MULTIPLICITY OF TOPOLOGICAL SYSTEMS

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Abstract. We define the topological multiplicity of an invertible topological system (X, T) as the minimal number k of real continuous functions $f_1, \cdots, f_k$ such that the functions $f_i \circ T^n$, $n \in \mathbb{Z}$, $1 \leq i \leq k$, span a dense linear vector space in the space of real continuous functions on X endowed with the supremum norm. We study some properties of topological systems with finite multiplicity. After giving some examples, we investigate the multiplicity of subshifts with linear growth complexity.

Contents

1. Introduction

The multiplicity of an invertible bounded operator $U : E \circlearrowleft$ on a normed vector space E is the minimal cardinality of subsets $F \subset E$, whose cyclic space (i.e. the vector space spanned by $U^k x$, $k \in \mathbb{Z}$, $x \in F$) is dense in E.

For an ergodic measure preserving system $(X, f, \mathcal{B}, \mu)$, the multiplicity $\mathrm{Mult}(\mu)$ of the Koopman operator, which is the operator of composition by f on the Hilbert space $L^2(\mu)$ is a dynamical invariant, which has been investigated in many works (see e.g. [Dan13] and the references therein).

Cyclicity, which corresponds to simple multiplicity (i.e. there is an element whose cyclic space is dense in the whole vector space), has been also established for operators of composition on the Hardy space $H^2(D)$ [BS97]. In this context a pioneering work of Birkhoff [Bir29] states that there is an entire function ϕ in the complex plane such that the set $\{\phi(\cdot + n),\ n \in \mathbb{N}\}$ is dense itself in the set of entire functions endowed with the uniform topology on compact subsets, i.e. the operator of translation by 1 is *hypercyclic*.

Quite surprisingly the corresponding topological invariant has not been studied in full generality. More precisely we consider here topological dynamical systems (X, T), where X

is a compact metrizable space and $T : X \circlearrowleft$ is a homeomorphism and we study the operator of composition by T on the Banach space $C(X)$ of real continuous functions endowed with the uniform topology. We call topological multiplicity of (X, T) the associated multiplicity and we denote it by $\mathrm{Mult}(T)$. We remark that our definitions and results can be extended to the noninvertible continuous map $T : X \circlearrowleft$. But for sake of simplicity, we focus on homeomorphisms $T : X \circlearrowleft$.

In this paper, we mostly focus on topological systems with finite multiplicity. We first show the following properties for such systems.

Theorem. *Let (X, T) be a topological system with finite multiplicity. Then the following properties are satisfied.*

1. *(X, T) has zero topological entropy.*
2. *(X, T) has finitely many ergodic measures.*

These properties are the main contents of Section 2. The property (1) is proven in Proposition 3.5 in two ways: one uses the variational principal of topological entropy; the other is purely topological. The property (2) is proven in Lemma 2.6 and Corollary 2.8. In fact we show more precisely that the number of ergodic measures is equal to the multiplicity of the operator induced on the quotient of $C(X)$ by the closure of coboundaries.

In Section 3, we relate the topological multiplicity with the dimension of cubical shifts, in which the action $T_* : \mathcal{M}(X) \circlearrowleft$ induced by T on the set $\mathcal{M}(X)$ of Borel probability measures on X may be affinely embedded. In Theorem 3.3, we show a necessary and sufficient condition for the existence of affinely embedding of $(\mathcal{M}(X), T_*)$ to the shift on $([0, 1]^d)^{\mathbb{Z}}$. Furthermore, we compare our result to Lindenstrauss-Tsukamoto conjecture for dynamical embedding (Corollary 3.4).

In Section 4, we state a generalized Banach version of a lemma due to Baxter [Bax71] which is a classical criterion of simplicity for ergodic transformations. The generalized Baxter's Lemma (Lemma 4.1) will play an important role on estimating the topological multiplicity in next sections.

For minimal Cantor systems a topological analogue of the rank of a measure preserving system has been defined and studied (see [DP22]). In Section 5, under this setting, we compare the topological multiplicity with the topological rank (Theorem 5.1).

In Section 6, we study some examples and estimate their topological multiplicity: minimal rotations on compact groups, Sturmian and Thue-Morse subshifts, homeomorphisms of the interval, etc. Among them, we show in Theorem 6.7 that even though the Thue-Morse subshift is a minimal uniquely ergodic system with simple mixed spectrum, its topological multiplicity is one.

Theorem. *The Thue-Morse subshift has simple topological spectrum.*

In Section 6, we estimate the topological multiplicity of subshifts with linear growth complexity, i.e. subshifts X such that the cardinality $p_X(n)$ of n-words in X satisfies $\liminf_{n\to\infty} \frac{p_X(n)}{n} < +\infty$. Such subshifts aroused a great deal of interest, specially recently [Bos92, CK19, CP23, DDMP21]. In [Bos92] it is proved that an aperiodic subshift X has at most k ergodic measures if $\liminf_n \frac{p_X(n)}{n} \leq k \in \mathbb{N}$. Our main related result states as follows (Theorem 7.1 and Theorem 7.6):

Theorem. *Let X be an aperiodic subshift with $\liminf_{n\to\infty} \frac{p_X(n)}{n} \leq k \in \mathbb{N}$. Then*

$$\mathrm{Mult}(T) \leq 2k \text{ and } \sum_{\mu \text{ ergodic}} \mathrm{Mult}(\mu) \leq 2k.$$

Except the results on multiplicity that we investigate, we propose several questions in the current paper.

2. Topological multiplicity, definition and first properties

2.1. **Multiplicity of a linear operator.** Let $(E, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$. We consider a linear invertible bounded operator $U : E \circlearrowleft$. A subset F of E is called a *generating family* of U when the vector space spanned by $U^k x$, $k \in \mathbb{Z}$, $x \in F$, is dense in E. In the following we denote by $\mathrm{span}(G)$ (resp. $\overline{\mathrm{span}}(G)$) the vector space spanned by a subset G of E (resp. its closure) and we then let $V_F^U := \overline{\mathrm{span}}\{U^k x : k \in \mathbb{Z}, x \in F\}$. Sometimes, we write V_F instead of V_F^U whenever the operator is fixed. The *multiplicity* $\mathrm{Mult}(U) \in \mathbb{N} \cup \{\infty\}$ of U is then the smallest cardinality of generating families of U. By convention we let $\mathrm{Mult}(U) = 0$ when E is reduced to $\{0\}$. A linear operator with multiplicity one is called *cyclic*.

We first study the equivariant map between two normed vector spaces with linear invertible bounded operators.

Lemma 2.1. *Let $U_i : E_i \circlearrowleft$, $i = 1, 2$ be two linear invertible bounded operators. Assume that there is a linear bounded operator $W : E_1 \to E_2$ satisfying $W \circ U_1 = U_2 \circ W$ then*

$$\mathrm{Mult}(U_2|_{\overline{\mathrm{Im}(W)}}) \leq \mathrm{Mult}(U_1).$$

Proof. One checks easily that if F is a generating family for U_1 then $W(F)$ is a generating family for the restriction of U_2 to the closure of the image of W. Therefore $\mathrm{Mult}(U_2|_{\overline{\mathrm{Im}(W)}}) \leq \mathrm{Mult}(U_1)$. □

A direct consequence of Lemma 2.1 is that the multiplicity is a spectral invariant : if U_i are linear invertible operators on E_i , $i = 1, 2$, satisfying $W \circ U_1 = U_2 \circ W$ for some invertible bounded linear operator $W : E_1 \to E_2$, then U_1 and U_2 have the same multiplicities.

When E' is a closed subspace of E we endow the quotient E/E' space with the norm $\|\overline{u}\|' = \inf\{\|u + v\|,\ v \in E'\}$. If E' is invariant by U we let $U^{E/E'}$ be the action induced by U on the quotient normed space E/E'. In this context, by applying Lemma 2.1 with $W : E \to E/E'$ being the natural projection, we get

$$\mathrm{Mult}(U^{E/E'}) \leq \mathrm{Mult}(U). \tag{2·1}$$

2.2. **Operator of composition : Topological and ergodic multiplicities.** Ergodic theory focuses on the study of invertible measure preserving systems $(X, f, \mathcal{B}, \mu)$. In particular the spectral properties of the unitary operator $U_f : L^2(\mu) \circlearrowleft$, $\phi \mapsto \phi \circ f$, are investigated. We let $\|f\|_2 := (\int_X |f(x)|^2 d\mu)^{1/2}$ be the L^2-norm of $f \in L^2(\mu)$.

Definition 2.2. *The* ***ergodic multiplicity*** $\mathrm{Mult}(\mu)$ *of an ergodic system* $(X, f, \mathcal{B}, \mu)$ *is the multiplicity of the restriction of* U_f *to the Hilbert space* $L^2_0(\mu) := \{f \in L^2(\mu),\ \int f\, d\mu = 0\}$*, that is to say,* $\mathrm{Mult}(\mu) = \mathrm{Mult}(U_f)$*.*

This quantity has been intensely studied in ergodic theory (see Danilenko's survey [Dan13]).

Next we consider here an invertible topological dynamical system (X, T), i.e. $T : X \circlearrowleft$ is a homeomorphism of a compact metric space X. We denote by $C(X)$ the Banach space of real continuous functions endowed with the topology of uniform convergence. We let $\|f\|_\infty := \sup_{x\in X} |f(x)|$ be the supremum norm of $f \in C(X)$.

Definition 2.3. *The* ***topological multiplicity*** $\mathrm{Mult}(T)$ *of* (X, T) *is the multiplicity of the operator of composition* $U_T : C(X) \circlearrowleft$*,* $\phi \mapsto \phi \circ T$*.*

Quite surprisingly this last notion seems to be new (note however that cyclicity of U_T has already been investigated in some cases). Let us first observe that the topological multiplicity bounds from above the ergodic multiplicity of ergodic T-invariant measures.

Lemma 2.4. *Let* (X, T) *be an invertible topological dynamical system. For any ergodic* T*-invariant measure* μ*, we have*

$$\mathrm{Mult}(\mu) \le \mathrm{Mult}(T).$$

Proof. Let F be a generating family with minimal cardinality of $U_T : C(X) \circlearrowleft$. Then the vector space spanned by F is dense in $(C(X), \|\cdot\|_\infty)$, therefore in $(L^2(\mu), \|\cdot\|_2)$. As $p : L^2(\mu) \to L^2_0(\mu)$, $f \mapsto f - \int f\ d\mu$ is continuous and $p \circ U_T = U_T \circ p$, the vector space spanned by $p(F)$ is dense in $L^2_0(\mu)$.

□

Let $\mathcal{M}(X)$ be the set of Borel probability measures endowed with the weak-$*$ topology. It is standard that $\mathcal{M}(X)$ is a compact metrizable space. The compact subset $\mathcal{M}(X, T) \subset \mathcal{M}(X)$ of Borel T-invariant probability measures of (X, T) is a simplex, whose extreme set is given by the subset $\mathcal{M}_e(X, T)$ of ergodic measures. A topological system with a unique (ergodic) invariant measure is said to be *uniquely ergodic*. Jewett-Krieger theorem states that every ergodic system has a uniquely ergodic model. Several proofs have been given of this theorem, e.g. see [DGS06, Section 29]. One may wonder if the multiplicity may be preserved:

Question 2.5. *Given an ergodic system with measure* μ*, is there a uniquely ergodic model* (X, T) *of it such that* $\mathrm{Mult}(T) = \mathrm{Mult}(\mu)$*?*

2.3. **The number of ergodic measures as a multiplicity.** Let (X, T) be an invertible topological dynamical system. A function $\psi \in C(X)$ is a called a *continuous* T*-coboundary*, if ψ is equal to $\phi \circ T - \phi$ for some $\phi \in C(X)$. In other terms the set $B_T(X)$ of continuous T-coboundaries is the image of $U_T - \mathrm{Id}$, in particular it is a vector space. Observe that $U_T(B_T(X)) = B_T(X)$. To simplify the notations we write $\tilde{U}_T$ for the action induced by U_T on the quotient Banach space $C(X)/\overline{B_T(X)}$ and $\underline{U}_T$ for the restriction of U_T to the closure $\overline{B_T(X)}$ of continuous coboundaries. By a standard application of Hahn-Banach theorem (see e.g. Proposition 2.13 in [Kat01]), a function ψ belongs to $\overline{B_T(X)}$ if and only if $\int \psi\, d\mu = 0$ for any $\mu \in \mathcal{M}(X,T)$ (resp. $\mu \in \mathcal{M}_e(X,T)$). It is well-known that unique ergodicity is equivalent to the decomposition $C(X) = \mathbb{R}\mathbb{1} \oplus \overline{B_T(X)}$ (see e.g. Lemma 1 in [LV97]), where $\mathbb{1}$ denotes the constant function equal to 1. In particular in the case of unique ergodicity, we have $C(X)/\overline{B_T(X)} \simeq \mathbb{R}\mathbb{1}$ and therefore $\mathrm{Mult}(\tilde{U}_T) = 1$. It may be generalized as follows.

Lemma 2.6. *Let (X,T) be an invertible topological dynamical system. We have*

$$\mathrm{Mult}(\tilde{U}_T) = \sharp\mathcal{M}_e(X,T).$$

Proof. We first show that $\mathrm{Mult}(\tilde{U}_T) \geq \sharp\mathcal{M}_e(X,T)$. Assume that :

- $\nu_1, \cdots, \nu_p$ are distinct ergodic measures,
- $\overline{F} = \{\overline{f_1}, \cdots, \overline{f_q}\} \in C(X)/\overline{B_T(X)}$ is a generating family of $\tilde{U}_T$.

For $1 \leq l \leq q$, let $f_l \in C(X)$ be a function (a priori not unique) such that $\overline{f_l} = f_l \mod \overline{B_T(X)}$. If $q < p$ then the p vectors

$$X_i = \left(\int f_l\, d\nu_i\right)_{l=1,2,\cdots,q}, i = 1, 2, \cdots, p$$

are linearly dependent in $\mathbb{R}^q$, i.e. there is $(c_i)_{1\leq i\leq p} \in \mathbb{R}^p \setminus (0, 0, \ldots, 0)$ such that

$$\sum_{1\leq i\leq p} c_i X_i = 0. \tag{2·2}$$

Let ν be the signed measure $\nu = \sum_{1\leq i\leq p} c_i\nu_i$. Then Equality (2·2) may be rewritten as follows:

$$\forall 1 \leq l \leq q,\ \int f_l\, d\nu = 0.$$

The measures ν_i being invariant for $1 \leq i \leq p$, so is ν. Therefore we get

$$\forall 1 \leq l \leq q, \forall k \in \mathbb{Z},\ \int f_l \circ T^k\, d\nu = 0. \tag{2·3}$$

But $V_{\overline{F}}^{\tilde{U}_T} = C(X)/\overline{B_T(X)}$, so that for any $\epsilon > 0$ and for any $g \in C(X)$, we may find $h \in \mathrm{span}(f_l \circ T^k,\ 1 \leq l \leq q, k \in \mathbb{Z})$ and $u \in B_T(X)$ with $\|g - (h + u)\|_\infty < \epsilon$. By 2·3 we have $\int h\, d\nu = 0$. As u is a coboundary, we have also $\int u\, d\nu = 0$. Therefore

$$\left|\int g\, d\nu\right| \leq \left|\int (h + u)\, d\nu\right| + \|g - (h + u)\|_\infty < \epsilon.$$

Since $\epsilon > 0$ and $g \in C(X)$ are chosen arbitrarily, we obtain $\int g\, d\nu = 0$, for any $g \in C(X)$, therefore $\nu = 0$. This contradicts the ergodicity of the measures ν_i for $1 \leq i \leq p$. Consequently we have $q \geq p$ and therefore $\mathrm{Mult}(\tilde{U}_T) \geq \sharp\mathcal{M}_e(X,T)$.

Let us show now the converse inequality. Without loss of generality we may assume that $p = \sharp\mathcal{M}_e(X,T) < \mathrm{Mult}(\tilde{U}_T) = q < \infty$. We let again:

- $\mathcal{M}_e(X,T) = \{\nu_1, \cdots, \nu_p\}$,
- $\overline{F} = \overline{f_1}, \cdots, \overline{f_q} \in C(X)/\overline{B_T(X)}$ a generating family of $\tilde{U}_T$ with minimal cardinality.

Then the q vectors

$$Y_l = \left(\int f_l \, d\nu_i\right)_{i=1,\cdots,p}, \ l = 1, \cdots, q,$$

are linearly dependent in $\mathbb{R}^p$, i.e. there is $(c_l)_{1\leq l\leq q} \in \mathbb{R}^q \setminus (0, 0 \ldots, 0)$ such that

$$\sum_{1\leq l\leq q} c_l Y_l = 0.$$

Let g be the function $g = \sum_{1\leq l\leq q} c_l f_l$. Then we have

$$\int g \, d\nu_i = 0, \ \forall 1 \leq i \leq p,$$

A previously mentioned, it implies that g lies in $\overline{B_T(X)}$. This contradicts the minimality of the generating family $\overline{F}$. □

Remark 2.7. *It follows from the proof of Lemma 2.6 that if $\mathcal{M}_e(X,T) = \{\nu_1, \cdots, \nu_p\}$, then $\overline{f_1}, \cdots, \overline{f_p}$ is a generating family of $\tilde{U}_T$ if and only if the matrix $A = (\int f_j \, d\nu_i)_{1\leq i,j\leq p} \in M_p(\mathbb{R})$ is invertible.*

By inequality (2·1) and Lemma 2.6 we get:

Corollary 2.8.

$$\sharp\mathcal{M}_e(X,T) \leq \mathrm{Mult}(T).$$

2.4. **Relating** $\mathrm{Mult}(T)$ **and** $\mathrm{Mult}(\underline{U}_T)$**.** It follows from definition of $\overline{B_T(X)}$ that the map

$$W : C(X) \to \overline{B_T(X)}, \ f \mapsto f \circ T - f$$

has dense image and commutes with U_T. By applying Lemma 2.1 with $U_1 = U_2 = U$ and $E_1 = C(X)$, $E_2 = \overline{B_T(X)}$, we obtain $\mathrm{Mult}(\underline{U}_T) \leq \mathrm{Mult}(T)$.

We show then in this subsection the following inequality.

Proposition 2.9.

$$\mathrm{Mult}(T) \leq \mathrm{Mult}(\tilde{U}_T) + \mathrm{Mult}(\underline{U}_T) - 1.$$

In particular if (X,T) is uniquely ergodic, $\mathrm{Mult}(T) = \mathrm{Mult}(\underline{U}_T)$ by Lemma 2.6. Let us now prove Proposition 2.9. For a family F of $C(X)$, we write $\overline{F}$ the subset of $C(X)/\overline{B_T(X)}$ consisting of $\overline{f} = f \mod \overline{B_T(X)}$ for $f \in F$. We start with a technical lemma.

Lemma 2.10. *Let (X,T) be an invertible dynamical system with $\sharp\mathcal{M}_e(X,T) < \infty$. If F is a family of $C(X)$ such that $\overline{F}$ is generating for $\tilde{U}_T$, then the constant function $\mathbb{1}$ belongs to V_F.*

Proof. Let $\mathcal{M}_e(X,T) = \{\nu_1, \cdots, \nu_p\}$. By Remark 2.7 the matrix $(\int f_i\, d\nu_j)_{1\le i,j\le p}$ is invertible. Then by replacing $F = \{f_1, \cdots, f_p\}$ by some invertible linear combinations we can assume

$$\forall 1 \le i,j \le p,\ \int f_i\, d\nu_j = \delta_{i,j},$$

where $\delta_{i,j}$ is equal to 1 if $i = j$ and 0 otherwise. Let $f = \sum_{i=1}^p f_i$. We have

$$\forall 1 \le j \le p, \int f\, d\nu_j = \sum_i \delta_{i,j} = 1,$$

$$\text{therefore, } \forall \nu \in \mathcal{M}(X,T), \int f\, d\nu = 1. \tag{2·4}$$

We claim that $\frac{1}{N}\sum_{n=0}^{N-1} f\circ T^n$ is converging uniformly to $\mathbb{1}$ as N goes to infinity. If not, there would exist a positive number ϵ, a sequence $(x_k)_{k\ge1}$ and an increasing sequence $(N_k)_{k\ge1}$ of positive integers such that

$$\left| \frac{1}{N_k}\sum_{n=0}^{N_k-1} f(T^n(x_k)) - 1 \right| > \epsilon,\ \forall k \ge 1. \tag{2·5}$$

After passing to a subsequence of $(N_k)_{k\ge1}$, we might assume that $\frac{1}{N_k}\sum_{n=0}^{N_k-1} \delta_{T^n(x_k)}$ is converging to a T-invariant measure μ in the weak-$*$ topology. It follows from (2·5) that

$$\left| \int f d\mu - 1 \right| > \epsilon > 0.$$

It is a contradiction to (2·4). Therefore, $\frac{1}{N}\sum_{n=0}^{N-1} f\circ T^n$ is converging uniformly to $\mathbb{1}$ as N goes to infinity, in particular $\mathbb{1} \in V_F$. □

Proof of Proposition 2.9. Let $\overline{\mathcal{F}} = \{\overline{f_1}, \cdots, \overline{f_p}\}$ and $\mathcal{G} = \{g_1, \cdots, g_q\}$ be generating families of $\tilde{U}_T$ and $\underline{U}_T$ with $p = \text{Mult}(\tilde{U}_T)$ and $q = \text{Mult}(\underline{U}_T)$. For $1 \le l \le p$, take $f_l \in C(X)$ be a function such that $\overline{f_l} = f_l \mod \overline{B_T(X)}$, then let $\mathcal{F} = \{f_1, \cdots, f_l\}$. One easily checks that $\mathcal{F} \cup \mathcal{G}$ is a generating family of U_T. By Lemma 2.6 we may write $\mathcal{M}_e(X,T) = \{\nu_1, \cdots, \nu_p\}$. As in the proof of Lemma 2.10 we may assume without loss of generality $\int f_i\, d\nu_j = \delta_{i,j}$ for any $1 \le i,j \le p$. Let $g_1' = g_1 + \mathbb{1}$, hence

$$\left(\int g_1'\, d\nu_i \right)_{1\le i\le p} = (1, \cdots, 1).$$

By Remark 2.7 the family $\{\overline{g_1'}, \overline{f_j} :\ 1 < j \le p\}$ is generating for $\tilde{U}_T$. By Lemma 2.10 the constant functions, therefore also g_1, belongs to $V_{\{g_1', f_j:\ 1<j\le p\}}$. Then $V_{\{g_1', f_j, g_i\ :\ 1<j\le p, 1<i\le q\}} = V_{\{g_1', f_j, g_i\ :\ 1<j\le p, 1\le i\le q\}}$ and $f_1 \in V_{\{g_1', f_j, g_i\ :\ 1<j\le p, 1\le i\le q\}}$. Consequently we get

$$V_{\{g_1', f_j, g_i\ :\ 1<j\le p, 1<i\le q\}} \supset V_{\mathcal{F}\cup\mathcal{G}} = C(X).$$

We conclude that $\text{Mult}(T) \le p + q - 1 = \text{Mult}(\tilde{U}_T) + \text{Mult}(\underline{U}_T) - 1$. □

3. Affine embedding of $(\mathcal{M}(X), T_*)$ in cubical shifts

For a topological system (X, T) we denote by T_* the action induced by T on the compact set $\mathcal{M}(X)$, i.e. $T_*\mu(\cdot) = \mu(T^{-1}\cdot)$ for all $\mu \in \mathcal{M}(X)$. Then $(\mathcal{M}(X), T_*)$ is also a topological system, which is called the *induced system* of (X, T).

For $d \in \mathbb{N}$ we let σ_d be the shift on the simplex $([0,1]^d)^{\mathbb{Z}}$. An *embedding* of (X,T) in $([0,1]^d)^{\mathbb{Z}}$ is a continuous injective map $\phi : X \to ([0,1]^d)^{\mathbb{Z}}$ satisfying $\phi \circ T = \sigma_d \circ \phi$. Existence of such embedding is related to the mean dimension theory (we refer to [Coo15] for an introduction). Such an embedding implies that the mean dimension of (X,T) is less than or equal to d. Moreover the topological dimension (i.e. Lebesgue covering dimension) d_n^T of the set of n-periodic points then also satisfy $\frac{d_n^T}{n} \leq d$. Conversely it has been shown that minimal systems with mean dimension less than $d/2$ can be embedded in the cubical shift σ_d [Lin99, GT20].

In this section we consider affine embedding of the induced system $(\mathcal{M}(X), T_*)$ in cubical shift σ_d, i.e. the embedding $\phi : \mathcal{M}(X) \to ([0,1]^d)^{\mathbb{Z}}$ is affine. In particular we will relate the embedding dimension d with the multiplicity of (X,T).

3.1. **Case of finite sets.** We first deal with the case of a finite set X. Then T is just a permutation of X and $\mathcal{M}(X)$ is a finite dimensional simplex. We classify the possible affine embedding of $(\mathcal{M}(X), T_*)$ in the following proposition.

Proposition 3.1. *Suppose X is a finite set and T is a a permutation of X. Let $\tau_1 \cdots \tau_k$ be the decomposition of T into disjoint cycles τ_i of length r_i for $1 \leq i \leq k$.*

(1) *If there is a nontrivial common factor of r_i for $1 \leq i \leq k$, then there is an affine embedding of $(\mathcal{M}(X), T_*)$ in $\big(([0,1]^k)^{\mathbb{Z}}, \sigma_k\big)$. Such k is sharp.*
(2) *If there is no nontrivial common factor of r_i for $1 \leq i \leq k$, then there is an affine embedding of $(\mathcal{M}(X), T_*)$ in $\big(([0,1]^{k-1})^{\mathbb{Z}}, \sigma_{k-1}\big)$. Such $k-1$ is sharp.*

Proof. For each $1 \leq i \leq k$ we fix a point $e_i \in X$ in each cycle τ_i, i.e. $\{T^j e_i : 0 \leq j \leq r_i\} = X$. Notice that there are continuous maps $a_e : \mathcal{M}(X) \to [0,1]$, $e \in X$, with $\sum_{e \in X} a_e = \mathbb{1}$ satisfying $\mu = \sum_{e \in X} a_e(\mu)\delta_e$ for all $\mu \in \mathcal{M}(X)$.
(1) Assume there is a nontrivial common factor p of r_i for $1 \leq i \leq k$. Then

$$\dim(\mathrm{Fix}(T_*^p)) = kp - 1,$$

where $\mathrm{Fix}(T_*^p) = \{\mu \in \mathcal{M}(X) : \ T_*^p\mu = \mu\}$. Since $p > 1$ and $\dim(\mathrm{Fix}(\sigma_{k-1}^p)) = kp - p$, the dynamical system $(\mathcal{M}(X), T_*)$ can not embed in $\big(([0,1]^{k-1})^{\mathbb{Z}}, \sigma_{k-1}\big)$.

Now we construct the embedding of $(\mathcal{M}(X), T_*)$ in $\big(([0,1]^k)^{\mathbb{Z}}, \sigma_k\big)$. We define firstly a dynamical embedding Ψ of the set of extreme points in $\mathcal{M}(X)$, which is identified with X through the map $x \mapsto \delta_x$, into $([0,1]^k)^{\mathbb{Z}}$ by letting

$$\forall i = 1, \cdots, k, \ \forall l \in \mathbb{Z}, \ \ (\Psi(T^l e_i))_i = \sigma^l\left((10^{r_i - 1})^\infty\right);$$

the other components $(\Psi(T^le_i))_j$, $j \neq i$, being chosen to be equal to the 0^∞ sequence. Then we may extend Ψ affinely from the set of extreme points on $\mathcal{M}(X)$ by letting

$$\Psi(\mu) = \sum_{e\in X} a_e(\mu)\Psi(\delta_e).$$

It is easy to check that Ψ is injective which deduces a dynamical embedding $(\mathcal{M}(X), T_*)$ in $\big(([0,1]^k)^{\mathbb{Z}}, \sigma_k\big)$.

(2) Assume there is no nontrivial common factor of r_i for $1 \le i \le k$. We have that $k-1$ numbers $q_i := (r_k, r_i)$, $1 \le i \le k-1$ are co-prime where (a, b) are the highest common factor of a and b. We define firstly a continuous map Ψ of the set of extreme points in $\mathcal{M}(X)$ into $([0,1]^{k-1})^{\mathbb{Z}}$ by letting

$$\forall l \in \mathbb{Z},\quad (\Psi(T^le_k))_j = \sigma^l\left((10^{r_i-1})^\infty\right), \forall 1 \le j \le k-1,$$

and

$$\forall 1 \le i \le k-1,\ \forall l \in \mathbb{Z},\quad (\Psi(T^le_i))_i = \sigma^l\left((10^{r_i-1})^\infty\right);\ \forall j \neq i, \forall l \in \mathbb{Z}, (\Psi(T^le_i))_j = 0^\infty.$$

Then we may extend Ψ affinely from the set of extreme points on $\mathcal{M}(X)$ by letting

$$\Psi(\mu) = \sum_{e\in X} a_e(\mu)\Psi(\delta_e).$$

It remains to show that Ψ is injective. Let $\mu = \sum_{e\in X} b_e\delta_e$ and $\mu' = \sum_{e\in X} b'_e\delta_e$. Suppose

$$\Psi(\mu) = \Psi(\mu') = \sum_{e\in X} a_e\Psi(\delta_e) = (c_{i,j})_{1\le i\le k-1, j\in\mathbb{Z}}.$$

Since $q_i := (r_k, r_i)$ then there are integers s_i and t_i such that $s_ir_i - t_ir_k = q_i$. Let

$$u_{i,l} = s_ir_i + l = t_ir_k + q_i + l.$$

It implies that

$$b'_{T^le_k} - b'_{T^{l+q_i}e_k} = b_{T^le_k} - b_{T^{l+q_i}e_k} = c_{i,l} - c_{i,u_{i,l}}.$$

Since $q_i, 1 \le i \le k-1$ are co-prime, there are integers $w_i, 1 \le i \le k-1$ such that $\sum_{i=1}^{k-1} w_iq_i = 1$. Since

$$\begin{aligned}(b_{e_k} - b_{T^{w_1q_1}e_k}) + \big(b_{T^{w_1q_1}e_k} - b_{T^{w_1q_1+w_2q_2}e_k}\big)\\ + \cdots + \big(b_{T^{w_1q_1+w_2q_2+\cdots+w_{k-2}q_{k-2}}e_k} - b_{T^{w_1q_1+w_2q_2+\cdots+w_{k-1}q_{k-1}}e_k}\big) = b_{e_k} - b_{Te_k},\end{aligned}$$

we have

$$b_{e_k} - b_{T^le_k} = b'_{e_k} - b'_{T^le_k},\ \forall l \in \mathbb{Z}.$$

Since $\sum_{e\in X} b_e = \sum_{e\in X} b'_e = 1$, we conclude that $b_{e_k} = b_{e'_k}$ and consequently $b_{e_i} = b_{e'_i}$ by $b_{e_k} + b_{e_i} = b'_{e_k} + b'_{e_i} = c_{i,0}$ for $1 \le i \le k-1$. It means that $\mu = \mu'$ and Ψ is injective. □

Remark 3.2. *For such a permutation T, we have* $\mathrm{Mult}(T) = \sharp\mathcal{M}_e(X,T) = k$*, with k being the number of cycles in the decomposition of T.*

3.1.1. *General case.* We consider now a general topological system and relates the dimension of the cubical shift in an affine embedding with the multiplicity of (X,T).

Theorem 3.3. *Let (X,T) be a topological system. If $\mathrm{Mult}(T)$ is equal to d, then there is an affine embedding of $(\mathcal{M}(X),T_*)$ in $(([0,1]^d)^{\mathbb{Z}},\sigma_d)$. Conversely if $(\mathcal{M}(X),T_*)$ embeds into $(([0,1]^d)^{\mathbb{Z}},\sigma_d)$ then*

- *either $\sharp\mathcal{M}_e(X,T)\leq d$ and $\mathrm{Mult}(T)\leq d$,*
- *or $\sharp\mathcal{M}_e(X,T)=d+1$ and $\mathrm{Mult}(T)=d+1$.*

Proof. Firstly, notice that any affine equivariant map $\Psi:(\mathcal{M}(X),T_*)\to(([0,1]^d)^{\mathbb{Z}},\sigma_d)$ is of the form

$$\Psi_f:\mu\mapsto\left(\int f\circ T^k\,d\mu\right)_{k\in\mathbb{Z}},$$

for some continuous function $f=(f_1,\cdots,f_d):X\to[0,1]^d$.

Assume the topological multiplicity $\mathrm{Mult}(X,T)$ is equal to d, i.e. there is a family $F=\{f_1,\cdots,f_d\}$ of continuous functions such that $V_F=C(X)$. Let us show the associated map Ψ_f is injective. Let $\mu_1,\mu_2\in\mathcal{M}(X)$ with $\Psi_f(\mu_1)=\Psi_f(\mu_1)$ i.e. $\int f_i\circ T^k\,d\mu_1=\int f_i\circ T^k\,d\mu_1$ for any $i=1,\cdots,d$ and any $k\in\mathbb{Z}$. Then by density of $\mathrm{span}(f_i\circ T^k,\ i,\ k)$ in $C(X)$ we have

$$\int g\,d\mu_1=\int g\,d\mu_1 \text{ for all } g\in C(X),$$

which implies $\mu_1=\mu_2$. Therefore we get the injectivity of Ψ_f.

Conversely, assume Ψ_f is injective for $f=(f_1,\cdots,f_d):X\to[0,1]^d$. Let $F=\{\mathbb{1},f_1,\cdots,f_d\}$. We claim that $V_F=C(X)$. Then if $\sharp\mathcal{M}_e(X,T)\leq d$, we get by injectivity of Ψ_f that there exists $A\subset\{1,2,\ldots,d\}$ with $\sharp A=\sharp\mathcal{M}_e(X,T)$ such that the matrix $\left(\int f_i d\nu\right)_{i\in A,\nu\in\mathcal{M}_e(X,T)}$ is invertible. Then by Remark 2.7 the family $\overline{F\setminus\{\mathbb{1}\}}$ is generating for $\tilde{U}_T$ and consequently $V_F=V_{F\setminus\{\mathbb{1}\}}$ by Lemma 2.10. If $\sharp\mathcal{M}_e(X,T)=d+1$ then we only get $\mathrm{Mult}(T)=d+1$.

It remains to show our claim. Assume to the contrary that $V_F\neq C(X)$. Then by Riesz Theorem there is a signed finite measure μ vanishing on each function in F. Let $\mu=\mu^+-\mu^-$ be the Jordan decomposition of μ (i.e. the measures μ^+ and μ^- are two finite positive measures which are mutually singular). Evaluating on the constant function $\mathbb{1}$, we get $\mu^+(X)=\mu^-(X)$. Then by rescaling, we may assume both μ^- and μ^+ belong to $\mathcal{M}(X)$. Finally we get $\Psi_f(\mu^+)=\Psi_f(\mu^-)$, and therefore $\mu^+=\mu^-$ by injectivity of Ψ_f contradicting therefore the mutual singularity of μ^+ and μ^-.

□

3.2. **Affine embeddings and Lindenstrauss-Tsukamoto conjecture.** Lindenstrauss and Tsukamoto [LT14] have conjectured that any topological system with mean dimension $\mathrm{mdim}(X,T)$ less than $d/2$ and such that the dimension d_n^T of the set of n-periodic points satisfies $\frac{d_n^T}{n}<d/2$ for any $n\in\mathbb{N}$ may be embedded in the shift over $([0,1]^d)^{\mathbb{Z}}$. As mentioned above it is known for minimal systems. We consider here affine systems, i.e. affine maps of

a simplex. Such maps are never minimal, as they always admits at least one fixed point.

The example below shows that Lindenstrauss-Tsukamoto conjecture does not hold true in the affine category. Recall that an ergodic system $(X, f, \mathcal{B}, \mu)$ has a *countable Lebesgue spectrum*, when there is a countable family $(\psi_n)_{n\in\mathbb{N}}$ in $L^2_0(\mu)$ such that $\psi_n \circ f^k$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$ form a Hilbert basis of $L^2_0(\mu)$.

Corollary 3.4. *There is an affine system with a unique periodic (fixed) point and zero-topological entropy (in particular mdim$(T) = 0$ and $d_n^T = 0$ for all $n \geq 1$) which does not embed affinely in $(([0,1]^k)^{\mathbb{Z}}, \sigma)$ for any $k \geq 1$.*

Proof. There exists an ergodic measure preserving system $(Y, \mathcal{A}, f, \mu)$ with zero entropy and countable Lebesgue spectrum [NP66, Par53] (in particular *totally ergodic*, i.e. f^n is ergodic for any $n \in \mathbb{Z}$). Then by Jewett-Krieger theorem there is a uniquely ergodic topological system (X, T) with measure ν realizing such a measure preserving system. All powers of T are uniquely ergodic as μ was chosen totally ergodic. Moreover the topological entropy of T, thus that of T_* is zero by Glasner-Weiss [GW95]. As the unique invariant measure ν has countable Lebesgue spectrum, the topological multiplicity of (X, T) is infinite by Lemma 2.4. We conclude with Theorem 3.3. □

3.3. **Application: zero topological entropy.** A classical result in ergodic theory states that any ergodic system $(X, f, \mathcal{B}, \mu)$ with positive entropy has a countable Lebesgue spectrum. In particular $[h(\mu) > 0] \Rightarrow [\mathrm{Mult}(\mu) = \infty]$. Then it follows from the variational principle for the topological entropy :

Proposition 3.5. *Any topological system (X, T) with* $\mathrm{Mult}(T) < \infty$ *has zero topological entropy.*

We may also give a purely topological proof of Proposition 3.5 based on mean dimension theory. More precisely we use the main result of [BS22], which states as follows:

Theorem 3.6. [BS22] *For any topological system (X, T) with positive topological entropy, the induced system $(\mathcal{M}(X), T_*)$ has infinite topological mean dimension. Therefore,*

$$h_{top}(T) > 0 \Leftrightarrow \mathrm{mdim}(T_*) > 0 \Leftrightarrow \mathrm{mdim}(T_*) = \infty.$$

Topological proof of Proposition 3.5. Assume $\mathrm{Mult}(T) = d$ is finite. Then by Theorem 3.3 the induced system $(\mathcal{M}(X), T_*)$ embeds in the cubical shift $\big(([0,1]^d)^{\mathbb{Z}}, \sigma\big)$. In particular the mean dimension of T_* is less than or equal to the mean dimension of the shift $\big(([0,1]^d)^{\mathbb{Z}}, \sigma\big)$, which is equal to d. By Theorem 3.6, it implies that T has zero topological entropy. □

4. Baxter's Lemma in Banach spaces

In [Bax71], Baxter gave a useful criterion to show simple spectrum of ergodic transformations. It may be extended more generally to bound the multiplicity of the spectrum (e.g. see Proposition 2.12 in [Que10]). We generalize this criterion for operators defined on a Banach space. It will be used in the next section to estimate the topological multiplicity in some examples.

Lemma 4.1. *Let B be a separable Banach space and $\mathcal{L}(B)$ be the set of bounded linear operator on B. We consider an invertible isometry $U \in \mathcal{L}(B)$. If $(\mathcal{F}_n)_n$ is a sequence of finite subsets in H satisfying for all $f \in B$*

$$\inf_{F_n \in V_{\mathcal{F}_n}} \|F_n - f\| \xrightarrow{n\to\infty} 0 \tag{4·1}$$

then there exists a family $\mathcal{F} \subset B$ with $\sharp\mathcal{F} \leq \sup_n \sharp\mathcal{F}_n$ and $B = V_{\mathcal{F}}$.

Classical proofs of Baxter's lemma strongly used the Hilbert structure. Here we use a Baire argument as in Lemma 5.2.10 [Fog02]. Note also that we do not require the sequence of vectors spaces $(V_{\mathcal{F}_n})_n$ to be nondecreasing. Observe finally that it is enough to assume (4·1) for f in S, where S spans a dense subset of B.

Proof. Let $m = \sup_n \sharp\mathcal{F}_n$. If $m = \infty$, it is trivial. Assume $m < \infty$. By passing to a subsequence, we assume $\sharp\mathcal{F}_n = m$ for all n. Let $B^{(m)}$ be the space of finite subsets of H whose cardinality is smaller than or equal to m. When endowed with the Hausdorff distance d_{Hau}, the space $B^{(m)}$ is a metric space, which is complete and separable. We assume the following claim, which we prove later on.

Claim 4.2. *For any $\epsilon > 0$ and $\mathcal{F} \in B^{(m)}$, the set*

$$O(\mathcal{F}, \epsilon) = \left\{ \mathcal{G} \in B^{(m)},\ \forall f \in \mathcal{F} \inf_{G \in V_{\mathcal{G}}} \|G - f\| < \epsilon \right\}$$

is open and dense.

Let $(g_q)_{q\in\mathbb{N}}$ be a countable dense family in B. Let $\mathcal{G}_q$ be the finite family $\{g_1, \cdots, g_q\}$. For any $p \in \mathbb{N}^*$ and any $q \in \mathbb{N}$ we consider the open and dense set

$$O_{p,q} := O(\mathcal{G}_q, 1/p) = \left\{ \mathcal{G} \in B^{(m)},\ \forall g \in \mathcal{G}_q \inf_{G \in V_{\mathcal{G}}} \|G - g\| < 1/p \right\}.$$

According to Baire's theorem, the intersection $\bigcap_{p,q} O_{p,q}$ is not empty. Clearly any family $\mathcal{F}$ in the intersection satisfies $V_{\mathcal{F}} = B$. It remains to prove Claim 4.2.

Proof of Claim 4.2. The set $O(\mathcal{F}, \epsilon)$ is open. We focus on the denseness property. Pick arbitrary $\delta > 0$ and $\mathcal{H} \in B^{(m)}$. We will show that there is $\mathcal{H}' \in O(\mathcal{F}, \epsilon)$ with $d_{Hau}(\mathcal{H}, \mathcal{H}') < \delta$. As the elements of $B^{(m)}$ with cardinality m are dense in $B^{(m)}$ we can assume without loss of generality that $\sharp\mathcal{H} = m$. By assumptions on the sequence $\mathcal{F}_n$, there exists n such that

$$\forall f \in \mathcal{F} \inf_{F_n \in V_{\mathcal{F}_n}} \|F_n - f\| < \epsilon, \tag{4·2}$$

$$\forall h \in \mathcal{H} \inf_{F_n \in V_{\mathcal{F}_n}} \|F_n - h\| < \delta. \tag{4·3}$$

We write $\mathcal{H} = \{h_1, h_2, \ldots, h_m\}$ and $\mathcal{F}_n = \{f_1, f_2, \ldots, f_m\}$. By (4·3) there are polynomials $(P_{i,j})_{1\leq i,j\leq m}$ in $\mathbb{R}[X]$ and a nonnegative integer p such that

$$\forall i = 1, \cdots, m, \left\| h_i - \sum_{j=1}^{m} U^{-p} P_{i,j}(U) f_j \right\| < \delta \tag{4·4}$$

Let $Q \in \mathbb{R}[X]$ be the polynomial given by the determinant of the matrix $M = (P_{i,j})_{1\leq i,j\leq m} \in M_m(\mathbb{R}[X])$. The spectrum $\mathrm{Sp}(U)$ of U is contained in the unit circle. In particular for arbitrarily small $\lambda \in \mathbb{R}$, the polynomial $Q(\cdot + \lambda)$ does not vanish on $\mathrm{Sp}(U)$. Hence by replacing $P_{i,j}$ by $P_{i,j}(\cdot + \lambda)$ we may assume that Q does not vanish on the spectrum of U. Then $Q(U) = \prod_{\lambda,\, Q(\lambda)=0}(U - \lambda \mathrm{Id})$ is invertible and its inverse may be approximated by polynomials in U and U^{-1}, because for λ with $Q(\lambda) = 0$ we have $(U - \lambda\mathrm{Id})^{-1} = -\sum_{k\in\mathbb{N}} \frac{U^k}{\lambda^{k+1}}$ for $|\lambda| > 1$ and $(U - \lambda\mathrm{Id})^{-1} = -\sum_{k\in\mathbb{N}} \frac{U^{-(k+1)}}{\lambda^k}$ for $|\lambda| < 1$ (these sequences are normally convergent in $\mathcal{L}(B)$ as we assume $\|U\| = \|U^{-1}\| = 1$). Let $\mathcal{H}' = \{h'_1, h'_2, \ldots, h'_m\}$ with $h'_i = \sum_{j=1}^m U^{-p}P_{i,j}(U)f_j$. We have

$$\begin{aligned}\mathcal{F}_n &= M(U)^{-1}U^p\mathcal{H}',\\ &= {}^{\mathrm{t}}\mathrm{com}\, M(U)U^pQ(U)^{-1}\mathcal{H}'.\end{aligned}$$

Then from the above observations we get

$$\mathcal{F}_n \subset V_{\mathcal{H}'}.$$

In particular $V_{\mathcal{F}_n} \subset V_{\mathcal{H}'}$ and it follows finally from (4·2) that $\mathcal{H}' \in O(\mathcal{F}, \epsilon)$, i.e.

$$\forall f \in \mathcal{F} \ \inf_{H'\in V_{\mathcal{H}'}} \|H' - f\| < \epsilon.$$

This completes the proof as we have $d_{Hau}(\mathcal{H}, \mathcal{H}') < \delta$ by (4·4), where $\mathcal{H}$ and δ have been chosen arbitrarily. □

□

5. Cantor systems with finite topological rank

Roughly speaking an ergodic measure preserving system is of finite rank r, when it may be obtained by cutting and staking with r Kakutani-Rohlin towers. For ergodic systems, Baxter's lemma implies that the ergodic multiplicity is less than or equal to the rank. Topological rank as been defined and studied for minimal Cantor systems (see e.g. [DDMP21] and the references therein). For such systems we show now with Lemma 4.1 that the same inequality holds for the topological quantities : the topological multiplicity is less than or equal to the topological rank.

Firstly we recall the definition of topological rank. Let (X, T) be a minimal Cantor system. A *Kakutani-Rohlin partition* of X is given by

$$\mathcal{T} = \{T^{-j}B(k) : 1 \leq k \leq d, 0 \leq j < h(k)\},$$

where $d, h(k), 1 \leq k \leq d$ are positive integers and $B(k), 1 \leq k \leq d$ are clopen subsets of X such that

$$\cup_{k=1}^d T^{-h(k)}B(k) = \cup_{k=1}^d B(k).$$

The *base* of $\mathcal{T}$ is the set $B(\mathcal{T}) = \cup_{k=1}^d B(k)$. A sequence of Kakutani-Rohlin partitions

$$\mathcal{T}_n = \{T^{-j}B_n(k) : 1 \leq k \leq d_n, 0 \leq j < h_n(k)\}, n \geq 1,$$

is *nested* if

(1) $\mathcal{T}_0$ is the trivial partition, i.e. $d_0 = 1, h_0 = 1$ and $B_0(1) = X$.
(2) $B(\mathcal{T}_{n+1}) \subset B(\mathcal{T}_n)$.
(3) $\mathcal{T}_{n+1} \succ \mathcal{T}_n$.
(4) $\sharp(\cap_{n\geq 0} B(\mathcal{T}_n)) = 1$.
(5) $\cup_{n\geq 1}\mathcal{T}_n$ spans the topology of X.

Moreover, it is *primitive* if for all $n \geq 1$ there exists $N > n$ such that for all $1 \leq k \leq d_N$ and for each $x \in T^{-(hN(k)-1)}B_N(k)$,

$$\{T^i(x) : 0 \leq i \leq h_N(k) - 1\} \cap B_n(j) \neq \emptyset, \forall 1 \leq j \leq d_n.$$

Following [DDMP21], a minimal Cantor system is of *topological rank* d if it admits a primitive sequence of nested Kakutani-Rohlin partitions with $d_n \leq d$ for all $n \in \mathbb{N}$.

Theorem 5.1. *Let (X, T) be a minimal Cantor system with topological rank d. Then* $\mathrm{Mult}(X, T) \leq d$.

Proof. Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be the primitive sequence of nested Kakutani-Rohlin partitions with $d_n \leq d$ for all $n \in \mathbb{N}$. Let

$$\mathcal{F}_n = \{\chi_{B_n(k)} : 1 \leq k \leq d_n\}.$$

Since $\cup_{n\geq 1}\mathcal{T}_n$ spans the topology of X, we have

$$\forall f \in C(X), \inf_{F_n\in V_{\mathcal{F}_n}} \|F_n - f\| \xrightarrow{n\to\infty} 0.$$

It follows from Lemma 4.1 and $d_n \leq d$ for all $n \in \mathbb{N}$ that $\mathrm{Mult}(X, T) \leq d$. □

Remark 5.2. *It was shown in* [DDMP21] *that the Thue-Morse subshift has topological rank* 3. *By Theorem 5.1, Thue-Morse subshift has therefore topological multiplicity at most* 3. *However, we will prove in Proposition 6.7 that Thue-Morse subshift has simple topological multiplicity.*

Examples of ergodic systems with rank r and multiplicity m have been built for any $1 \leq m \leq r$ in [KL97]. We then propose the following question.

Question 5.3. *Can one build for any $1 \leq m \leq r$ a minimal Cantor system with topological multiplicity m and topological rank r?*

6. Examples of finite topological multiplicity

An invertible dynamical system is called *topological simple* or *have simple topological spectrum* if $\mathrm{Mult}(T) = 1$.

6.0.1. *Minimal rotation on compact groups.* Let G be a compact abelian group. Denote by $\hat{G}$ the dual group of G and by λ the Haar measure on G. For $f \in C(G)$, we write $\hat{f}$ the Fourier transformation of f.

Proposition 6.1. *Any minimal translation τ on a compact abelian group G is topologically simple.*

Proof. We claim that any $f \in C(G)$ with $\hat{f}(\chi) \neq 0$ for all $\chi \in \hat{G}$ is cyclic, i.e. the vector space spanned by $f \circ \tau^k$, $k \in \mathbb{N}$ is dense in $C(X)$. As characters of a compact abelian group separates points it is enough to show by Stone-Weierstrass theorem that any character belongs to the complex vector space spanned by $f \circ \tau^k$, $k \in \mathbb{N}$. But for all $\chi \in \hat{G}$ we have

$$\hat{f}(\chi)\chi = \chi * f = \int f(\cdot - y)\chi(y)\, d\lambda(y).$$

Then the function f being uniformly continuous, there are functions of the form $\sum_k f(\cdot - y_k)\chi(y_k)$ arbitrarily close to $\hat{f}(\chi)\chi$ for the supremum norm. By minimality of τ, there are integers $l_k \in \mathbb{N}$ such that $f \circ \tau^{l_k}$ and $f(\cdot - y_k)$ are arbitrarily closed. It concludes the proof. □

6.0.2. *Sturmian subshift.* A word $u \in \{0,1\}^{\mathbb{Z}}$ is called *Sturmian* if it is recurrent under the shift σ, and the number of n-words in u equals $n+1$ for each $n \geq 1$. Take the shift-orbit closure $X_u = \overline{O_\sigma(u)}$. The corresponding subshift (X_u, σ) is called a *Sturmian subshift*. Sturmian sequences are symbolic representation of circle irrational rotations.

We first recall some standard notations in symbolic dynamics. For a subset Y of $\mathcal{A}^{\mathbb{Z}}$ with $\mathcal{A}$ being a finite alphabet we let $\mathcal{L}_n(Y)$ be the number of n-words appearing in the sequences of Y. Then for $w \in \mathcal{L}_n(Y)$ we let $[w]$ be the associated cylinder defined as $[w] := \{(x_n)_{n\in\mathbb{Z}} \in Y : \ x_0 \cdots x_{n-1} = w\}$. The indicator function of a subset E of X will be denoted by χ_E.

Proposition 6.2. *Any Sturmian subshift has simple topological spectrum.*

Proof. Let u be a Sturmian sequence. Let

$$F_n = \operatorname{span}\{\chi_{[w]} : w \in \mathcal{L}_n(u)\}.$$

It follows that

$$C(X_u) = V_{\cup_n F_n}.$$

Notice $\dim(F_n) = \sharp\mathcal{L}_n(u) = n+1$. We let $f : X_u \to \mathbb{R}$ be the continuous function defined as $f : x = (x_n)_n \mapsto (-1)^{x_0}$. Let

$$G_n = \operatorname{span}\{\mathbb{1}, f \circ \sigma^k : 0 \leq k \leq n-1\}.$$

Clearly, $G_n \subset F_n$. To prove that (X_u, σ) has simple topological spectrum, it is sufficient to show $\dim(G_n) = n+1$. Thus it is enough to show the functions $\{\mathbb{R}\mathbb{1}, f\circ\sigma^k : 0 \leq k \leq n-1\}$ are linearly independent. If not, for some n there exists a nonzero vector $(a_0, a_1, \dots, a_n)$ such that

$$a_0(-1)^{x_0} + a_1(-1)^{x_1} + \cdots + a_{n-1}(-1)^{x_{n-1}} + a_n = 0, \forall x \in X_u.$$

Since $\sharp\mathcal{L}_{n-1}(u) > \sharp\mathcal{L}_{n-2}(u)$, we can find distinct $x, x' \in X_u$ such that $x|_0^{n-2} = x'|_0^{n-2}$ but $x_{n-1} \neq x'_{n-1}$. It follows that $a_{n-1} = 0$. Since for each $0 \leq k \leq n-2$ we can always find y, y' such that $y|_0^{k-1} = y'|_0^{k-1}$ but $y_k \neq y'_k$, we obtain that $a_{n-2} = a_{n-3} = \cdots = a_0 = 0$. Finally, we get $a_n = 0$. This is a contradiction. Therefore we conclude that $\dim(G_n) = n+1$, then $F_n = G_n$. Then by Lemma 4.1, (X_u, σ) has simple topological spectrum. □

6.0.3. *Homeomorphism of the interval.* Estimating the multiplicity of non-zero dimensional systems is difficult in general. Below we focus on homeomorphisms of the interval (see [Jav19] for related results on the circle). We use the following result due to Atzmon and Olevskii [AO96]. We denote by $C_0(\mathbb{R})$ the set of continuous map on $\mathbb{R}$ with zero limits in $\pm\infty$. For $f \in C_0(\mathbb{R})$ and $n \in \mathbb{Z}$ we let $f_n = f(\cdot + n)$ be the translation of f by n.

Theorem 6.3. [AO96] *There exists $g \in C_0(\mathbb{R})$ such that the vector space spanned by g_n, $n \in \mathbb{N}$ is dense in $C_0(\mathbb{R})$.*

In particular the operator $V : C_0(\mathbb{R}) \circlearrowleft$, $f \mapsto f(\cdot + 1)$, is cyclic. A Borel set S of $\mathbb{R}$ is called a *set of uniqueness* if the sets $S_n := (S + 2\pi n) \cap [-\pi, \pi]$, $n \in \mathbb{Z}$ satisfy the following properties:

(1) S_n, $n \in \mathbb{Z}$, are pairwise disjoint,
(2) $\mathrm{Leb}(S_n \cap U) > 0$ for any $n \in \mathbb{Z}$ and any open set U of $[-\pi, \pi]$,
(3) $\mathrm{Leb}(S) < \infty$,

where Leb denotes the Lebesgue measure on $\mathbb{R}$.

Atzmon and Olevskii proved for any set of uniqueness S (such sets exist!) the conclusion of Theorem 6.3 holds true with g being the the Fourier transform of the indicator function of S. Let us just remark that if S is a set of uniqueness then

$$S^l = \bigcup_{n \in \mathbb{Z}} (S_{nk+l} + 2\pi k), 1 \leq l \leq k,$$

are k disjoints sets of uniqueness. Let $C_0(\mathbb{R}; \mathbb{C})$ be the set of continuous map on $\mathbb{C}$ with zero limits in infinity.

Lemma 6.4. *The operator $V : C_0(\mathbb{R}; \mathbb{C})^k \circlearrowleft$, $(f_i)_{1 \leq i \leq k} \mapsto (f_i(\cdot + 1))_{1 \leq i \leq k}$ is cyclic. In particular, the operator $U : C_0(\mathbb{R})^k \circlearrowleft$, $(f_i)_{1 \leq i \leq k} \mapsto (f_i(\cdot + 1))_{1 \leq i \leq k}$ is cyclic.*

Proof. Let S, S^l, $1 \leq l \leq k$, be sets of uniqueness as above. By following [AO96] we show that the vector space generated by the translates of $g := (\widehat{\chi_{S^l}})_{1 \leq l \leq k}$ is dense in $C_0(\mathbb{R}; \mathbb{C})^k$ with $\widehat{\chi_{S^l}}$ be the Fourier transform of the indicator function χ_{S^l} of S^l. It follows that the translates of $Re(g)$ is dense in $C_0(\mathbb{R})^k$. Let $\mu = (\mu_l)_{1 \leq l \leq k}$ be a complex bounded measure with

$$\langle V^n(g), \mu \rangle = \sum_{1 \leq l \leq k} \int (\widehat{\chi_{S^l}})_n d\mu_l = 0,$$

for all $n \in \mathbb{Z}$. It is enough to prove $\mu_l = 0$ for all $1 \leq l \leq k$. By Plancherel-Parseval formula we have

$$\int (\widehat{\chi_{S^l}})_n d\mu_l = \int \widehat{(\widehat{\chi_{S^l}})_n}(t) \hat{\mu}_l(-t)\, dt = -\int \chi_{S^l}(t) e^{-int} \hat{\mu}_l(t)\, dt.$$

Therefore we have

$$\sum_{1 \leq l \leq k} \int \chi_{S^l}(t) \hat{\mu}_l(t) e^{-int}\, dt = 0,$$

for all n. But this term is just the n^{th} coefficient of the function of $L^1([-\pi, \pi])$ given by $\sum_{m \in \mathbb{Z}} (\sum_{1 \leq l \leq k} \chi_{S^l} \hat{\mu}_l)(\cdot + 2\pi m)$, which should therefore be 0. As the sets $(S^l + 2\pi m) \cap$

$[-\pi,\pi], m\in\mathbb{Z}$, are pairwise disjoint, each term of the previous sum should be zero ; that is $(\chi_{S^l}\hat{\mu}_l)(x+2\pi k)=0$ for all m,l and for Lebesgue almost every $x\in[-\pi,\pi]$. By Property (2) in the definition of a set of uniqueness, we conclude $\widehat{\mu_l}=0$. Therefore $\mu_l=0$ for each $1\leq l\leq k$ and consequently the translates of g is dense in $C_0(\mathbb{R};\mathbb{C})^k$. ☐

Proposition 6.5. *Let $f:[0,1]\circlearrowleft$ be a homeomorphism of the interval. Then*

$$\mathrm{Mult}(f)=\sharp\mathcal{M}_e([0,1],f).$$

Proof. We first deal with the case of an increasing homeomorphism. The ergodic measures of f are the Dirac measures at these fixed points. Notice that f has at least two fixed points, 0 and 1. If it has infinitely many fixed points, then $\mathrm{Mult}(U_f)\geq\sharp\mathcal{M}_e([0,1],f)=\infty$. Now assume it has finitely many fixed points. Let $2\leq k+1<+\infty$ be the number of fixed points. Since $\varphi(x)-\varphi\circ f(x)=0$ for any continuous function $\varphi\in C(X)$ and any fixed point x, the space $\overline{B_f([0,1])}$ is the set of real continuous maps on the interval which vanishes at the fixed points. It follows that the operator $\underline{U_f}$ is spectrally conjugate to $V:C_0(\mathbb{R})^k\circlearrowleft$, $(f_i)_{1\leq i\leq k}\mapsto(f_i(\cdot+1))_{1\leq i\leq k}$. By Lemma 6.4 we have $\mathrm{Mult}(\underline{U_f})=1$. It follows then from Proposition 2.9 and Proposition 2.6 that

$$\sharp\mathcal{M}_e([0,1],f)\leq\mathrm{Mult}(U_f)\leq\sharp\mathcal{M}_e([0,1],f)+\mathrm{Mult}(\underline{U_f})-1=\sharp\mathcal{M}_e([0,1],f). \tag{6·1}$$

It remains to consider the case of a decreasing homeomorphism f. Let $0<a<1$ be the unique fixed point of f. Then $f^2:[0,a]\circlearrowleft$ is an increasing homeomorphism. Let $0=x_1<x_2<\cdots<x_k=a$ be the fixed points of $f^2|_{[0,a]}$. Then the ergodic measures of f are the atomic periodic measures δ_a and $\frac{1}{2}\left(\delta_{x_i}+\delta_{f(x_i)}\right)$ for $i=1,\cdots,k-1$. In particular we have $k=\sharp\mathcal{M}_e([0,1],f)$. From the previous case there is a generating family $\mathcal{G}=\{g_1,\cdots,g_k\}$ for $f^2:[0,a]\circlearrowleft$. Let $h\in C([0,1])$. For any $\epsilon>0$, there are $N\in\mathbb{N}$, $a_{l,n}$ and $b_{l,n}$, for $l=1,\cdots,k$ and $|n|\leq N$, (depending on ϵ), such that

$$\|h-\sum_{l,n}a_{l,n}g_l\circ f^{2n}\|_{[0,a],\infty}<\epsilon, \tag{6·2}$$

and

$$\|h\circ f^{-1}-\sum_{l,n}b_{l,n}g_l\circ f^{2n}-h(a)\|_{[0,a],\infty}<\epsilon, \tag{6·3}$$

where $\|g\|_{[0,a],\infty}=\sup_{x\in[0,a]}|g(x)|$. We consider the extension $\tilde{g_l}$ of g_l to $[0,1]$ with $\tilde{g_l}=g_l(a)$ on $[a,1]$. We check now that $\tilde{\mathcal{G}}=\{\tilde{g}_1,\cdots,\tilde{g}_k\}$ is generating for f. It follows from (6·2) and (6·3) at $x=a$ that

$$\left|h(a)-\sum_{l,n}a_{l,n}g_l(a)\right|<\epsilon\text{ and }\left|\sum_{l,n}b_{l,n}g_l(a)\right|<\epsilon. \tag{6·4}$$

Observe that

$$h=h|_{[0,a]}+(h\circ f^{-1}|_{[0,a]})\circ f|_{[a,1]}.$$

Combining (6·4) with (6·2), we obtain that

$$\begin{aligned}&\|h-\sum_{l,n} a_{l,n}\tilde{g}_l\circ f^{2n}-\sum_{l,n} b_{l,n}\tilde{g}_l\circ f^{2n+1}\|_{[0,a],\infty}\\ =&\|h-\sum_{l,n} a_{l,n}g_l\circ f^{2n}-\sum_{l,n} b_{l,n}g_l(a)\|_{[0,a]\infty}<2\epsilon.\end{aligned}$$

Similarly, combining (6·4) with (6·3) we get

$$\begin{aligned}&\|h-\sum_{l,n} a_{l,n}\tilde{g}_l\circ f^{2n}-\sum_{l,n} b_{l,n}\tilde{g}_l\circ f^{2n+1}\|_{[a,1],\infty}\\ =&\|h-\sum_{l,n} a_{l,n}g_l(a)-\sum_{l,n} b_{l,n}g_l\circ f^{2n+1}\|_{[a,1]\infty}\\ \leq&|h(a)-\sum_{l,n} a_{l,n}g_l(a)|+\|h-\sum_{l,n} b_{l,n}g_l\circ f^{2n+1}-h(a)\|_{[a,1]\infty}<2\epsilon.\end{aligned}$$

Therefore we have

$$\|h-\sum_{l,n} a_{l,n}\tilde{g}_l\circ f^{2n}-\sum_{l,n} b_{l,n}\tilde{g}_l\circ f^{2n+1}\|_{\infty}<2\epsilon.$$

We conclude that $\tilde{\mathcal{G}}$ is generating for f as $\epsilon>0$ and $h\in C([0,1])$ have been chosen arbitrarily.

□

Question 6.6. *What is the topological multiplicity of a Morse-Smale diffeomorphism?*

6.0.4. *Thue-Morse Subshift.* We give now an example of a uniquely ergodic system with mixed spectrum and simple topological multiplicity. The *Thue-Morse subshift* X_ζ is the bilateral subshift associated to the substitution $\zeta(0)=01$ and $\zeta(1)=10$, i.e. $X_\zeta=X_u$ with u being the infinite word of $\{0,1\}^{\mathbb{Z}}$ given by $\cdots u_2u_1u_0u_0u_1u_2\cdots$ with $u_0u_1u_2\cdots$ being the unique fixed point of the subtitution $v_0v_1v_2\cdots\mapsto\zeta(v_0)\zeta(v_1)\zeta(v_2)\cdots$. This subshift is known to be a minimal uniquely ergodic system with simple mixed spectrum (for the Koopman operator U_σ on $L^2_0(\nu)$ with ν being the unique invariant probability measure) [Mic76, Kwi81]. The continuous part of its spectrum is singular with respect to the Lebesgue measure [Kak72]. The map $\tau: X_\zeta\to X_\zeta$, $(x_n)_{n\in\mathbb{Z}}\mapsto(1-x_n)_{n\in\mathbb{Z}}$ defines an involution of X_ζ. As τ commutes with σ, the measure ν is also τ-invariant.

Theorem 6.7. *The Thue-Morse subshift is topologically simple.*

For any $n\in\mathbb{N}$ we let $\mathcal{F}_n=\{\chi_{[\zeta^n(i)]}:i\in\{0,1\}\}$ and $f_n=\chi_{[\zeta^n(0)]}$. In order to prove Theorem 6.7, we first show the following lemma which states that the space $V_{\mathcal{F}_n}$ is cyclic.

Lemma 6.8.

$$V_{\mathcal{F}_n}=V_{\{f_n\}}.$$

Proof. Fix $n\in\mathbb{N}$. Notice that the (simple) point spectrum of U_σ consists in the powers of 2. In particular , the system $(\zeta^n(X),\sigma^{2^n},\mathcal{B},\nu)$ is ergodic, then the restriction of σ to $\zeta^n(X)$ is uniquely ergodic. Consequently the Birkhoff sum $\frac{1}{p}\sum_{0\leq k<p} f_n\circ\sigma^{k2^n}$ is converging uniformly to $\int f_n\,d\nu = \nu([\zeta^n(0)])$ on $\zeta^n(X)$, when p goes to infinity. But $\nu([\zeta^n(0)]) = \nu([\zeta^n(1)]) \neq 0$. It follows that $\chi_{[\zeta^n(X)]} \in V_{\{f_n\}}$. Since $\chi_{\zeta^n(X)} = f_n + \chi_{[\zeta^n(1)]}$, we conclude that the continuous function $\chi_{[\zeta^n(1)]}$ belongs to $V_{\{f_n\}}$.

□

We are now in a position to prove Theorem 6.7. We make use of the following notation. For a point $x = (x_n)_{n\in\mathbb{Z}}$, we write it as

$$x = \ldots x_{-2}x_{-1}.x_0x_1\ldots.$$

Proof of Proposition 6.7. According to Lemma 6.8, it is enough to check the assumptions of Lemma 4.1 with $(\mathcal{F}_n)_n$. As the sequence of vectors spaces, $(V_{\mathcal{F}_n})_{n\in\mathcal{N}}$ is nondecreasing, one only needs to show $\overline{\bigcup_n V_{\mathcal{F}_n}} = C(X_\zeta)$. If not, there would be distinct probability measures $\mu^\pm$ with

$$\mu^+(\sigma^k[\zeta^n(i)]) = \mu^-(\sigma^k[\zeta^n(i)])\ \forall k,n\,\forall i = 0,1.$$

Let $E = \{\zeta^\infty(i).\zeta^\infty(j) : i,j\in\{0,1\}\}$. Then for every n,

$$P_n := \{\sigma^k[\zeta^n(i)],\ i=0,1,\ 0\leq k<2^n\}$$

is a partition of $X_\zeta\setminus O_\sigma(E)$ where $O_\sigma(E)$ is the orbit of E under σ, i.e. $O_\sigma(E) = \{\sigma^k(x) : x\in E, k\in\mathbb{Z}\}$. For any open set $U\supset O_\sigma(E)$, we will show that $\mu^+|_{X_\zeta\setminus U} = \mu^-|_{X_\zeta\setminus U}$. Obviously, we have

$$\mathrm{diam}(P_n(x)) \xrightarrow{n\to\infty} 0, \forall x\in X_\zeta\setminus U.$$

Thus $\{P_n\cap(X_\zeta\setminus U)\}_{n\in\mathbb{N}}$ generates Borel σ-algebra on $X_\zeta\setminus U$. Therefore, we have $\mu^+|_{X_\zeta\setminus U} = \mu^-|_{X_\zeta\setminus U}$. Since U is chosen arbitrarily, we obtain that

$$\mu^+|_{X_\zeta\setminus O_\sigma(E)} = \mu^-|_{X_\zeta\setminus O_\sigma(E)}. \tag{6·5}$$

Let

$$E_i = \{\zeta^\infty(j).\zeta^\infty(i) : j=0,1\} \text{ and } E^j = \{\zeta^\infty(j).\zeta^\infty(i) : i=0,1\},$$

which form a partition of E. Then by $\mu^+([\zeta^n(i)]) = \mu^-([\zeta^n(i)])$ for all n, we have $\mu^+(E_i) = \mu^-(E_i)$. Similarly, we have $\mu^+(E^j) = \mu^-(E^j)$. Observe that

$$\zeta^\infty(0).\zeta^\infty(1) = \bigcap_n [\zeta^{2n}(0).\zeta^{2n}(1)].$$

It follows that $\mu^+(\zeta^\infty(0).\zeta^\infty(1)) = \mu^-(\zeta^\infty(0).\zeta^\infty(1))$. Thus we get

$$\mu^+|_E = \mu^-|_E.$$

Similarly, we obtain $\mu^+|_{\sigma^k(E)} = \mu^-|_{\sigma^k(E)}$ for all $k\in\mathbb{Z}$. It implies that $\mu^+|_{O_\sigma(E)} = \mu^-|_{O_\sigma(E)}$. Combining this with (6·5), we conclude that $\mu^+ = \mu^-$ which is a contradiction.

□

7. Subshifts with linear growth complexity

We consider a subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ with letters in a finite alphabet $\mathcal{A}$. For $x \in \mathcal{A}^{\mathbb{Z}}$ we denote by $x = (x_n)_{n\in\mathbb{Z}}$ for $x_n \in \mathcal{A}$. Let $\mathcal{L}_n(X) \subset \mathcal{A}^n$ be the finite words of X of length n, i.e. $\mathcal{L}_n(X) = \{x_k x_{k+1} \dots x_{k+n-1} : x \in X, k \in \mathbb{Z}\}$. The word complexity of X is given by

$$\forall n \in \mathbb{N}, \quad p_X(n) = \sharp\mathcal{L}_n(X).$$

We suppose that X is aperiodic and has linear growth, that is, for some $k \in \mathbb{N}^*$

$$\liminf_n \frac{p_X(n)}{n} \leq k. \tag{7·1}$$

Boshernitzan [Bos92] showed that such a subshift admits at most k ergodic measures. By Theorem 5.5 in [DDMP21] such subshifts, when assumed to be moreover minimal, have topological rank less than or equal to $(1+k\sharp\mathcal{A}^2)^{2(k+2)}$. We show in this section the following upper bound on the topological multiplicity.

Theorem 7.1. *Any aperiodic subshift X with $\liminf_{n\to\infty} \frac{p_X(n)}{n} \leq k$ has topological multiplicity less than or equal to $2k$.*

One may wonder if the upper bound in Theorem 7.1 is sharp.

Question 7.2. *Is an aperiodic subshift X with $\liminf_{n\to\infty} \frac{p_X(n)}{n} = 1$ topologically simple?*

In order to prove Theorem 7.1, we define some notations. Let Q_n be the subset of $\mathcal{L}_n(X)$ given by words w such that there are several letters $a \in \mathcal{A}$ with $wa \in \mathcal{L}_{n+1}(X)$. We also let Q'_{n+1} be the $(n+1)$-words wa as above. Clearly, we have

$$\sharp Q_n \leq p_X(n+1) - p_X(n) \text{ and } \sharp Q'_{n+1} = \sharp Q_n + p_X(n+1) - p_X(n). \tag{7·2}$$

Through this section, we always assume the subshift is aperiodic and satisfies the linear growth (7·1).

Lemma 7.3. *The subset of integers*

$$\mathcal{N} = \{n \in \mathbb{N} : p_X(n+1) < (k+1)(n+1) \text{ and } p_X(n+1) - p_X(n) \leq k\}.$$

is infinite.

Proof. By (7·1), we have

$$\liminf_n (p_X(n) - (k+1)n) = -\infty.$$

It follows that

$$\mathcal{M} = \left\{ n \in \mathbb{N} : p_X(n+1) - (k+1)(n+1) \leq \min\{0, \min_{1\leq m\leq n}\{p_X(m) - (k+1)m\}\} - 1 \right\}$$

is an infinite set. For any $n \in \mathcal{M}$, we have

$$p_X(n+1) - p_X(n) \leq (k+1)(n+1) - (k+1)n - 1 = k.$$

On the other hand, for any $n \in \mathcal{M}$, we get

$$p_X(n+1) \leq (k+1)(n+1) - 1.$$

This implies that $\mathcal{M} \subset \mathcal{N}$. Therefore, the set $\mathcal{N}$ is infinite. □

Lemma 7.4 ([Bos84], Lemma 4.1)**.** *For any $n \in \mathcal{N}$ and $m \geq (k+2)(n+1)$, any word $w \in \mathcal{L}_m$ contains a subword in Q_n.*

For the sake of completeness we provide a proof here.

Proof. We prove it by contradiction. Assume to the contrary that all $(m-n+1)$ n-subwords of w do not belong to Q_n. That means that each of these n-blocks determines uniquely the next letter. Since $m - n + 1 \geq 2(k+1)n > p_X(n)$, at least one n-word appears more than one time as a subword of w. Therefore X contains a periodic point. This contradicts our assumption. □

Now we show that any cylinder of length less than n can be decomposed as the cylinders of elements in Q'_{n+1} after translations.

Lemma 7.5. *Let $n \in \mathcal{N}$. Any cylinder $[w]$ with length of w less than n may be written uniquely as a finite disjoint union of sets of the form $\sigma^p[q'_{n+1}]$ with $p \in \mathbb{N}$, $q'_{n+1} \in Q'_{n+1}$, such that $\sigma^t[q'_{n+1}] \cap [q_n] = \emptyset$ for any $0 < t < p$ and any $q_n \in Q_n$.*

Remark that by Lemma 7.4 the integers p belongs to $[0, (k+2)(n+1)]$.

Proof. Let $[w]$ be a cylinder associated to a word $w \in \mathcal{L}_l(X)$ with $l < n$. For $x \in [w]$, we let $K_x \in \mathbb{Z}$ be the largest integer j less than l such that $x_{j-n+1} \cdots x_j$ belongs to Q_n. Then the word $w^x_{n+1} = x_{K_x - n+1} \cdots x_{K_x+1}$ belongs to Q'_{n+1}. Observe also that by Lemma 7.4 we have $n - 1 - K_x \leq (k+2)(n+1)$. Let W_{n+1} be the collection of these words w^x_{n+1} over $x \in [w]$. By definition of K_x and Q_n the word w^x_{n+1} completely determines the $l - 1 - K_x$ next letters, that is to say,

$$[w^x_{n+1}] = [x_{K_x - n+1} \cdots x_{l-1}].$$

As x belongs to $[w]$ we have in particular $\sigma^{n-1-K_x}[w^x_{n+1}] \subset [w]$ and finally

$$[w] = \coprod_{w^x_{n+1} \in W_{n+1}} \sigma^{n-1-K_x} w^x_{n+1}.$$

We complete the proof. □

Proof of Theorem 7.1. By (7·2) and the definition of $\mathcal{N}$ we have for $n \in \mathcal{N}$:

$$\begin{aligned} \sharp Q'_{n+1} &= \sharp Q_n + p_X(n+1) - p_X(n), \\ &\leq 2(p_X(n+1) - p_X(n)), \\ &\leq 2k. \end{aligned}$$

For $n \in \mathcal{N}$ we let $F_n = \{\chi_{[q'_{n+1}]},\ q'_{n+1} \in Q'_{n+1}\}$. By Lemma 7.5, any cylinder $[w]$ with length less than n is a finite disjoint union of $\sigma^p[q'_{n+1}]$. In particular $\chi_{[w]}$ lies in V_{F_n}. We may therefore apply Lemma 4.1 to $(F_n)_{n \in \mathcal{N}}$ and we get

$$\mathrm{Mult}(X, \sigma) \leq \sup_{n \in \mathcal{N}} \sharp Q'_{n+1} \leq 2k.$$

□

7.1. **Multiplicity of invariant measures.** It follows from Theorem 7.1 and Lemma 2.4 that any ergodic measure has (ergodic) multiplicity bounded by $2k$. In fact we may refine this result as follows:

Theorem 7.6. *Let X be an aperiodic subshift with $\liminf_{n\to\infty}\frac{p_X(n)}{n}\leq k$. Then*

$$\sum_{\mu\in\mathcal{M}_e(X,\sigma)}\mathrm{Mult}(\mu)\leq 2k.$$

In order to prove Theorem 7.6, we first recall some notations and then show two lemma for general aperiodic subshifts. Let (Y,σ) be an aperiodic subshift. For two finite words w and v, we denote by $N(w|v)$ the number of times that w appears as a subword of v. Also, We define $d(w|v)=N(w|v)/|v|$. For a generic point x of a measure μ, we have

$$\lim_{n\to\infty} d(w|x_1^n)=\mu([w]),$$

where $x_1^n=x_1x_2\dots x_n$. For a finite word v, we denote by $v^{\otimes m}=\underbrace{vv\dots v}_{m\text{ times}}$. For a finite word w, we denote by

$$\nu_w=\frac{1}{|w|}\sum_{k=0}^{|w|-1}\delta_{\sigma^k(\bar{w})},$$

where $|w|$ is the length of w and $\bar{w}\in\mathcal{A}^{\mathbb{Z}}$ is the periodization of w, i.e. $w^{\otimes\infty}$.

Let w_n be a word of length n. Foy any n, we put $\ell_n=\ell(w_n):=\min\{1\leq\ell<n:[w_n]\cap\sigma^\ell([w_n])\neq\emptyset\}$ and $L_n:=1+\sharp\{1\leq\ell<n:[w_n]\cap\sigma^\ell([w_n])\neq\emptyset\}$, with the convention $\min\emptyset=n$. Let $v_n=v(w_n)$ be the first ℓ_n-subword of w_n. It follows that $w_n=v_n^{\otimes K_n}\hat{v}_n$ with $\hat{v}_n\neq v_n$ being a prefix of v_n. Then $K_n=\lfloor n/\ell_n\rfloor\geq L_n$. Observe that for any $x\in X$, $p\geq\ell_n$ and any word u of length less than ℓ_n we have

$$\begin{aligned}N(u|x_1^p)&\geq N(u|w_n)\frac{N(w_n|x_1^p)}{L_n},\\&\geq N(u|v_n)K_n\frac{N(w_n|x_1^p)}{L_n},\\&\geq N(u|v_n)N(w_n|x_1^p).\end{aligned}\tag{7·3}$$

In the next two lemmas we assume that

- the subshift (Y,σ) is aperiodic;
- $w_n\in\mathcal{L}_n(Y)$ for $n\in\mathbb{N}$;
- $\nu_{w_n}\xrightarrow[n\in\mathcal{N}]{n\to+\infty}\nu\in\mathcal{M}_e(Y,\sigma)$, i.e. ν_{w_n} is weakly converging to an ergodic measure ν when n goes to infinity along a subsequence $\mathcal{N}$.

Lemma 7.7. *Under the above assumption, we have*

$$\ell_n=|v_n|\xrightarrow[n\in\mathcal{N}]{n\to+\infty}+\infty$$

and

$$\nu_{v_n} \xrightarrow[n\in\mathcal{N}]{n\to+\infty} \nu.$$

Proof. We argue by contradiction. Assume $(\ell_n)_{n\in\mathbb{N}}$ has a bounded infinite subsequence $\mathcal{N}'$ of $\mathcal{N}$. Then there are finite words v and $\hat{v}_n$ with $|\hat{v}_n| < |v|$ such that $w_n = v^{\otimes K_n}\hat{v}_n$ for $n \in \mathcal{N}''$ where $\mathcal{N}''$ is some infinite subsequence of $\mathcal{N}'$. Observe firstly that the length of w_n goes to infinity. As a consequence, K_n goes also to infinity as n goes to infinite along $\mathcal{N}''$. But then X should contain the periodic point $\overline{v}$ associated to v which is a contradiction to the aperiodicity of (Y, σ). Therefore $\ell_n \xrightarrow[n\in\mathcal{N}]{n\to+\infty} +\infty$.

Let us check now that $\nu_{v_n} \xrightarrow[n\in\mathcal{N}]{n\to+\infty} \nu$. Let $\nu' = \lim_{k\to\infty} \nu_{v_{n_k}}$ be a weak limit of $(\nu_{v_n})_{n\in\mathcal{N}}$ with a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathcal{N}$. For any word u with $|u| < \ell_n$, by (7·3) we have

$$N(u|w_n) \geq N(u|v_n)K_n,$$

and consequently

$$\begin{aligned} d(u|w_n) \geq &\quad d(u|v_n)\tfrac{K_n|v_n|}{(K_n+1)|v_n|}, \\ \geq &\quad \tfrac{1}{2}d(u|v_n). \end{aligned}$$

For any cylinder $[u]$, By letting n got infinity we get that

$$\nu([u]) \geq \frac{1}{2}\nu'([u]).$$

It implies that $\nu - \frac{1}{2}\nu'$ is an σ-invariant measure. It follows from the ergodicity of ν that $\nu = \nu'$. □

Lemma 7.8. *For any ergodic measure $\mu \neq \nu$, we have*

$$\lim_{n\in\mathcal{N}, n\to\infty} |v_n|\mu([w_n]) = 0.$$

Proof. Assume $\limsup_{n\in\mathcal{N}, n\to\infty} |v_n|\mu([w_n]) > 0$. By passing to an infinite subsequence $\mathcal{N}'$ of $\mathcal{N}$ we have $\lim_{n\in\mathcal{N}', n\to\infty} |v_n|\mu([w_n]) = b > 0$. By Lemma 7.7 the sequence ν_{v_n}, $n \in \mathcal{N}'$, is converging to the measure ν.

Let x be a generic point of μ. Then we have for any n

$$\lim_{p\to\infty} \frac{1}{p}\sum_{\ell=0}^{p-1} \chi_{[w_n]}(\sigma^\ell(x)) = \lim_{p\to\infty} \frac{N(w_n|x_1^p)}{p} = \mu([w_n]). \tag{7·4}$$

In particular for any n we can choose $P_n \in \mathbb{N}$ such that for $p \geq P_n$

$$\frac{N(w_n|x_1^p)}{p} \geq \frac{\mu([w_n])}{2}. \tag{7·5}$$

Pick an arbitrary cylinder $[u]$. by Lemma 7.7 there exists an integer N such that for $n > N$ we have

$$N(u|v_n) \geq \frac{1}{2}\nu([u])|v_n|. \tag{7·6}$$

It follows from (7·3) that

$$d(u|x_1^p) \geq \frac{1}{4}\nu([u])|v_n|\mu(w_n).$$

By letting p, then $n \in \mathcal{N}'$ go to infinity, we get for any cylinder $[u]$

$$\mu([u]) \geq \frac{b}{4}\nu([u]).$$

It implies that $\mu - \frac{b}{4}\nu$ is an σ-invariant measure which is a contradiction to the ergodicity of μ. □

We recall now briefly the proof of Boshernitzan that an aperiodic subshift of linear growth has finite many ergodic measures. Let $\mathcal{N}$ be the infinite set as in Lemma 7.3. For any $n \in \mathcal{N}$, one can choose (not uniquely) an ordered k-tuple of n-words $K_n := \{q_{n,1}, \dots, q_{n,k}\}$ which coincides with Q_n. By passing to a subsequence $\mathcal{N}'$ of $\mathcal{N}$, we can make each of the sequences of $\nu_{q_{n,i}}$ weakly convergences to some measures $\mu_i \in \mathcal{M}(X,T)$. Boshernitzan showed that

$$\mathcal{M}_e(X,T) \subset \{\mu_1, \mu_2, \dots, \mu_k\}.$$

Since μ_i may coincide with the other μ_j for $j \neq i$, we define $I_i = \{1 \leq j \leq k : \mu_j = \mu_i\}$.

We will use the following complement of Lemma 7.5.

Lemma 7.9. *In the decomposition of a cylinder $[w]$ given by Lemma 7.5, for any term $\sigma^p[q'_{n+1}]$ with $|v(q'_{n+1})| < n+1$ we have $p \leq |v(q'_{n+1})|$.*

Proof. We argue by contradiction. To simplify the notations we write $v_n = v(q'_{n+1})$. Assume $|v_n| < n+1$ and $p > |v_n|$. By definition of v_n we have

$$\emptyset \neq \sigma^p[q'_{n+1}] \cap \sigma^{p-|v_n|}[q'_{n+1}].$$

But it follows from Lemma 7.5 that $\sigma^p[q'_{n+1}]$ does not intersect $\sigma^l\left(\bigcup_{q_n \in Q_n}[q_n]\right)$ for $0 < l < p$, therefore with $q_n \in Q_n$ being the prefix of q'_{n+1} we get the contradiction

$$\sigma^p[q'_{n+1}] \cap \sigma^{p-|v_n|}[q'_{n+1}] \subset \sigma^p[q'_{n+1}] \cap \sigma^{p-|v_n|}[q_n] = \emptyset.$$

Thus we have $p \leq |v(q'_{n+1})|$. □

For a given i we let $(q^l_{n,i})_{l \in \mathcal{Q}_{n,i}}$ be the elements of Q'_{n+1} with prefix $q_{n,i}$, where $\mathcal{Q}_{n,i}$ is a subset of $\mathcal{A}$ for each $n \in \mathcal{N}', 1 \leq i \leq k$. Note that $\nu_{q^l_{n,i}}$ is also converging to μ_i for any l when n goes to infinity along $\mathcal{N}'$. Finally we let $v^l_{n,i} = v_n(q^l_{n,i})$ for each $n \in \mathcal{N}', 1 \leq i \leq k$ and $l \in \mathcal{Q}_{n,i}$.

Proof of Theorem 7.6. Pick an arbitrary cylinder $[w]$. Let

$$[w] = \coprod_{j,l} \coprod_{\in P^l_{n,j}} \sigma^p[q^l_{n,j}],$$

be the decomposition of $[w]$ given by Lemma 7.5. Recall that $P_{j,l}$ is a subset of $[0,(k+2)(n+1)]$ for any l and by Lemma 7.9 we have also $P_{j,l} \subset [0, |v^l_{n,j}|-1]$ if $|v^l_{n,j}| < n+1$.

For each $n \in \mathcal{N}'$, we decompose $\{(j,l) : 1 \le j \le k, l \in \mathcal{Q}_{n,j}\}$ into three set $J_{n,i}$, $J'_{n,i}$ and $J''_{n,i}$, where $J_{n,i} := \{(j,l) : j \in I_i\}$, $J'_{n,i} := \{(j,l) : j \notin I_i, |v^l_{n,j}| = n+1\}$ and $J''_{n,i}$ is the rest. Then for $(j,l) \in J'_{n,i}$ we have

$$\mu_i\left(\coprod_{p\in P^l_{n,j}} \sigma^p[q^l_{n,j}]\right) \le (k+2)(n+1)\mu_i([q^l_{n,j}]) = (k+2)|v^l_{n,j}|\mu_i([q^l_{n,j}]). \tag{7·7}$$

On the other hand, for $(j,l) \in J''_{n,i}$, we have

$$\mu_i\left(\coprod_{p\in P^l_{n,j}} \sigma^p[q^l_{n,j}]\right) \le |v^l_{n,j}|\mu_i([q^l_{n,j}]). \tag{7·8}$$

By summing up (7·7) and (7·8), we have

$$\mu_i\left(\coprod_{(j,l)\in J'_{n,i}\cup J''_{n,i}}\ \coprod_{p\in P^l_{n,j}} \sigma^p[q^l_{n,j}]\right) \le 2k(k+2) \sum_{(j,l)\in J'_{n,i}\cup J''_{n,i}} |v^l_{n,j}|\mu_i([q^l_{n,j}]). \tag{7·9}$$

Combining this with Lemma 7.8, we obtain

$$\lim_{n\to\infty} \mu_i\left(\coprod_{(j,l)\in J'_{n,i}\cup J''_{n,i}}\ \coprod_{p\in P^l_{n,j}} \sigma^p[q^l_{n,j}]\right) = 0.$$

Therefore we have

$$\begin{aligned}\left\|\chi_{[w]} - \sum_{(n,j)\in J_{n,i},\ p\in P^l_{n,j}} \chi_{[q^l_{n,j}]}\circ\sigma^{-p}\right\|^2_{L^2(\mu_i)} &= \left\|\chi_{[w]} - \chi_{\coprod_{(n,j)\in J_{n,i},\ p\in P^l_{n,j}} \sigma^p[q^l_{n,j}]}\right\|^2_{L^2(\mu_i)}, \\ &= \mu_i\left(\coprod_{(j,l)\in J'_{n,i}\cup J''_{n,i}} \coprod_{p\in P^l_{n,j}} \sigma^p[q^l_{n,j}]\right) \xrightarrow[n\in\mathcal{N}']{n\to+\infty} 0.\end{aligned}$$

Thus we can apply Lemma 4.1 in $L^2_0(\mu_i)$ with $F_n = \{\chi_{[q^l_{n,j}]}\ :\ (j,l) \in J_{n,i}\}$ to get

$$\mathrm{Mult}(\mu_i) \le \liminf_{n\in\mathcal{N}',n\to\infty} \sharp J_{n,i}.$$

By summing it up, we conclude that

$$\begin{aligned}\sum_{\mu\in\mathcal{M}_e(X,\sigma)} \mathrm{Mult}(\mu) \le\ & \textstyle\sum_i \liminf_{n\in\mathcal{N}',n\to\infty} \sharp J_{n,i}, \\ \le\ & \liminf_{n\in\mathcal{N}',n\to\infty} \sharp Q'_{n+1}, \\ \le\ & 2k.\end{aligned}$$

□

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