mathbf{i}^n), k_n(\mathbf{i}^n)) = (K_n, k_n) \right\} \leq \mathfrak{C}(r) e^{\frac{K_n - k_n}{r-1}}.$$

Each node of level n of the tree represents a strongly bounded subcurve of $f^n \circ \sigma$. The last item gives estimates of the valence of the tree in terms of the labels. Again the proof goes by induction from Lemma 6.

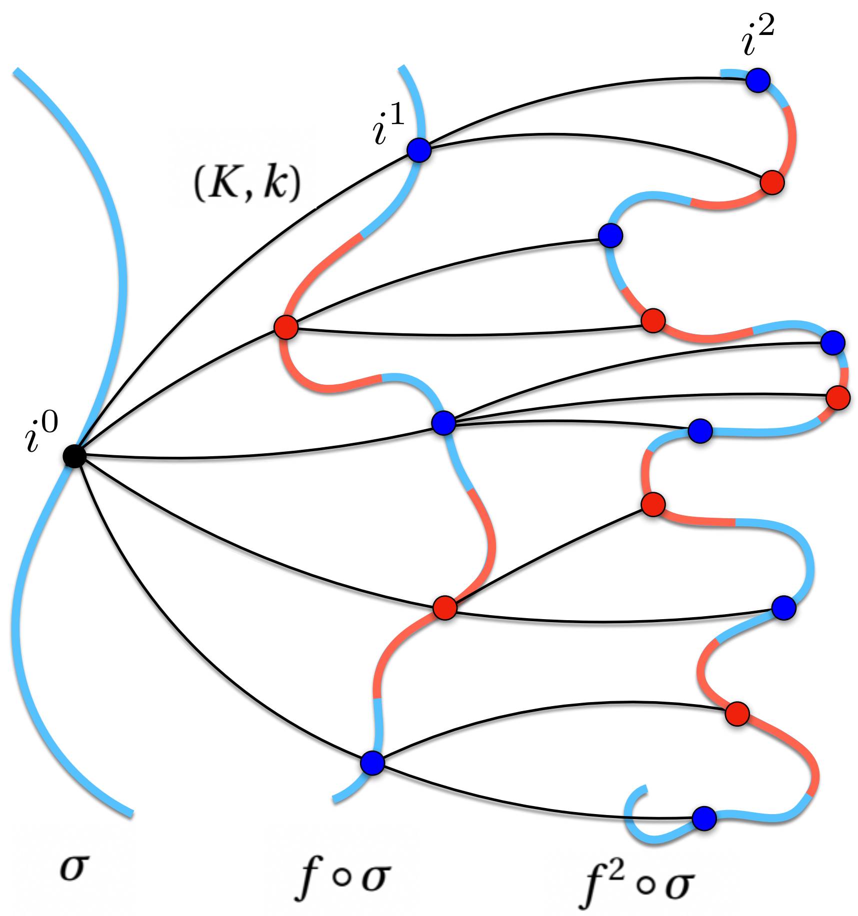


Figure 6.1: The tree and the corresponding strongly bounded subcurves in σ, $f \circ \sigma$ and $f^2 \circ \sigma$. Red subcurves correspond to pieces with bounded geometry. Picture by Joshua Park.

We end this chapter by introducing the notion of geometric times which plays a key role in the proof of Theorem 27 below. Let $x \in \mathrm{Ima}(\sigma)$. Roughly speaking a geometric time of x is a time n for which the curve $f^n \circ \sigma$ looks *nice* around $f^n x$. More precisely an integer n is said to be a **geometric time** of x with respect to σ when $f^n x$ belongs to a subcurve with bounded geometry of $f^n \circ \sigma$.

Part III

Applications to the ergodic theory of C^r smooth systems

Chapter 7

Measures of maximal entropy in small dimensions

Related personal works : [25],[26],[34],[31]

Contents

We first introduce Lyapunov exponents which are closely related to entropy for smooth systems. Let $(M, \|\cdot\|)$ be a Riemannian compact manifold and let $f : M \circlearrowleft$ be a C^1 map. The Lyapunov exponents estimate the exponential growth in n of the differential of f. More precisely, we let

$$\forall v \in T_xM,\ x \in X,\ \chi(x,v) := \limsup_n \frac{1}{n} \log \|d_x f^n(v)\|.$$

There is a (measurable in x) flag $\{0\} = F_x^{r(x)} \subsetneq \cdots \subsetneq F_x^2 \subsetneq F_x^1 = T_xM$ and real numbers $-\infty \leq \chi_{r(x)-1}(x) < \cdots < \chi_2(x) < \chi_1(x) < +\infty$, such that $\chi(x,v) = \chi_i(x)$ for all $v \in F_x^i \setminus F_x^{i-1}$ for $i = 1, \cdots, r(x)-1$. The numbers $(\chi_i(x))_i$ are called the (upper) Lyapunov exponents of x. Oseledets theorem states that for any $\mu \in \mathcal{M}(M,f)$ the limsup defining Lyapunov exponents are in fact limits for μ almost every x. The maximal Lyapunov exponent may be also defined as $\chi_1(x) = \limsup_n \frac{1}{n} \log \|d_x f^n\|$. Here we let with $\chi_1^+(x) = \max(\chi_1(x), 0)$:

$$\forall \mu \in \mathcal{M}(M,f),\ \chi^+(\mu) = \int \chi_1^+(x)\, d\mu(x).$$

It is easily checked that $R(f) = \sup_x \chi_1^+(x) = \sup_\mu \chi^+(\mu)$.

The algebra ΛTM of exterior powers of TM inherits a Riemannian structure from $(M, \|\cdot\|)$. We let χ_Λ^+ be the maximal nonnegative Lyapunov exponent of $\Lambda df : \Lambda TM \circlearrowleft$:

$$\chi_\Lambda^+(x) = \limsup_n \frac{1}{n} \log^+ \|\Lambda d_x f^n\|.$$

When x is a Lyapunov regular point, $\chi_\Lambda^+(x)$ is just the sum of the positive pointwise Lyapunov exponents, i.e. $\chi_\Lambda^+(x) = \sum_i \chi_i^+(x)$. For an invariant measure μ we let then

$$\chi_\Lambda^+(\mu) = \int \chi_\Lambda^+(x)\, d\mu(x).$$

7.1 Measures of maximal entropy for C^r, $r \geq 1$, interval maps with large entropy

For piecewise monotone interval maps, Hofbauer [59] has built a coding of the dynamic by a topological Markov chain with countable states, which preserves the entropy. In particular he showed in this setting the finiteness of ergodic measures of maximal entropy.

In his PhD thesis, Buzzi has generalized this construction to C^r interval maps. With Ruette [45], they showed that any C^r interval map with $h_{top}(f) > \frac{2R(f)}{r}$ admits measures of maximal entropy and there are only finitely many ergodic one's. In the other hand there are counter-examples in finite smoothness for interval maps with small topological entropy :

Theorem 13. *[39] For any $r \geq 1$, there are C^r smooth interval maps $f : [0,1] \circlearrowleft$ without measure of maximal entropy or with infinitely many one's satisfying $h_{top}(f) = \frac{R(f)}{r} > 0$.*

In [25] we improve Buzzi-Ruette estimates to get the following sharp statement:

Theorem 14. *[25] Any C^r interval map $h_{top}(f) > \frac{R(f)}{r} > 0$ admits a finite (positive) number of measures of maximal entropy.*

As in [45], the proof is based on the Markov representation of the Buzzi-Hofbauer diagram and an estimate of the entropy escaping to infinity. In the same setting, I showed in [26] that the topological entropy is continuous for the C^r topology at any interval map with $h_{top}(f) > \frac{R(f)}{r}$:

Theorem 15. *[26] Let f be a C^r interval map with $h_{top}(f) > \frac{R(f)}{r}$, then*

$$\lim_{g \xrightarrow{C^r} f} h_{top}(g) = h_{top}(f).$$

7.2 Measures of maximal entropy for C^r for surface diffeomorphisms with large entropy

In a *tour de force*, Sarig has generalized the construction of Markov partitions to the non-uniformly hyperbolic setting for a surface diffeomorphism [83] (see [6] in higher dimensions). In this way he obtained a coding of the non-uniformly hyperbolic points by a topological Markov chain with countable states preserving the entropy. With Crovisier and Buzzi, they show that any C^r, $r > 1$, surface diffeomorphism f with $h_{top}(f) > \frac{R(f)}{r}$ has at most finitely many maximal measures when the topological entropy of f is larger than $\frac{R(f)}{r}$ [44]. In the C^∞ case, this solves a long-standing conjecture of Newhouse. More recently to study the statistical properties of the measures of maximal entropy they show the following entropic continuity of the maximal Lyapunov exponent.

Before stating their result, let us recall some notations. For a C^1 surface diffeomorphism $f : M \circlearrowleft$, recall that $F : PTM \circlearrowleft$ is the induced map on the projective tangent bundle and that $\phi_f : PTM \to \mathbb{R}$ denotes the continuous map $(x, v) \mapsto \log \|d_x f(v)\|$. Observe now that if μ is an f-invariant measure such that μ almost every point x has exactly one positive Lyapunov exponent, there is a unique lift $\hat{\mu}^+$ of μ supported on the unstable Oseledets bundle. Note that we have in this case $\int \phi_f \, d\hat{\mu}^+ = \chi^+(\mu)$.

Theorem 16. *[43] Let $(f_k : M \circlearrowleft)_k$ be a sequence of C^∞, surface diffeomorphisms converging to f (in the C^∞ topology). Suppose that there are ergodic F_k-invariant measures $\hat{\nu}_k^+$, $k \in \mathbb{N}$, converging to $\hat{\mu}$.*

Then there are F-invariant measures $\hat{\mu}_0$ and $\hat{\mu}_1^+$ with $\hat{\mu} = (1-\beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$, $\beta \in [0, 1]$, such that

$$\limsup_{k \to +\infty} h(\nu_k) \leq \beta h(\mu_1). \tag{7.1}$$

In the above statement we mean implicitly that the projection μ_1 of $\hat{\mu}_1^+$ on M has exactly one positive exponent.

As a consequence of the above theorem, if $f_k = f$ for all k with $h_{top}(f) > 0$ and $(\nu_k)_k$ is a sequence of ergodic f-invariant measures converging to μ with $\lim_k h(\nu_k) = h_{top}(f)$, then $\hat{\mu} = \hat{\mu}_1^+$, $\mu_1 = \mu$ and $\beta = 1$, therefore

$$\lim_k \chi^+(\nu_k) = \lim_k \int \phi_{f_k} \, d\hat{\nu}_k^+ = \int \phi_f \, d\hat{\mu}_1^+ = \chi^+(\mu).$$

We prove recently the following C^r version of Theorem 16. We follow almost the same strategy as [43], except that to prove Inequality (7.1) we cut the orbit of typical ν_k points x along geometric times of the unstable manifold rather than hyperbolic times.

Theorem 17. *[34] Let $(f_k)_k$ be a sequence of C^r, with $r > 1$, surface diffeomorphisms converging to f (in the C^r topology). Suppose that there are ergodic F_k-invariant measures $\hat{\nu}_k^+$, $k \in \mathbb{N}$, converging to $\hat{\mu}$.*

Then for any $\alpha > \frac{R(f)}{r}$ there are F-invariant measures $\hat{\mu}_0$ and $\hat{\mu}_1^+$ with $\hat{\mu} = (1-\beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$, $\beta \in [0,1]$, such that

$$\limsup_{k\to+\infty} h(\nu_k) \leq \beta h(\mu_1) + (1-\beta)\alpha.$$

As consequences of Theorem 17 we get the analogous statements of Theorem 14 and Theorem 15 for surface diffeomorphisms.

Corollary 5 (Existence of maximal measures)**.** *Let f be a C^r, with $r > 1$, surface diffeomorphism with $h_{top}(f) > \frac{R(f)}{r}$. Then f admits a measure of maximal entropy.*

Proof. We let $f_k = f$ for all k. Let ν_k be a sequence of ergodic measures with $\lim_k h(\nu_k) = h_{top}(f)$. By applying Theorem 17 with $\frac{R(f)}{r} < \alpha < h_{top}(f)$, we get after extracting a subsequence $h_{top}(f) = \limsup_k h(\nu_k) \leq \beta h(\mu_1) + (1-\beta)\alpha$ for some $\mu_1 \in \mathcal{M}(M, f)$. Therefore $\beta = 1$ and $h(\mu_1) = h_{top}(f)$. □

One proves similarly the upper semi-continuity of the topological entropy at f with $h_{top}(f) > \frac{R(f)}{r}$. Also by a celebrated result of Katok [62] any C^{1+} surface diffeomorphism admits hyperbolic horseshoes with entropy arbitrarily close to the topological entropy. As a consequence, the topological entropy is lower semi-continuous on the set of C^r surface diffeomorphisms. Therefore we get:

Corollary 6 (Continuity of the topological entropy)**.** *Let $(f_k)_k$ be a sequence of C^r, with $r > 1$, surface diffeomorphisms converging in the C^r topology to f with $h_{top}(f) > \frac{R(f)}{r}$. Then we have*

$$h_{top}(f) = \lim_k h_{top}(f_k).$$

Finally we mention that C^r counter-examples f with $h_{top}(f) < \frac{R(f)}{r}$ have been built by Buzzi in [40].

7.3 Equidistribution of periodic points

For expansive homeomorphisms with the strong specification property, Bowen proved in [13] the equality between the topological entropy and the exponential growth rate of periodic points, together with the equidistribution of periodic points along the unique invariant measure of maximal entropy. In particular these properties hold true for topologically transitive subshifts of finite type and Axiom A systems. Therefore, for C^{1+} surface diffeomorphisms, by Katok's horseshoe theorem, the exponential growth in n of saddle (hyperbolic) n-periodic points is larger than or equal to the topological entropy.

We get in [31] the following result :

Theorem 18. *Let $f : M \to M$ be a C^∞ diffeomorphism of a compact surface M (resp. a C^∞ interval map) with positive topological entropy $h_{top}(f) > 0$.*

Then for any $\delta \in]0, h_{top}(f)[$ the set Per_n^δ of saddle (resp. repelling) n-periodic points with Lyapunov exponents δ-away from zero grows exponentially in n as the topological entropy. Moreover these periodic points are equidistributed along measures of maximal entropy, i.e.:

- $\limsup_{n\to+\infty} \frac{1}{n} \log \sharp \mathrm{Per}_n^\delta = h_{top}(f)$,
- *any weak-star limit of* $\left(\frac{1}{\sharp \mathrm{Per}_{n_k}^\delta} \sum_{x \in \mathrm{Per}_{n_k}^\delta} \delta_x \right)_k$ *is an f-invariant measure of maximal entropy, for all increasing sequences of positive integers $(n_k)_k$ satisfying*
$$\lim_{k\to+\infty} \frac{1}{n_k} \log \sharp \mathrm{Per}_{n_k}^\delta = h_{top}(f).$$

V. Kaloshin [61] has shown that in C^∞ Newhouse domains (i.e. C^∞ open sets with a dense subset of diffeomorphisms having an homoclinic tangency) generic C^∞ smooth surfaces diffeomorphisms with arbitrarily fast growth of saddle periodic points (see [5] for real-analytic examples). For these examples we can therefore not replace the sets Per_n^δ with the sets Per_n of all n-periodic points in our Main Theorem.

It follows also from Sarig's coding that there is an integer p such that for any $0 < \delta < h_{top}(f)$:

- $\lim_{n\to+\infty,\ p|n} \frac{1}{n} \log \sharp \mathrm{Per}_n^\delta = h_{top}(f)$,
- any weak-star limit of $\left(\frac{1}{\sharp \mathrm{Per}_n^\delta} \sum_{x \in \mathrm{Per}_n^\delta} \delta_x \right)_{n,\ p|n}$ is a measure of maximal entropy.

The proof of Theorem 18, which consists in showing the asymptotic Per^δ-expansiveness, is based on another more involved Reparametrization Lemma of the differential map acting on the bundle $\coprod_{x\in M} T_xM \times T_xM$.

Chapter 8

Symbolic extensions in intermediate smoothness

Related personal works : [21],[22],[24],[28],[36]

Contents

The theory of symbolic extensions, which was developed for general topological systems, is particularly relevant in the study of smooth dynamical systems. Indeed the existence of (principal) symbolic extensions highly depends on the smoothness of the system :

- C^∞ systems are asymptotically h-expansive (Corollary 4), therefore always admit principal symbolic extensions by Corollary 1,
- there is an open set O of C^1 diffeomorphisms such that generic diffeomorphism $f \in O$ has no symbolic extension in dimension ≥ 3 [4]. In the conservative setting, C^1 generic system with no dominated spliitting have no symbolic extension [41, 54].
- there is an open set O of C^r smooth, $+\infty > r > 1$, diffeomorphisms such that generic diffeomorphisms $f \in O$ has no principal symbolic extension in dimension ≥ 2 [54].

8.1 The case of interval maps

Downarowicz and Maass have established the existence of symbolic extensions for C^r interval maps in [53]. For a f-invariant probability measure μ, we recall that $\chi^+(\mu) = \int \chi^+(x)\, d\mu(x)$ with $\chi^+(x)$ being the positive Lyapunov exponent at x given by $\chi^+(x) = \limsup_n \frac{1}{n} \log^+ |(f^n)'(x)|$.

Theorem 19. *[53] Any C^r interval map with $r > 1$ admits a symbolic extension π with $h^\pi = \frac{\chi^+}{r-1}$.*

In [21], I have built sharp examples on the interval :

Theorem 20. *For any $r > 1$, there is an interval map $f_r : [0,1] \circlearrowleft$ fixing 0 such that for any symbolic extension $\pi : (Y,S) \to [0,1]$ of f_r, we have*

$$h^\pi(\delta_0) = \frac{\log \|f'\|_\infty}{r-1} > 0.$$

In the proof we follow the strategy of [54] by accumulating horseshoes at finer and finer scales. However the construction is here explicit (we do not use Baire arguments).

8.2 Symbolic extensions for C^r surface diffeomorphisms

I proved the existence of symbolic extensions for C^2 surface diffeomorphisms in [22]. The proof is based on a reparametrization lemma of the whole dynamical ball as in Lemma 5, where the reparametrizations θ_n are straightening maps of the finite time stable and unstable distributions. In [24] we improved the main result of [22] as follows:

Theorem 21. *Any C^r diffeomorphism with $r > 1$ on a compact Riemannian manifold of dimension 2 or 3 admits a symbolic extension π with $h^\pi = \frac{\chi_\Lambda^+}{r-1}$.*

Ideas of the proof. We sketch the proof for surface diffeomorphisms. We have to check that $\frac{\chi^+}{r-1}$ is a superenvelope, i.e. for some entropy structure $(h_k)_k$ for any invariant measure μ and for all $\gamma > 0$ there is k_μ and δ_μ such that for any invariant measure ν which is δ_μ-close to μ :

$$\left(h - h_{k_\mu}\right)(\nu) \leq \frac{\chi^+(\mu) - \chi^+(\nu)}{r-1} + \gamma$$

In fact by using functional analysis arguments we may only consider ergodic measures ν. Then by using Newhouse works [75, 76] we may compare the local entropy $\left(h - h_{k_\mu}\right)(\nu)$ with the one-dimensional local growth of ν typical unstable manifolds. We conclude by applying the Reparametrization Lemma 7, where in the estimate $\prod_{0\leq l<n} e^{\frac{K(g^l x)-k_l}{r-1}}$ for the number of reparametrizations, the terms $\left(\prod_{0\leq l<n} e^{k_l}\right)^{1/n}$ and $\left(\prod_{0\leq l<n} e^{K(g^l x)}\right)^{1/n}$ are respectively closed to $e^{\chi^+(\nu)}$ and $e^{\chi^+(\mu)}$ for large n. Moreover the scale ε in Lemma 7 corresponds to the integer k_μ. □

8.3 ... in higher dimensions

The reparametrization lemma proved in [22] is stronger in some sense : it reparametrizes two-dimensional dynamical balls. For example we use it with

Fisher in [36] to prove the existence of symbolic extensions for partially hyperbolic sytems with a two dimensional center bundle.

In higher dimensions we conjecture :

Conjecture 2. *Any C^r smooth system with $r > 1$ admits a symbolic extension π with $h^\pi = \frac{\chi_\Lambda^+}{r-1}$.*

To prove the conjecture in higher dimensions, we attempt now to extend the tools from Chapter 6 to k-discs σ with $k > 1$. In [28] we prove this conjecture for skew product circle maps :

Theorem 22. *[28] Let $f : \mathbb{T}^d \circlearrowleft$ be a C^r, $r > 1$, map of the d-torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ of the form $f : (x_1, x_2, \cdots, x_d) \mapsto (f_1(x_1), f_2(x_1, x_2), \cdots, f_d(x_1, \cdots, x_d))$.*

Then f admits a symbolic extension π with $h^\pi = \frac{\chi_\Lambda^+}{r-1}$.

Chapter 9

SRB and physical measures

Related personal works : [33],[30]

Contents

9.1 SRB and physical measures

One fundamental problem in dynamics consists in understanding the statistical behaviour of the system. Given a topological system (X, f) we are more precisely interesting in the asymptotic distribution of the empirical measures $\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^kx}\right)_n$ for typical points x with respect to a reference measure. In the setting of differentiable dynamical systems the natural reference measure to consider is the Lebesgue measure on the manifold.

The basin of a f-invariant measure μ is the set $\mathcal{B}(\mu)$ of points whose empirical measures are converging to μ in the weak-$*$ topology. By Birkhoff's ergodic theorem the basin of an ergodic measure μ has full μ-measure. An invariant measure is said **physical** when its basin has positive Lebesgue measure. We may wonder when such measures exist and then study their basins.

In the works of Y. Sinai, D. Ruelle and R. Bowen [85, 13, 81] these questions have been successfully solved for uniformly hyperbolic systems. A SRB measure of a C^{1+} system is an invariant probability measure with at least one positive Lyapunov exponent almost everywhere, which has absolutely continuous conditional measures on unstable manifolds [89]. Physical measures may neither be SRB measures nor sinks (as in the famous figure-eight attractor), however hyperbolic ergodic SRB measures are physical measures. For uniformly hyperbolic systems, there is a finite number of such measures and their basins cover a full Lebesgue subset of the manifold. For two Borel subsets A and B of M we write $A \overset{o}{\subset} B$ (resp. $A \overset{o}{=} B$) when we have $\mathrm{Leb}(A \setminus B) = 0$ (resp. $\mathrm{Leb}(A\Delta B) = 0$).

Theorem 23. *[82] Let $f : M \circlearrowleft$ be a C^{1+} diffeomorphism of a compact smooth manifold M. We assume that there is an open set U such that $\Lambda = \bigcap_{n\in\mathbb{N}} f^n U$ is a compact invariant hyperbolic set.*

Then there are finitely many ergodic hyperbolic SRB measures $(\mu_i)_{i\in I}$ with

$$U \overset{o}{\subset} \bigcup_{i\in I} \mathcal{B}(\mu_i).$$

There are essentially two methods to build SRB measures in the uniformly hyperbolic case. The first one uses symbolic dynamics through Markov partitions. In the second method, called the geometrical method, one builds SRB measures as limits of $\left(\frac{1}{n}\sum_{k=0}^{n-1} f_*^k \mathrm{Leb}_{D_u}\right)_n$, where D_u is a local unstable disc and Leb_{D_u} denotes the normalized Lebesgue measure on D_u induced by its inherited Riemannian structure as a submanifold of M.

9.2 Entropy and exponents physically

The following well-known inequality due to Ruelle relates the entropy of a measure with the sum of its positive exponents :

Theorem 24. *[81] For a C^1 system (M, f), we have*

$$\forall \mu \in \mathcal{M}(M, f),\ h(\mu) \leq \chi_\Lambda^+(\mu).$$

For a C^∞ map $f : M \circlearrowleft$ we showed in [30] that from the physical viewpoint the entropy is larger than or equal to the maximal pointwise exponent χ_Λ^+. For $x \in M$ we let $pw(x)$ be the set of accumulation points in $\mathcal{M}(X)$ of the empirical measures $\left(\frac{1}{n}\sum_{0\leq k<n} \delta_{f^k x}\right)_n$.

Theorem 25. *[30] Let $f : M \circlearrowleft$ be a C^∞ map. Then for Lebesgue almost every point x there exists $\mu_x \in pw(x)$ with*

$$h(\mu_x) \geq \chi_\Lambda^+(x).$$

Of course the inequality does not hold true for all x, e.g. when x is a periodic point with a positive Lyapunov exponent. However the set of such points has zero Lebesgue measure. Note that in general we only have $\chi_\Lambda^+(x) \leq \chi_\Lambda^+(\mu_x)$. For C^1 systems with a dominated splitting, Catsiegeras, Cerminara and Enrich have established a similar result [46].

Ideas of the proof. The proof of Theorem 25 is based on a control of the distorsion given by Lemma 5 as follows. Let $\sigma : (0,1)^k \to M$ be an embedded k-disc. Consider the projective action $P\Lambda^k(df)$ on the projective exterior bunddle $P\Lambda^k TM$ and the associated disc $P\Lambda^k(d\sigma)$. Then we can apply Lemma 5 with the potential $\phi : v \in P\Lambda^k TM \mapsto \|\Lambda^k df(v)\|$. We get reparametrizations of the dynamical ball such that the Bowen-like property implies that

σ has bounded distorsion when restricted to the image of a reparametrization. Then we follows the geometrical method to build SRB measures by considering $\mu_n = \frac{1}{n}\sum_{0\leq k<n} f^k_*\mathrm{Leb}_\sigma$, $n \in \mathbb{N}^*$. Following an argument due to Misiurewicz, for any finite partition P there is a weak limit μ of $(\mu_n)_n$ satisfying $h(\mu, P) \geq \limsup_n \frac{1}{n}\int \log \mathrm{Leb}_\sigma(P^n(x))\, d\mathrm{Leb}_\sigma$. The bounded distorsion property allows us to relate the right member of this last inequality with the Lyapunov exponents along σ. One may conclude the proof by choosing carefully σ. □

As a direct consequence of Theorem 25 we obtain the following lower bound on the entropy of a physical measure.

Corollary 7. *Let μ be a physical measure of a C^∞ map $f : M \circlearrowleft$. Then*

$$h(\mu) \geq \overline{\chi^+_\Lambda|_{\mathcal{B}(\mu)}},$$

where $\overline{\chi^+_\Lambda|_{\mathcal{B}(\mu)}}$ is the essential supremum of χ^+_Λ on $\mathcal{B}(\mu)$.

9.3 SRB measures for C^r surface diffeomorphisms

Ledrappier and Young have shown that the equality case in Ruelle inequality characterizes SRB measures for $\mathcal{C}^{1+}$ systems.

Theorem 26. *Let $f : M \circlearrowleft$ be a C^{1+} diffeomorphism. Let μ be a measure such that μ almost every point has a positive exponent. Then μ is a SRB measure if and only if $h(\mu) = \chi^+_\Lambda(\mu)$.*

Beyond the uniformly hyperbolic case (Theorem 23) existence of SRB measures and description of their basins are also known for large classes of partially hyperbolic systems [12, 2, 3]. Corresponding results have been established for unimodal maps with negative Schwartzian derivative [63]. SRB measures have been also deeply investigated for parameter families such as the quadratic family and Henon maps [60, 7, 9, 8]. For C^r, $r > 1$, surface diffeomorphisms we show:

Theorem 27. *[33] Let $f : M \circlearrowleft$ be a C^r, $r > 1$, surface diffeomorphism. There are countably many ergodic SRB measures $(\mu_i)_{i\in I}$ with $\Lambda := \{\chi^+(\mu_i),\ i \in I\} \subset \left]\frac{R(f)}{r}, +\infty\right[$, such that we have :*

- $\left\{\chi > \frac{R(f)}{r}\right\} \overset{o}{=} \{\chi \in \Lambda\}$,
- $\{\chi = \lambda\} \overset{o}{\subset} \bigcup_{i,\chi^+(\mu_i)=\lambda} \mathcal{B}(\mu_i)$ *for all* $\lambda \in \Lambda$.

In particular, for C^∞ surface diffeomorphisms, Lebesgue almost every point x with $\chi^+(x) > 0$ lies in the basin of an ergodic SRB measure. In a forthcoming paper, for any $1 < r < +\infty$, we build C^r examples of C^r diffeomorphisms satisfying $\chi(x) > 0$ for x in a set of positive Lebesgue measure disjoint with the basins of ergodic SRB measures. In his famous ICM's talk [86], Viana conjectured that when there is a set of positive Lebesgue measure with non zero Lyapunov exponents, then there is an SRB measure. The above results answer Viana's conjecture in the case of C^r, $r > 1$, surface diffeomorphisms. In the C^∞ case the existence of SRB measures has also been obtained in [42] by using techniques from [43].

We explain now in few lines the main ideas to build a SRB measure under the assumptions of the Main Theorem. As in the proof of Theorem 25 we first consider a smooth C^r embedded curve D such that

$$\chi(x, v_x) := \limsup_n \frac{1}{n} \log \|d_x f^n(v_x)\| > b > \frac{R(f)}{r}$$

for (x, v_x) in the unit tangent space $T^1 D$ of D with x in a subset B of D with positive Leb_D-measure. For x in B we let $E(x)$ be the set of geometric times of x with respect to D. We show that $E(x)$ has positive upper asymptotic density for x in a subset A of B with positive Leb_D-measure by using the global reparametrization lemma stated in Lemma 8. Let $F : \mathbb{P}TM \circlearrowleft$ be the map induced by f on the projective tangent bundle $\mathbb{P}TM$. We build a SRB measure by considering a weak limit μ of a sequence of the form $\left(\frac{1}{\sharp F_n} \sum_{k \in F_n} F_*^k \mu_n\right)_n$ where :

- $(F_n)_n$ is a Fölner sequence, so that any weak limit μ is F-invariant,
- for all n, the measure μ_n is the probability measure induced by Leb_D on $A_n \subset A$, the Leb_D-measure of A_n being not exponentially small,
- the sets $(F_n)_n$ are in some sense *filled with* the geometric set $E(x)$ for $x \in A_n$.

Finally we check with some Fölner Gibbs property that the limit empirical measure μ projects to a SRB measure on M by using the Ledrappier-Young entropic characterization.

To prove the covering property of the basins, we argue by contradiction by assuming there is a set $A \subset \left\{\chi^+ > \frac{R(f)}{r}\right\}$ of positive Lebesgue measure disjoint from the basins of the ergodic SRB measures. Then we follow the previous construction by pushing the Lebesgue measure on a smooth curve D with $\mathrm{Leb}_D(A) > 0$, so that the limit empirical measure is an SRB measure. We get a contradiction by using the absolute continuity of Pesin stable holonomy at $f^n x$, where x is a Leb_D-density point of B and $n \in E(x)$ is a geometric time.

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