ENTROPY OF PHYSICAL MEASURES FOR C^∞ DYNAMICAL SYSTEMS

DAVID BURGUET

Abstract. For a C^∞ map f on a compact manifold M we prove that for a Lebesgue randomly picked point x there is an empirical measure from x with entropy larger than or equal to the top Lyapunov exponent of $\Lambda\, df : \Lambda TM \circlearrowleft$ at x. This contrasts with the well-known Ruelle inequality. As a consequence we give some refinement of Tsujii's work [23] relating physical and Sinai-Ruelle-Bowen measures.

Introduction

Entropy is a master invariant in dynamical systems, which estimates the dynamical complexity by counting the separated orbits. For a differentiable system other dynamical quantities of high interest are the Lyapunov exponents. They are given by the exponential growth rate of the derivative. Heuristically the first derivative controls the separation of points (as in the mean value inequality) so that the entropy is always less than or equal to the (sum of positive) Lyapunov exponents. This inequality, due to Ruelle [20], holds at any invariant measure. Moreover the case of equality characterizes the so-called Sinai-Ruelle-Bowen measures for $C^{1+\alpha}$ systems.

Here we use a slightly different framework. We do not consider entropy and Lyapunov exponent defined on invariant measures but on points. For the entropy we let $h(x)$ be the supremum entropy of the empirical measures at a given point x. We may also define a pointwise positive Lyapunov exponent, denoted by $\chi_\Lambda^+(x)$, by considering the limsup in the exponential growth of the derivative at x acting on the exterior algebra bundle (see Section 1 for the precise definitions). We then aim to compare $h(x)$ and $\chi_\Lambda^+(x)$ "physically", i.e. for Lebesgue almost every point x. For a C^∞ system we prove quite surprisingly the entropy is physically bounded from below by the sum of positive Lyapunov exponents, i.e.

$$h \geq \chi_\Lambda^+ \text{ Lebesgue almost surely.}$$

In [25] Yomdin introduced tools of semi-algebraic geometry in order to control the local volume growth of C^∞ smooth systems. In particular it allows him to show that Shub's entropy conjecture holds true in this setting. Using a similar approach we manage to control locally not only the

Date: June 2018.
2010 *Mathematics Subject Classification.* Primary 37C40, 37D25.
Key words and phrases. Entropy, Lyapunov exponent, SRB measure, physical measure.

volume growth but also the distortion (see also [5] and [6]). The resulting Reparametrization Lemma of dynamical balls is the key argument in the proof of our Main Theorem.

The paper is organized as follows. In the first section we recall the notions of physical, physical-like and Sinai-Ruelle-Bowen measures. We also introduce the *strong Lyapunov exponents* which provide a new way to estimate the exponential growth of the derivative at a point. Our Main Theorem and its Corollaries are stated and discussed in Section 2. The last two sections are devoted to the proof. Finally we present a counter-example in finite smoothness in the appendix*.

1. Background

1.1. **Physical measures.** Let (M, f) be a topological system, i.e. $f : M \to M$ is a continuous map on a compact metrizable space M. Fix a metric d on M. We let $\mathcal{M}(M)$ (resp. $\mathcal{M}(M, f)$) be the set of Borel probability measures (resp. f-invariant). Endowed with the weak-$*$ topology these sets are compact metrizable spaces. When $(\phi_n : M \to \mathbb{R})_{n\in\mathbb{N}}$ is a dense countable family of the set of real continuous functions on X for the usual supremum norm then the following convex metric $\mathfrak{d}$ on $\mathcal{M}(M)$ is compatible with the weak-$*$ topology:

$$\forall \mu, \nu \in \mathcal{M}(M),\ \mathfrak{d}(\mu, \nu) := \sum_n \frac{\left|\int \phi_n \, d\nu - \int \phi_n \, d\mu\right|}{2^n(1 + \sup_x |\phi_n(x)|)}.$$

We will also consider the set $\mathcal{KM}(M)$ of all nonempty closed subsets of $\mathcal{M}(M)$ with the associated Hausdorff metric $\mathfrak{d}^H$.

The basin $\mathcal{B}_\mu$ of an invariant measure $\mu \in \mathcal{M}(M, f)$ is the set of points $x \in M$ whose empirical measures $\mu_n^x := \frac{1}{n}\sum_{0\leq k<n} \delta_{f^k x}$ is converging to μ, when n goes to infinity. According to Birkhof ergodic theorem the set $\mathcal{B}_\mu$ has full μ-measure for an ergodic measure μ. In the following we will always consider a C^∞ smooth compact Riemannian manifold $(M, \|\cdot\|)$ and its induced Riemannian distance d. The (normalized) volume form inherited from the Riemannian structure will be called the Lebesgue measure and is denoted by Leb. An invariant measure is said **physical** when its basin has positive Lebesgue measure. From the works of Sinai, Ruelle and Bowen [22, 20, 4] it is known that any C^2 Axiom A attractor admits finitely many ergodic physical measures such that the union of their basin has full Lebesgue measure in the basin of attraction.

We recall now the concept of physical-like measures [8, 9]. For $x \in M$ we let $p\omega(x) \subset \mathcal{M}(M, f)$ be the accumulation points of the empirical measures $(\mu_n^x)_n$ at x. An invariant measure $\mu \in \mathcal{M}(M, f)$ is said **physical-like** when for any $\epsilon > 0$ the set $\{x,\ \mathfrak{d}(\mu, p\omega(x)) < \epsilon\}$ has positive Lebesgue measure (in

*The example given in the appendix is on the interval. Following the same approach we build in a forthcoming work [7] such counter-examples of surface diffeomorphisms.

particular the physical measures are physical-like). The set $\mathcal{PL} = \mathcal{PL}(Leb)$ of physical-like measures is the smallest compact subset of measures containing $p\omega(x)$ for Lebesgue almost every point $x \in M$. In other terms if one considers the closed-set valued map

$$\begin{aligned} p\omega : \quad X \to & \quad \mathcal{KM}(M), \\ x \mapsto & \quad p\omega(x) \end{aligned}$$

and its essential range $\overline{\mathrm{Im}}_{\mathrm{Leb}}(p\omega)$ then we have (see Appendix B)

$$\mathcal{PL} = \bigcup_{K \in \overline{\mathrm{Im}}_{\mathrm{Leb}}(p\omega)} K.$$

Instead of the Lebesgue measure we may consider any other Borel measure m and define similarly $\mathcal{PL}(m)$ as the smallest compact subset of measures containing $p\omega(x)$ for m-almost every point $x \in M$. We let $PL(m)$ be the set with full m-measure given by points $x \in M$ with $p\omega(x) \in \overline{\mathrm{Im}}_m(p\omega)$. Therefore we have $p\omega(x) \subset \mathcal{PL}(m)$ for all $x \in PL(m)$.

1.2. **Standard Lyapunov exponents.** In this section we recall some background on Lyapunov exponents (see [1] for further details). Let M be a standard Borel space and let $f : M \to M$ be a Borel system. We consider a measurable fiber bundle $\pi : V \to X$ over M of dimension d equipped with a measurable Riemannian metric $\|\cdot\|_x$ on each fiber $V_x = \pi^{-1}(\{x\})$, together a bundle morphism $F : V \to V$ with $\pi \circ F = f \circ \pi$. The (forward upper) Lyapunov exponent of (x, v) for $x \in M$ and $v \in V_x$ is defined as follows

$$\chi^F(x, v) := \limsup_{n \to +\infty} \frac{1}{n} \log \|F^n(v)\|_{f^n x}.$$

The function $\chi(x, \cdot) := \chi^F(x, \cdot)$ admits only finitely many values $\chi_1(x) > ... > \chi_{p(x)}(x)$ on $TM \setminus \{0\}$ and generates a flag $0 \subsetneq V_{p(x)}(x) \subsetneq \cdots \subsetneq V_1 = T_xM$ with $V_i(x) = \{v \in TM, \ \chi(x, v) \leq \chi_i(x)\}$. In particular, $\chi(x, v) = \chi_i(x)$ for $v \in V_i(x) \setminus V_{i+1}(x)$. The function p as well the functions χ_i and the vector spaces $V_i(x)$, $i = 1, ..., p(x)$ depend Borel measurably on x. In the following we work with the Lyapunov exponents $(\chi_j)_{1 \leq j \leq d}$ counted (still nondecreasingly) with multiplicity, i.e. $\forall j \, \exists i_j \leq p, \ \chi_j = \chi_{i_j}$ and $\sharp\{j, \ i_j = i\} = \dim(V_i) - \dim(V_{i+1})$.

For a sequence of real numbers $(a_n)_n$ we let $\lim_n^{\searrow} a_n$ the limit in n of the sequence $(a_n)_n$ when the sequence is converging to $\inf_n a_n$. By the subadditive ergodic theorem we have for all $\mu \in \mathcal{M}(M, f)$:

$$\int \chi_1^+ \, d\mu \quad = \quad \overset{\searrow}{\lim_n} \frac{1}{n} \int \log^+ |||F^n|||_x \, d\mu(x). \tag{1.1}$$

For any positive integer k less than or equal to d we let χ^k be the top Lyapunov exponent of the bundle morphism $\Lambda^k F$ induced by F on the k-exterior bundle $\Lambda^k V$ (in particular $\chi^1 = \chi_1$). We also consider the top

Lyapunov exponent $\chi_\Lambda = \max_k \chi^k$ of the action ΛF on the exterior algebra bundle $\Lambda V = \oplus_k \Lambda^k V$. We have then for all $\mu \in \mathcal{M}(M, f)$,

$$\int \chi_\Lambda^+ \, d\mu = \overset{\searrow}{\lim_n} \frac{1}{n} \int \log^+ \max_k \left\| \Lambda^k F^n \right\|_x d\mu(x).$$

A point is said **(forward) Lyapunov regular** when $\chi^k(x) = \sum_{i=1}^k \chi_i(x)$ for all $1 \leq k \leq d$. By Oseledets theorem [16] the set of Lyapunov regular points has full μ-measure for any Borel f-invariant probability measure μ (but it may have zero Lebesgue measure, see [17]). Moreover we have $\chi_\Lambda^+(x) = \sum_j \chi_j^+(x)$ when x is Lyapunov regular and therefore we get for all $\mu \in \mathcal{M}(M, f)$,

$$\int \chi_\Lambda^+ \, d\mu = \int \sum_j \chi_j^+ \, d\mu.$$

1.3. **Empirical Lyapunov exponent.** Now we assume that M is a compact metric space, $f : M \to M$ a continuous map, V a continuous Riemannian bundle and $F : V \to V$ a continuous bundle morphism. By (1.1) the function $\mu \mapsto \int \chi_1^+ d\mu$ is then upper semicontinuous on $\mathcal{M}(M, f)$ as a nonincreasing limit of continuous functions. We introduce a new kind of pointwise positive Lyapunov exponents, called **empirical maximal exponent**. First we let for all $p \geq 1$ and for all $x \in M$

$$\lambda_p(x) := \limsup_n \frac{1}{n} \sum_{l=0}^{n-1} \log^+ \|F^p\|_{f^l x}.$$

Clearly we have $\frac{1}{p}\lambda_p(x) \geq \chi^+(x)$ by submultiplicativity of the subordinate norm. Moreover the sequence $(\lambda_p(x))_p$ is a subadditive sequence. Then we let for all $x \in M$

$$\lambda(x) := \overset{\searrow}{\lim_p} \frac{1}{p} \lambda_p(x) \geq \chi^+(x).$$

Lemma 1. *With the above notations, we have for all $x \in M$*

$$\sup_{\mu \in p\omega(x)} \int \chi_1^+ \, d\mu = \lambda(x).$$

Proof. Let $x \in M$. Let $\mu = \lim_k \mu_{n_k}^x \in p\omega(x)$ for an increasing sequence of integers $(n_k)_k$. For all n and p we have

$$\int \log^+ \|F^p\|_y \, d\mu_n^x(y) = \frac{1}{n} \sum_{l=0}^{n-1} \log^+ \|F^p\|_{f^l x}.$$

Taking the limit over $n = n_k$ when k goes to infinity we get

$$\int \frac{\log^+ \|F^p\|_y}{p} \, d\mu(y) \leq \frac{\lambda_p(x)}{p}$$

and by taking the limit when p goes to infinity we have finally

$$\sup_{\mu\in p\omega(x)} \chi^+(\mu) \leq \lambda(x).$$

Let us now show $\sup_{\mu\in p\omega(x)} \chi^+(\mu) \geq \lambda(x)$. For any p there exist a subsequence $(n_{k,p})_k$ such that

$$\lambda_p(x) = \lim_k \frac{1}{n_{k,p}} \sum_{l=0}^{n_{k,p}-1} \log^+ |||F^p|||_{f^l x}.$$

Then if $\mu_p \in p\omega(x)$ is a weak limit of $(\mu^x_{n_{k,p}})_k$ we have

$$\int \log^+ |||F^p|||_y \, d\mu_p(y) = \lambda_p(x). \tag{1.2}$$

For any $\mu \in \mathcal{M}(M,f)$ and for any $z \in X$, the sequences $\left(\int \log^+ |||F^p|||_y \, d\mu(y)\right)_p$ and $(\lambda_p(z))_p$ being both subadditive the terms $\chi^+(\mu)$ and $\lambda(z)$ are respectively the limits of the nonincreasing sequences $\left(\frac{\int \log^+ |||F^{p_k}|||_y \, d\mu(y)}{p_k}\right)_k$ and $\left(\frac{\lambda_{p_k}(z)}{p_k}\right)_k$ for any increasing sequence of integers $(p_k)_k$ with $p_k \mid p_{k+1}$ for all k.

Fix such a sequence $(p_k)_k$. We get :

$$\begin{aligned} \sup_{\mu\in p\omega(x)} \chi^+(\mu) &= \sup_{\mu\in p\omega(x)} \inf_p \int \frac{\log^+ |||F^p|||_y}{p} \, d\mu(y), \\ &= \sup_{\mu\in p\omega(x)} \inf_k \int \frac{\log^+ |||F^{p_k}|||_y}{p_k} \, d\mu(y) \end{aligned}$$

By Proposition 2.4 in [2] we may invert the supremum and the infimum in the right-hand side term :

$$\begin{aligned} \sup_{\mu\in p\omega(x)} \chi^+(\mu) &= \inf_k \sup_{\mu\in p\omega(x)} \int \frac{\log^+ |||F^{p_k}|||_y}{p_k} \, d\mu(y), \\ &\geq \inf_k \int \frac{\log^+ |||F^{p_k}|||_y}{p_k} \, d\mu_{p_k}(y), \\ &\geq \inf_k \frac{\lambda_{p_k}(x)}{p_k}, \end{aligned}$$

where the last inequalities follow from (1.2). Finally we get

$$\sup_{\mu\in p\omega(x)} \chi^+(\mu) \quad \geq \quad \inf_p \frac{\lambda_p(x)}{p} = \lambda(x).$$

$\square$

We let λ_Λ be the empirical Lyapunov exponent associated to the bundle morphism $\Lambda F : \Lambda V \to \Lambda V$. Then for all $x \in M$, we have

$$\sup_{\mu \in p\omega(x)} \int \chi_\Lambda^+ \, d\mu = \lambda_\Lambda(x) \geq \chi_\lambda^+(x).$$

A point x is said to be $p\omega$**-regular** when we have $\lambda_\Lambda(x) = \chi_\lambda^+(x)$. For an ergodic measure μ, almost every point x with respect to μ lies in the basin $\mathcal{B}_\mu$ of μ (in other terms $p\omega(x) = \mu$) and $\chi_\Lambda^+(x) = \int \chi_\Lambda^+ \, d\mu$. Using the ergodic decomposition it follows then from Lemma 1 :

Lemma 2. *$p\omega$-regular points have full measure with respect to any invariant measure.*

1.4. **Essential supremum of Lyapunov exponents.** In the present paper we are interested in empirical measures with Lebesgue typical initial conditions and we do not assume there exists an invariant measure absolutely continuous with respect to Leb. In particular it may happen that the set of Lyapunov regular points has not full Lebesgue measure (see e.g. [17] for the eight attractor). We will never assume Lyapunov regularity in the present paper.

We denote by $\overline{\lambda_\Lambda}$ the essential supremum of λ_Λ with respect to Leb. We also let $\overline{\chi_\Lambda^+}$ (resp. $\overline{\chi^k}$ for $k = 1, \cdots, d$) be the essential supremum of χ_Λ^+ (resp. χ^k) with respect to the Lebesgue measure, in particular $\overline{\chi_\Lambda^+} = \max(0, \overline{\chi^1}, \cdots, \overline{\chi^d})$. As the set $PL := PL(\mathrm{Leb})$ has full Lebesgue measure we have

$$\overline{\lambda_\Lambda} \leq \sup_{x \in PL} \lambda_\Lambda(x)$$

and then it follows from Lemma 1 and $\lambda_\Lambda \geq \chi_\Lambda^+$ that

$$\overline{\chi_\Lambda^+} \leq \sup_{\mu \in \mathcal{PL}} \chi_\Lambda^+(\mu).$$

In general the equality does not hold (see Remark 6), where we have $0 = \overline{\chi_\Lambda^+} < \chi_\Lambda^+(\delta_S) = \sup_{\mu \in \mathcal{PL}} \chi_\Lambda^+(\mu)$ with S being the associated saddle hyperbolic point.

Based on Yomdin's theory and the volume growth estimates due to Newhouse, Koslowski [11] showed the following integral formula for the topological entropy of a C^∞ smooth system :

$$h_{top}(f) = \lim_n \frac{1}{n} \log \int \max_k \|\Lambda^k d_x f^n\| \, d\,\mathrm{Leb}(x).$$

By Jensen's inequality we have for all integers n

$$\log \int \max_k \|\Lambda^k d_x f^n\| \, d\,\mathrm{Leb}(x) \geq \int \log \max_k \|\Lambda^k d_x f^n\| \, d\,\mathrm{Leb}(x).$$

According to Borel-Cantelli Lemma, for all $\gamma > 0$, the set $\{x \in M, \ \max_k \|\Lambda^k d_x f^n\| \geq e^{n(\overline{\chi_\Lambda^+} - \gamma)}\}$ has Lebesgue measure larger than $e^{-n\gamma}$ for infinitely many n.

Therefore we conclude that

$$h_{top}(f) \geq \limsup_n \frac{1}{n} \int \log \max_k \|\Lambda^k d_x f^n\| \, d\,\mathrm{Leb}(x) \geq \overline{\chi_\Lambda^+}. \tag{1.3}$$

1.5. **Ruelle inequality.** From now on we consider a compact manifold M of dimension d, a C^1 map $f : M \to M$ and the Lyapunov exponent with respect to the derivative cocycle $df : TM \to TM$. We recall that for a C^1 diffeomorphism Ruelle's inequality [20] the metric entropy of an invariant measure is bounded from above by the sum of its positive Lyapunov exponents. With the previous notations, this inequality may be written as follows :

Theorem 1 (Ruelle)**.** *Let $f : M \to M$ be a C^1 map on a compact manifold, then for all $\mu \in \mathcal{M}(M, f)$,*

$$h(\mu) \leq \int \chi_\Lambda^+ \, d\mu.$$

Equivalently, for all $x \in M$,

$$\sup_{\mu \in p\omega(x)} h(\mu) \leq \lambda_\Lambda(x).$$

An ergodic measure $\mu \in \mathcal{M}(M, f)$ is said **hyperbolic** when any of its Lyapunov exponent $\int \chi_j \, d\mu$, $j = 1, \cdots, d$ is nonzero. For a surface diffeomorphism, any ergodic measure with positive entropy is hyperbolic by Ruelle's inequality.

1.6. **Sinai-Ruelle-Bowen measures.** For a $C^{1+\alpha}$ diffeomorphism f of M, an invariant measure μ is said to be a **Sinai-Ruelle-Bowen measure** (SRB measure for short) when μ-almost every point has a positive Lyapunov exponent and the disintegration of μ along the unstable manifolds is absolutely continuous with respect to the volume on the unstable manifolds inherited from the Riemanian structure on M.

From Pesin theory any ergodic hyperbolic SRB measure is physical [18]. For an invariant measure μ of a $C^{1+\alpha}$ diffeomorphism we let T_μ be the set of (forward) Lyapunov regular points x in the basin $\mathcal{B}_\mu$ of μ with $\chi_i(x) = \chi_i(\mu)$ for all i. In particular any point x in T_μ satisfies $\chi_\Lambda^+(x) = \sum_i \chi_i^+(x) = \chi_\Lambda^+(\mu)$ and therefore any such point is $p\omega$-regular. Tsujii showed that there exists an SRB measure when the union of T_μ over all ergodic hyperbolic measures has positive Lebesgue measure. He also proved that an ergodic hyperbolic measure μ is an SRB measure if and only if T_μ has positive Lebesgue measure.

Ledrappier and Young [12] (see also [19] for the noninvertible version) gave a thermodynamical characterization of SRB measures : they are exactly the invariant measures with a positive Lyapunov exponent almost everywhere satisfying the so-called Pesin formula (equality case in Ruelle's inequality) :

$$h(\mu) = \int \sum_j \chi_j^+ \, d\mu.$$

In particular any SRB measure has positive entropy. It is thus hyperbolic when considering a surface diffeomorphism. The set of SRB measures is a face of the Choquet simplex of invariant measures, i.e. the ergodic components of a SRB measure are also SRB measures. As a direct consequence of the aforementioned results we have for any $C^{1+\alpha}$ surface diffeomorphism :

$$\begin{aligned}\sup_{\mu \text{ SRB}} h(\mu) &\leq \sup_{\mu \text{ physical}} h(\mu),\\ &\leq \sup_{\mu \text{ physical}} \int \chi^1 \, d\mu,\\ &\leq \overline{\chi^1}.\end{aligned}$$

Question. Do we have $\sup_{\mu \in \mathcal{PL}} h(\mu) \leq \overline{\chi^1}$ for a C^1 (resp. $C^{1+\alpha}$, C^∞) surface diffeomorphism?

2. Statements

We aim to compare the entropy of physical-like measures with the (strong) positive sum of Lyapunov exponents for C^∞ systems.

Main Theorem. *Let $f : M \to M$ be a C^∞ map. Then for Lebesgue almost every point x there exists $\mu_x \in p\omega(x)$ with*

$$h(\mu_x) \geq \chi_\Lambda^+(x).$$

Of course the inequality does not hold true for all x, e.g. when x is a periodic point with a positive Lyapunov exponent. However the set of such points has zero Lebesgue measure.

Remark 2. *For a C^2 Axiom A diffeomorphism $f : M \to M$, there are finitely many ergodic physical measures whose basins cover a set of full Lebesgue measure. Such measures also satisfies Pesin formula. In this case we have moreover $\chi_\Lambda^+(x) = \int \log \mathrm{Jac}(df|_{E_u})(x)\, d\mu(x)$ for $x \in \mathcal{B}_\mu$ by continuity of $x \mapsto \mathrm{Jac}(df|_{E_u})(x)$. Therefore for Lebesgue almost every point x we get $h(\mu_x) = \chi_\Lambda^+(x)$ with $p\omega(x) = \{\mu_x\}$.*

As a direct consequence of the Main Theorem we obtain the following lower bound on the entropy of a physical measure.

Corollary 3. *Let μ be a physical measure of a C^∞ map $f : M \to M$. Then*

$$h(\mu) \geq \overline{\chi_\Lambda^+|_{\mathcal{B}_\mu}},$$

where $\overline{\chi_\Lambda^+|_{\mathcal{B}_\mu}}$ is the essential supremum of χ_Λ^+ on $\mathcal{B}_\mu$.

The Main Theorem and Corollary 3 are wrong in finite smoothness. We give in the Appendix A an example of a C^r smooth interval map for any finite $r \geq 1$ with a Dirac physical measure at a source such that the essential supremum of the Lyapunov exponent on its basin is positive.

We recover Inequality (1.3) obtained from Kozlovski integral formula. More precisely we have :

Corollary 4. *Let $f : M \to M$ be a C^∞ map. Then*

$$\max_{\mu \in \mathcal{PL}} h(\mu) \geq \overline{\chi_\Lambda^+}.$$

Proof. For any $\epsilon > 0$ the set $\{\chi_\Lambda^+ > \overline{\chi_\Lambda^+} - \epsilon\}$ has positive Lebesgue measure, so that there exists a point x in this set with $p\omega(x) \subset \mathcal{PL}$ which satisfies the conclusion of the Main Theorem, i.e. there exist $\mu_x \in p\omega(x)$ with

$$h(\mu_x) \geq \chi_\Lambda^+(x) > \overline{\chi_\Lambda^+} - \epsilon.$$

We conclude by upper semicontinuity of the metric entropy for C^∞ maps [15] and by compactness of $\mathcal{PL}$. □

For C^∞ maps we get the following refinement of Tsujii's theorem.

Corollary 5. *Let $f : M \to M$ be a C^∞ map.*

1. *Assume the set of $p\omega$-regular points in $\{\chi_\Lambda^+ > 0\}$ has positive Lebesgue measure. Then f admits an SRB measure.*
2. *Let μ be a physical measure such that the set of $p\omega$-regular points in $\{\chi_\Lambda^+ > 0\} \cap \mathcal{B}_\mu$ has positive Lebesgue measure. Then μ is an SRB measure.*

We recall Tsujii's results only deal with diffeomorphisms but under the weaker $C^{1+\alpha}$ smoothness assumption. Contrarily to Tsujii's statement we do not assume in the second item neither ergodicity nor hyperbolicity of the physical measure μ.

Proof. We only prove the first item. The proof of the second one follows the same lines. According to the Main Theorem, for Lebesgue almost every x in $\{\chi_\Lambda^+ = \lambda_\Lambda > 0\}$ there is an SRB measure $\mu_x \in p\omega(x)$ satisfying

$$h(\mu_x) \geq \chi_\Lambda^+(x).$$

Moreover it follows from Ruelle's inequality and Lemma 1 that

$$\lambda_\Lambda(x) \geq \chi_\Lambda^+(\mu_x) \geq h(\mu_x).$$

Since we have $\chi_\Lambda^+(x) = \lambda_\Lambda(x)$ the measure μ_x satisifes Pesin's entropy formula and is therefore an SRB measure. □

Unlike the Main Theorem, which is false in finite smoothness, we conjecture Corollary 5 holds true for any $C^{1+\alpha}$ map. It can be deduced from the Reparametrization Lemma in [5] the case of $C^{1+\alpha}$ interval maps and surface diffeomorphisms. However as it involves stronger technicalities we prefer to only consider C^∞ maps in the present paper.

Remark 6. *The C^∞ assumption does not imply that the basin of an ergodic physical measure contains a positive Lebesgue measure subset of $p\omega$-regular points. If we consider again the C^∞ eight attractor of Bowen* [17, 13] *the strong Lyapunov exponent $\lambda_\Lambda(x)$ is equal to the unstable Lyapunov exponent of the saddle physical measure, whereas according to our Main Theorem we have $\chi_\Lambda^+(x) = 0$ for Lebesgue almost every point in the basin.*

3. Some technical lemmas

For a C^2 Anosov surface diffeomorphism one build SRB measures as follows. One takes the inherited Lebesgue probability measure μ on a local unstable disc and then checks that the limit ν of $\left(\frac{1}{n}\sum_{0\leq k<n} f^k\mu\right)_n$ disintegrates absolutely continuously on unstable manifolds with respect to the Lebesgue measure. Here we follow somehow a similar approach by considering the Lebesgue probability measure μ on a smooth disc with Lebesgue typical exponential growth. Then we estimate the entropy of ν by using a *Reparametrization Lemma* of dynamical balls.

3.1. **Lyapunov exponent along smooth leaves.** In the Lemma below we select the appropriate smooth disc.

Lemma 3. *Let $1 \leq k \leq d$ and let $a < \overline{\chi^k}$. We consider a Borel subset E of $\{a < \chi^k\}$ with positive Lebesgue measure. Then there exist a compact subset F of E and a foliation box U with respect to a C^∞ smooth k-foliation $\mathcal{F}$ with* $\mathrm{Leb}(U \cap F) > 0$ *such that*

$$\forall x \in U \cap F,\ \chi^k(x, T_x\mathcal{F}) > a,$$

where $T_x\mathcal{F}$ denotes any unit-norm element of $\Lambda^k(TM)$ generating the tangent space at x of the $\mathcal{F}$-leaf containing x.

Proof. Let F be a compact subset of E with $\mathrm{Leb}(F) > 0$ such that $x \mapsto V_i(x)$ is continuous on F for all i, where $(V_i(x))_i$ denotes the Lyapunov flag at x of $\Lambda^k df$ acting on $\Lambda^k TM$ (in particular recall $V_1(x) = \Lambda^k T_x M$). Let x be a Lebesgue density point of F and let $u \in V_1(x) \setminus V_2(x)$. We may assume that u is a monomial exterior product and we then let $\mathfrak{U}$ be the associated k-vector subspace of T_xM. We denote the exponential map at x by $\exp_x : T_xM \to M$. Then for a small enough neighborhood U of x the vector $\left(\Lambda^k d\exp_x(u)\right)_y$ belongs to $V_1(y) \setminus V_2(y)$ for all $y \in U \cap F$. Finally this vector generates the tangent space at y of the foliation $\mathcal{F} = \exp_x(\mathcal{F}_x)$ where $\mathcal{F}_x$ is the foliation in $\mathfrak{U}$-directed k-planes of T_xM. $\square$

3.2. **Entropy computation.** We state now a technical entropy computation due to Misiurewicz [14] in its elementary proof of the variational principle for the entropy, which we will use to bound from below the entropy of ν. For a probability space $(X, \mathcal{B}, \mu)$ and a finite measurable partition P of X we denote the static entropy of P as follows

$$H_\mu(P) := -\sum_{A\in P} \mu(A)\log\mu(A).$$

Lemma 4. [14]*Let (X, f) be a Borel system and let P be a finite measurable partition of X. We consider a sequence $(\mu_n)_n$ of probability Borel measures on X and the associated sequence $(\nu_n)_n$ given for all $n > 0$ by*

$\nu_n = \frac{1}{n}\sum_{0\leq k<n} f^k\mu_n$. *Then we have with* $P^n = \bigvee_{k=0}^{n-1} f^{-k}P$

$$\forall m>0,\ \frac{1}{m}H_{\nu_n}(P^m) \geq \frac{1}{n}\left(H_{\mu_n}(P^n) - 3m\log \sharp P\right).$$

3.3. **Local distortion.** The key argument which allows to control the distortion is given by the following lemma whose proof relies on tools of semi-algebraic geometry. For $x \in M$, $n \in \mathbb{N}$ and $\alpha > 0$ we let $B_f(x,n,\alpha)$ be the dynamical ball at x of length n and size α :

$$B_f(x,n,\alpha) := \{y \in M,\ \mathsf{d}(f^l x, f^l y) < \alpha \text{ for } l = 0,...,n-1\}.$$

Reparametrization Lemma. *Let $f : M \to M$ be a C^∞ map. Let $a \in \mathbb{R}$, $\gamma \in \mathbb{R}^+ \setminus \{0\}$ and let k be a positive integer with $k \leq d$. For some $\alpha > 0$, for all $x \in M$ and for all $\sigma : [0,1]^k \to M$ of class C^∞ with $\|d\sigma\| \leq 1$ and $\Lambda^k d_t\sigma \neq 0$ for all $t \in [0,1]^k$, there exists for large enough n (depending on σ but not on x) a family of reparametrizations $(\theta_i^n : [0,1]^k \circlearrowleft)_{i\in I_n}$ with the following properties:*

- $\bigcup_{i\in I_n} \mathrm{Im}(\theta_i^n) \Supset^{\dagger} \left\{t \in [0,1]^k,\ \frac{\|\Lambda^k d_t(f^n\circ\sigma)\|}{\|\Lambda^k d_t\sigma\|} \geq e^{na} \text{ and } \sigma(t) \in B(x,n,\alpha)\right\}$,
- $\forall i \in I_n,\ \|d(f^n \circ \sigma \circ \theta_i^n)\| \leq 1$,
- $\forall i \in I_n\ \forall t,t' \in \mathrm{Im}(\theta_i^n),\ \frac{\|\Lambda^k d_t(f^n\circ\sigma)\|}{\|\Lambda^k d_{t'}(f^n\circ\sigma)\|} \leq 2$,
- $\sharp I_n \leq e^{\gamma n}$.

Roughly speaking, the preimage under σ of any dynamical ball of length n with small enough radius may be covered by an exponentially small number of pieces, where the distorsion of $f^n \circ \sigma$ is bounded.

Such Reparametrizations Lemmas first appear in the pioneering work of Yomdin [25] (see also [10]) in his proof of Shub's entropy conjecture for C^∞ systems. In Yomdin's earlier form the control of the distortion given by the third item did not appear. Moreover the reparametrized set was the whole dynamical ball (here this is the case when f is a local diffeomorphim by choosing a small enough). Others similar forms of the Reparametrization Lemma were used succesfully by the author to study symbolic extensions and exponential growth of periodic points for C^r surface diffeomorphisms [5, 6]. The technical proof could be skipped at a first reading.

We first establish a version of the Reparametrization Lemma for a C^∞ nonautonomous system $\mathfrak{F} = (\mathfrak{f}_l : B \to \mathbb{R}^d)_{l\in\mathbb{N}}$ on the unit Euclidean ball B of $\mathbb{R}^d$. For $m \in \mathbb{N}$ we let $\mathfrak{F}_m$ be the finite sequence of C^∞ maps $\mathfrak{F}_m := (\mathfrak{f}_l)_{0\leq l<m}$. In this context we define the dynamical ball $B_{\mathfrak{F}_m}$ as follows

$$B_{\mathfrak{F}_m} := \{y \in B,\ \mathfrak{f}_l \circ \cdots \circ \mathfrak{f}_0(y) \in B \text{ for } 0 \leq l < m\}.$$

We then put $\mathfrak{f}^{m+1} = \mathfrak{f}_m \circ \cdots \circ \mathfrak{f}_0 : B_{\mathfrak{F}_m} \to \mathbb{R}^d$ (let also $\mathfrak{f}^0$ be the identity map of $\mathbb{R}^d$).

† By $\bigcup_{i\in I} A_i \Supset B$ we mean that $\bigcup_{i\in I} A_i \supset B$ and $A_i \cap B \neq \emptyset$ for all $i \in I$.

Let $\mathcal{A}=(a_l)_{l\in\mathbb{N}}$ be an infinite sequence of integers. For the corresponding finite sequences $\mathcal{A}_m:=(a_0,...,a_{m-1})$ we also consider the following dynamical ball induced by $\mathfrak{F}_m$ on the k-exterior bundle of the tangent space $T\mathbb{R}^d$ endowed with the norm induced by the Euclidean norm:

$$B^k(\mathcal{A}_m):=\{(y,v)\in\Lambda^k(T\mathbb{R}^d),\ \|v\|=1,\ y\in B_{\mathfrak{F}_m}$$
$$\text{and } \forall l=0,...,m-1,\ \lceil\log\|\Lambda^k d_{\mathfrak{f}^l y}\mathfrak{f}_l(v_l)\|\rceil=a_l\},$$

with the notations $v_l=\frac{\Lambda^k d_y\mathfrak{f}^l(v)}{\|\Lambda^k d_y\mathfrak{f}^l(v)\|}$, $l=0,...,m-1$, and $\lceil\cdot\rceil$ for the ceiling function.

For a C^∞ smooth k-disc $\mathfrak{s}:[0,1]^k\to\mathbb{R}^d$ we aim to reparametrize the set $C_{\mathfrak{s}}(\mathcal{A}_n)$ defined as follows :

$$C_{\mathfrak{s}}(\mathcal{A}_m)=\left\{t\in[0,1]^k,\ \left(\mathfrak{s}(t),\frac{\Lambda^k d_t\mathfrak{s}}{\|\Lambda^k d_t\mathfrak{s}\|}\right)\in B^k(\mathcal{A}_m)\right\}.$$

Proposition 7. *Let $r>2$ be an integer. Assume $\|d^s\mathfrak{f}_l\|\leq 1$ for all $s=2,\cdots,r$ and for all $l\in\mathbb{N}$. Then for all $m\in\mathbb{N}$ there exists a family of reparametrizations $(\phi_i^m:[0,1]^k\circlearrowleft)_{i\in\mathcal{I}(\mathcal{A}_m)}$ with the following properties :*

(1) $\bigcup_{i\in\mathcal{I}(\mathcal{A}_m)}\mathrm{Im}(\phi_i^m)\supset C_{\mathfrak{s}}(\mathcal{A}_m)$,

(2) $\forall i\in\mathcal{I}(\mathcal{A}_m)\ \forall s=0,...,r$,
$$\|d^s\left(\mathfrak{f}^m\circ\mathfrak{s}\circ\phi_i^m\right)\|\leq 1,$$

(3) $\forall i\in\mathcal{I}(\mathcal{A}_m)\ \forall s=1,...,r-1$,
$$\|d^s\left(t\mapsto\Lambda^k d_{\phi_i^m(t)}(\mathfrak{f}^m\circ\mathfrak{s})\right)\|\leq\frac{1}{2}\max_{u\in[0,1]^k}\|\Lambda^k d_{\phi_i^m(u)}(\mathfrak{f}^m\circ\mathfrak{s})\|,$$

(4) $\sharp\mathcal{I}(\mathcal{A}_m)\leq C(r,d)^m\prod_{l=0}^{m-1}\max\left(1,\|d_0\mathfrak{f}_l\|^{k/r},\left(\frac{\max(1,\|\Lambda^k d_0\mathfrak{f}_l\|)}{e^{a_l}}\right)^{\frac{k}{r-1}}\right)$ *with* $C(r,d)$ *being a universal function in r and d.*

Proof. We argue by induction on m. Assume the family $(\phi_i^m:[0,1]^k\circlearrowleft)_{i\in\mathcal{I}(\mathcal{A}_m)}$ is already built for $\mathcal{A}_m=(a_0,\cdots,a_{m-1})$. We proceed to the inductive step by building the required family of reparametrizations with respect to $\mathcal{A}_{m+1}=(a_0,\cdots,a_m)$. From the formula for the derivatives of a composition and the induction hypothesis we get therefore for any $\phi=\phi_i^m$:

$$\begin{aligned}\|d^{r-1}\left(t\mapsto\Lambda^k d_{\phi(t)}(\mathfrak{f}^{m+1}\circ\mathfrak{s})\right)\| &= \|d^{r-1}\left(t\mapsto\Lambda^k d_{\mathfrak{f}^m\circ\mathfrak{s}\circ\phi(t)}\mathfrak{f}_{m+1}\circ\Lambda^k d_{\phi(t)}(\mathfrak{f}^m\circ\mathfrak{s})\right)\|,\\ &\leq A(r,d)\max_{0\leq s\leq r-1}\|d^s\left(\Lambda^k d\mathfrak{f}_{m+1}\right)\|\max_t\|\Lambda^k d_{\phi(t)}(\mathfrak{f}^m\circ\mathfrak{s})\|,\\ &\leq A(r,d)\max\left(1,\|\Lambda^k d_0\mathfrak{f}_{m+1}\|\right)\max_t\|\Lambda^k d_{\phi(t)}(\mathfrak{f}^m\circ\mathfrak{s})\|\end{aligned}$$

and

$$\begin{aligned}\|d^r\left(\mathfrak{f}^{m+1}\circ\mathfrak{s}\circ\phi\right)\| &= \|d^{r-1}\left(d_{\mathfrak{f}^m\circ\mathfrak{s}\circ\phi}\mathfrak{f}_{m+1}\circ d_\phi(\mathfrak{f}^m\circ\mathfrak{s})\right)\|,\\ &\leq A(r,d)\max\left(1,\|d_0\mathfrak{f}_{m+1}\|\right),\end{aligned}$$

for some universal function A in r and d.

We use now the following lemma which is a slightly different version of the Main Lemma in [10].

Lemma 5. *Let $G_0 : [0,1]^e \to \mathbb{R}^{e'}$ and $G_1 : [0,1]^e \to \mathbb{R}^{e''}$ be respectively C^r and C^s maps. We denote by $B_{e'}$ and $B_{e''}$ the unit Euclidean balls of $\mathbb{R}^{e'}$ and $\mathbb{R}^{e''}$. Then there exists a family $(\psi_j : [0,1]^e \circlearrowleft)_{j\in\mathcal{J}}$ such that :*

- $\bigcup_{j\in\mathcal{J}} \mathrm{Im}(\psi_j) \Supset G_0^{-1}(B_{e'}) \cap G_1^{-1}(B_{e''})$,
- $\forall j \in \mathcal{J}\ \forall k = 0, ..., r,\ \|d^k\left(G_0 \circ \psi_j\right)\| \leq 1$,
- $\forall j \in \mathcal{J}\ \forall k = 0, ..., s,\ \|d^k\left(G_1 \circ \psi_j\right)\| \leq 1/12$,
- $\sharp\mathcal{J} \leq B(r, s, e, e', e'') \times \max\left(1, \|d^r G_0\|^{e/r}, \|d^s G_1\|^{e/s}\right)$ *for some universal function B.*

The proof follows the same lines, the unique difference being that one applies the Algebraic Lemma in [10] simultaneously to the interpolating polynomials of G_0 and G_1 with respective (maybe distinct) degrees r and s.

To conclude the inductive step we apply Lemma 5 for every $i \in \mathcal{I}(\mathcal{A}_m)$ with the C^{r-1} map $G_1 : s \mapsto \frac{\Lambda^k d_{\phi_i^m(s)}(\mathfrak{f}^{m+1}\circ\mathfrak{s})}{e^{a_m}\max_{u\in[0,1]^k}\|\Lambda^k d_{\phi_i^m(u)}(\mathfrak{f}^m\circ\mathfrak{s})\|}$ and the C^r map $G_0 = \mathfrak{f}^{m+1}\circ\mathfrak{s}\circ\phi_i^m$ (for any $t \in \mathrm{Im}(\phi_i^m) \cap C_{\mathfrak{s}}(\mathcal{A}_{m+1})$ we have $\|G_1(t)\| \leq 1$). We let ψ_j, $j \in \mathcal{J} = \mathcal{J}(\phi_i^m)$, be the resulting reparametrizations. The maps $\phi_{i,j}^{m+1} = \phi_i^m \circ \psi_j$ over all $(i,j) \in \mathcal{I}(\mathcal{A}_{m+1}) := \{(i,j),\ i \in \mathcal{I}(\mathcal{A}_m) \text{ and } j \in \mathcal{J}(\phi_i^m) \text{ with } \mathrm{Im}(\phi_i^m\circ\psi_j)\cap C_{\mathfrak{s}}(\mathcal{A}_{m+1}) \neq \emptyset\}$ give the required family of reparametrizations for the $(m+1)^{th}$ step. Let us just check the new reparametrizations $\phi_{i,j}^{m+1}$ satisfies (3) for any $s = 1, ..., r-1$:

$$\begin{aligned}
\|d^s\left(t\mapsto \Lambda^k d_{\phi_{i,j}^{m+1}(t)}(\mathfrak{f}^{n+1}\circ\mathfrak{s})\right)\| &\leq e^{a_m}\max_{u\in[0,1]^k}\|\Lambda^k d_{\phi_i^m(u)}(\mathfrak{f}^m\circ\mathfrak{s})\|\|d^s(G_1\circ\psi_j)\|,\\
&\leq \frac{e^{a_m-1}}{4}\max_{u\in[0,1]^k}\|\Lambda^k d_{\phi_i^m(u)}(\mathfrak{f}^m\circ\mathfrak{s})\|,\\
&\leq \frac{e^{a_m-1}}{2}\min_{u\in[0,1]^k}\|\Lambda^k d_{\phi_i^m(u)}(\mathfrak{f}^m\circ\mathfrak{s})\|.
\end{aligned}$$

Since we have $\mathrm{Im}(\phi_{i,j}^{m+1}\circ\psi_j)\cap C_{\mathfrak{s}}(\mathcal{A}_{m+1}) \neq \emptyset$ there exists $v \in [0,1]^k$, with $\frac{\|\Lambda^k d_{\phi_{i,j}^{m+1}(v)}(\mathfrak{f}^{m+1}\circ\mathfrak{s})\|}{\|\Lambda^k d_{\phi_{i,j}^{m+1}(v)}(\mathfrak{f}^{m}\circ\mathfrak{s})\|} \geq e^{a_m-1}$ and therefore

$$\begin{aligned}
\|d^s\left(t\mapsto \Lambda^k d_{\phi_{i,j}^{m+1}(t)}(\mathfrak{f}^{m+1}\circ\mathfrak{s})\right)\| &\leq \frac{e^{a_m-1}}{2}\|\Lambda^k d_{\phi_{i,j}^{m+1}(v)}(\mathfrak{f}^m\circ\mathfrak{s})\|,\\
&\leq \frac{1}{2}\|\Lambda^k d_{\phi_{i,j}^{m+1}(v)}(\mathfrak{f}^{m+1}\circ\mathfrak{s})\|,\\
&\leq \frac{1}{2}\max_{u\in[0,1]^k}\|\Lambda^k d_{\phi_{i,j}^{m+1}(u)}(\mathfrak{f}^{m+1}\circ\mathfrak{s})\|.
\end{aligned}$$

This concludes the proof of Proposition 7. $\square$

A sequence $\mathcal{A}_m = (a_0, \cdots, a_{m-1})$ is said **A-admissible** for $A \in \mathbb{R}$ when

$$B^k(\mathcal{A}_m) \cap \{(y,v) \in \Lambda^k(T\mathbb{R}^d),\ \|v\| = 1 \text{ and } \|\Lambda^k d_y \mathfrak{f}^m(v)\| \geq e^{mA}\} \neq \emptyset.$$

In particular we have then $\sum_{l=0}^{m-1} a_l \geq mA$.

Let F be the real function $\mathbb{R}^+ \ni t \mapsto t\left[t^{-1}\log(t^{-1}) + (1-t^{-1})\log(1-t^{-1})\right]$, in particular $F(t) \leq t\log 2$ for all t and $\lim_{t\to+\infty} \frac{F(t)}{t} = 0$. By a standard combinatorial argument (see e.g. Lemma 8 in [5]) we have :

Lemma 6. *Let $A \in \mathbb{R}$. Assume $\left|\log^+\|\Lambda^k d_0\mathfrak{f}_l\| - \log^+\|\Lambda^k d_y \mathfrak{f}_l\|\right| < 1$ for all $l \in \mathbb{N}$ and $y \in B$. Then the number k_m of A-admissible sequences $\mathcal{A}_m$ is bounded from above as follows*

$$\frac{\log k_m}{m} \leq F(\lambda^k(\mathfrak{F}_m) + 2 - A),$$

where $\lambda^k(\mathfrak{F}_m) := \frac{1}{m}\sum_{l=0}^{m-1}\log^+\|\Lambda^k d_0 \mathfrak{f}_l\|$

We are now in position to prove the Reparametrization Lemma.

Proof of the Reparametrization Lemma. Without loss of generality we can assume $a < -1$. Fix then $\gamma > 0$ and $x \in X$ and take a positive integer p precised later on. Let $\mathbb{N}^* \ni n = p(m-1) + q$ with $m, q \in \mathbb{N}^*$ and $0 < q \leq p$. As in the previous works [25, 10, 5] we may replace[‡] σ by $\mathfrak{s} = \alpha^{-1}\sigma(\alpha\cdot)$ for $\alpha > 0$ and the local dynamic of f around x of time n by the nonautonomous system $\mathfrak{F}_m = (\mathfrak{f}_l)_{0\leq l<m}$ defined on the unit Euclidean ball B of $\mathbb{R}^d$ by $\mathfrak{f}_l = \alpha^{-1}f^p(f^{pl}x + \alpha\cdot)$ for $0 \leq l < m-1$ and $\mathfrak{f}_{m-1} = \alpha^{-1}f^q(f^{p(m-1)}x + \alpha\cdot)$. We assume here without loss of generality that M is the d-torus $\mathbb{R}^d/\mathbb{Z}^d$ and α is less than 1 (in general, without an affine structure, one should conjugate f with the exponential map at $f^l x$ to get a map $\mathfrak{f}_l$ on $B \subset \mathbb{R}^d$ and take α less than the radius of injectivity of $(M, \|\cdot\|)$. Moreover one has to replace the Euclidean norm by the Riemanian norms along the orbit of x, in the nonautonomous system).

We may take $\alpha > 0$ so small that $\left|\log^+\|\Lambda^k d_0\mathfrak{f}_l\| - \log^+\|\Lambda^k d_y\mathfrak{f}_l\|\right| < 1$ for all $0 \leq l < m$ and $y \in B$. We have $p/2 \leq n/m (\leq p)$ once $m \geq 2$. Therefore in this case a an/m-admissible sequence $\mathcal{A}_m$ is $ap/2$-admissible. It follows then from Lemma 6 that the number k_m of *an/m-admissible* sequences $\mathcal{A}_m$ satisfies

$$\frac{\log k_m}{m} \leq F(\lambda^k(\mathfrak{F}_m) + 2 - ap/2),$$

Moreover we have

$$B_f(x,n,\alpha) \subset B_{\mathfrak{F}_m}$$

[‡]Of course we only reparametrize in this a way the subset $\sigma(\alpha[0,1]^k)$. But one can reparametrize similarly $\sigma(C_\alpha)$ for any subcube C_α of $[0,1]^k$ of size α and we only need $\lceil\alpha^{-1}\rceil^d$ such subcubes to cover $[0,1]^k$.

and for all $t \in \alpha[0,1]^k$

$$\frac{\|\Lambda^k d_t(f^n \circ \sigma)\|}{\|\Lambda^k d_t \sigma\|} = \frac{\|\Lambda^k d_{\alpha^{-1}t}(\mathfrak{f}^m \circ \mathfrak{s})\|}{\|\Lambda^k d_{\alpha^{-1}t}\mathfrak{s}\|}.$$

Therefore we get

$$\bigcup \{\alpha C_{\mathfrak{s}}(\mathcal{A}_m),\ \mathcal{A}_m \ an/m\text{-admissible }\} \supset \left\{ t \in \alpha[0,1]^k,\ \frac{\|\Lambda^k d_t(f^n \circ \sigma)\|}{\|\Lambda^k d_t \sigma\|} \geq e^{na} \text{ and } \sigma(t) \in B(x,n,\alpha) \right\}.$$

For $\gamma > 0$ we take r such that

$$\max(1, \|df\|^{k/r}) \times \left(\frac{\max\left(1, \|\Lambda^k df\|\right)}{e^a} \right)^{\frac{k}{r-1}} < e^{\gamma/6}.$$

We consider then an integer p so large that

$$p > \frac{6\left(2k + \log C(r,d)\right)}{\gamma}$$

and

$$\sup_{x > p\gamma/3\log 2} \frac{F(x)}{x-2} < \frac{\gamma}{6k\max(\log\|df\|, |a|)}.$$

This last constraint allows to control the number k_m of an/m-admissible sequences $\mathcal{A}_m$ (observe $\lambda^k(\mathfrak{F}_m) \leq pk\log^+\|df\|$):

$$\begin{aligned}
\frac{\log k_m}{m} &\leq F\left(\lambda^k\left(\mathfrak{F}_m\right) + 2 - ap/2\right), \\
&\leq \max\left(\left(\lambda^k(\mathfrak{F}_m) - ap/2\right) \sup_{x > p\gamma/3\log 2} \frac{F(x)}{x-2}, \sup_{x \leq p\gamma/3\log 2} F(x) \right), \\
&\leq \max\left(\frac{p\gamma\left(k\log^+\|df\| + |a|/2\right)}{6k\max(\log^+\|df\|, |a|)}, p\gamma/3 \right), \\
\frac{\log k_m}{m} &\leq p\gamma/3.
\end{aligned}$$

Moreover, for any an/m-admissible sequence $\mathcal{A}_m = (a_0, \cdots, a_{m-1})$ we have $\frac{\max(1, \|\Lambda^k d_0 \mathfrak{f}_l\|)}{e^{a_l}} \geq 1/e^2$ for any $0 \leq l < m$ and therefore

$$\begin{aligned}
&C(r,d)^m \prod_{l=0}^{m-1} \max\left(1, \|d_0\mathfrak{f}_l\|^{k/r}, \left(\frac{\max(1,\|\Lambda^k d_0\mathfrak{f}_l\|)}{e^{a_l}}\right)^{\frac{k}{r-1}}\right)\\
&\leq (e^{2k}C(r,d))^m \prod_{l=0}^{m-1} \max\left(1, \|d_0\mathfrak{f}_l\|^{k/r}\right) \times \prod_{l=0}^{m-1} \left(\frac{\max(1,\|\Lambda^k d_0\mathfrak{f}_l\|)}{e^{a_l}}\right)^{\frac{k}{r-1}},\\
&\leq (e^{2k}C(r,d))^m \left(\max(1,\|df\|^{k/r}) \times \left(\frac{\max(1,\|\Lambda^k df\|)}{e^{a}}\right)^{\frac{k}{r-1}}\right)^n,\\
&\leq e^{2k}C(r,d)e^{\gamma n/3},
\end{aligned}$$

where the last inequality follows from $m \leq 1 + \frac{n}{p} \leq 1 + \frac{\gamma n}{6(2k+\log C(r,d))}$).

We fix finally $\alpha > 0$ so small that $\|d^s\mathfrak{f}_l\| \leq \alpha^{s-1}\|d^s f\| \leq 1$ for all $s = 2, \cdots, r$ and for all $l \in \mathbb{N}$. The reparametrizations $(\phi_i^m)_{i\in\mathcal{I}(\mathcal{A}_m)}$ built in Proposition 7 with respect to $\mathfrak{F}_p$ over all an/m-admissible sequences $\mathcal{A}_m$ then satisfies the conclusion of the Reparametrization Lemma after a rescaling of size α:

- $$\begin{aligned}
&\bigcup\left\{\alpha\phi_i^m([0,1]^k),\ i\in\mathcal{I}(\mathcal{A}_m) \text{ and } \mathcal{A}_m\ an/m\text{-admissible }\right\}\\
&\supset \bigcup\{\alpha C_{\mathfrak{s}}(\mathcal{A}_m),\ \mathcal{A}_m\ an/m\text{-admissible }\}\\
&\supset \left\{t\in\alpha[0,1]^k,\ \frac{\|\Lambda^k d_t(f^n\circ\sigma)\|}{\|\Lambda^k d_t\sigma\|} \geq e^{na} \text{ and } \sigma(t)\in B(x,n,\alpha)\right\}.
\end{aligned}$$

  By taking a subfamily we may assume the image of each reparametrization has a non empty intersection with this last set.
- $\forall \mathcal{A}_m,\ i\in\mathcal{I}(\mathcal{A}_m)$,
  $$\begin{aligned}
  \|d(f^n\circ\sigma\circ\alpha\phi_i^m)\| &= \alpha\|d\left(\mathfrak{f}^m\circ\mathfrak{s}\circ\phi_i^m\right)\|,\\
  &\leq \alpha < 1.
  \end{aligned}$$
- $\forall \mathcal{A}_m,\ i\in\mathcal{I}(\mathcal{A}_m)$, we get from Lemma 7 (with the notation $\|\Lambda^k d_\phi g\| := \max_{u\in[0,1]^k}\|\Lambda^k d_{\phi(u)}g\|$ for maps $\phi : [0,1]^k \circlearrowleft$ and $g : [0,1]^k \to \mathbb{R}^d$ or M):
  $$\begin{aligned}
  \|d\left(t\mapsto \Lambda^k d_{\alpha\phi_i^m(t)}(f^n\circ\sigma)\right)\| &= \|d\left(t\mapsto\Lambda^k d_{\phi_i^m(t)}(\mathfrak{f}^m\circ\mathfrak{s})\right)\|,\\
  &\leq \frac{1}{2}\|\Lambda^k d_{\phi_i^m}(\mathfrak{f}^m\circ\mathfrak{s})\|,\\
  &\leq \frac{1}{2}\|\Lambda^k d_{\alpha\phi_i^m}(f^n\circ\sigma)\|.
  \end{aligned}$$

  Then it follows from the mean value inequality :

$$\forall t,t'\in[0,1]^k,\ \|\Lambda^k d_{\alpha\phi_i^m(t)}(f^n\circ\sigma) - \Lambda^k d_{\alpha\phi_i^m(t')}(f^n\circ\sigma)\| \leq \frac{1}{2}\|\Lambda^k d_{\alpha\phi_i^m}(f^n\circ\sigma)\|$$

and by the triangular inequality

$$\|\Lambda^k d_{\alpha\phi_i^m(t)}(f^m \circ \sigma)\| \geq \|\Lambda^k d_{\alpha\phi_i^m(t')}(f^m \circ \sigma)\| - \frac{1}{2}\|\Lambda^k d_{\alpha\phi_i^m}(f^m \circ \sigma)\|.$$

Finally we get by taking the maximum over $t' \in [0,1]^k$:

$$\|\Lambda^k d_{\alpha\phi_i^m(t)}(f^m \circ \sigma)\| \geq \frac{1}{2}\|\Lambda^k d_{\alpha\phi_i^m}(f^m \circ \sigma)\|.$$

•

$$\begin{aligned}
&\sharp\{\phi_i^m,\ i \in \mathcal{I}(\mathcal{A}_m) \text{ and } \mathcal{A}_m\ an/m-\text{admissible}\} \\
&\leq \sum_{\mathcal{A}_m\ an/m-\text{admissible}} \sharp\mathcal{I}(\mathcal{A}_m), \\
&\leq e^{2k}C(r,d)re^{\gamma n/3}k_m, \\
&\leq e^{2k}C(r,d)re^{\gamma n/3}e^{\gamma mp/3}, \\
&\leq e^{\gamma n} \text{ for } n \text{ large enough (with } p \text{ staying fixed).}
\end{aligned}$$

□

Remark 8. *In the proof of the Main Theorem below, we will only need to apply the Reparametrization Lemma for $a > 0$.*

4. Proof of the Main Theorem

For $1 \leq k \leq d$ and $a < \overline{\chi^k}$ we let $PL_a^k = PL(\mathrm{Leb}_{\{\chi^k > a\}})$ be the set of points x in M with $p\omega(x) \subset \mathcal{PL}(\mathrm{Leb}_{\{\chi^k > a\}})$. Recall $\mathrm{Leb}_{\{\chi^k > a\}}(PL_a^k) = 1$.

Proposition 9. *For any $1 \leq k \leq d$ and $a < \overline{\chi^k}$ we have*

$$\forall x \in PL_a^k\ \exists \mu_x \in p\omega(x), \quad h(\mu_x) \geq a.$$

We first prove the Main Theorem assuming the above Proposition 9. Let A be a countable and dense subset of $\mathbb{R}^+$. The countable intersection E over $1 \leq k \leq d$ and $a_k \in A$ of the sets $PL_{a_k}^k \cup \{\chi^k \leq a_k\}$ has full Lebesgue measure. Fix $x \in E$ and let us show that there exists $\mu_x \in p\omega(x)$ with $h(\mu_x) \geq \chi_\Lambda^+(x)$. We may assume $\chi_\Lambda^+(x) > 0$. Take k with $\chi^k(x) = \chi_\Lambda^+(x)$. For any $a_k \in A$ with $a_k < \chi^k(x)$ we have $h(\mu_x) \geq a_k$ for some $\mu_x \in p\omega(x)$, according to Proposition 9. Since A is dense in $\mathbb{R}^+$ and the metric entropy is upper semicontinuous we conclude that

$$\sup_{\mu \in p\omega(x)} h(\mu) = \max_{\mu \in p\omega(x)} h(\mu) \geq \chi_\Lambda^+(x).$$

Proof of Proposition 9. Fix x in PL_a^k. For all $\epsilon > 0$ the set $E = \{y,\ \chi^k(y) > a \text{ and } \mathfrak{d}^H(p\omega(y), p\omega(x)) < \epsilon/2\}$ has positive Lebesgue measure (by definition of PL_a^k). Let F be the subset of E and let U be the $\mathcal{F}$-foliation box given both by Lemma 3. Fix $\gamma, \epsilon > 0$. As the foliation is smooth, there is by Fubini's theorem a leaf L of $\mathcal{F}$ intersecting F in a set of positive Lebesgue measure (for the Lebesgue measure Leb_L induced on the smooth leaf L). Let

$\mathcal{V}$ be a finite cover of $p\omega(x)$ by balls V of radius $\frac{\epsilon}{2}$ centered at $x_V \in p\omega(x)$. We put for all integers n and for all $V \in \mathcal{V}$

$$B_n^V(= B_n^V(x)) := \quad \{y \in L \cap F \subset U,\ \|\Lambda^k df^n(T_y\mathcal{F})\| \geq e^{na} \\ \text{and } \mathfrak{d}(\mu_n^y, V) < \epsilon/2\}.$$

By Borel-Cantelli Lemma we have $Leb_L(B_n^{V'}) \geq e^{-n\gamma}$ for some $V' \in \mathcal{V}$ and for n in an infinite subset $I_{\epsilon,\gamma}$ of positive integers. Indeed if not we should have $\mathrm{Leb}_L(\limsup_n B_n^V) = 0$ for all $V \in \mathcal{V}$, but as by Lemma 4 we have $L \cap F \subset \{y,\ \chi^k(y, T_y\mathcal{F}) > a \text{ and } \mathfrak{d}^H(p\omega(y), p\omega(x)) < \epsilon/2\} \subset \bigcup_{V\in\mathcal{V}} \limsup_n B_n^V$, it would contradict $\mathrm{Leb}_L(F) > 0$. For $n \in I_{\epsilon,\gamma}$ we let μ_n be the probability measure induced on $B_n^{V'}$ by the Lebesgue measure Leb_L on L and $\nu_n := \frac{1}{n}\sum_{l=0}^{n-1} f^l\mu_n = \int \mu_n^y\, d\mu_n(y)$. By convexity of the metric $\mathfrak{d}$ we have $\mathfrak{d}(\nu_n, p\omega(x)) \leq \mathfrak{d}(\nu_n, x_{V'}) < \epsilon$.

Lemma 7. *With the above notations, any weak limit $\nu = \nu_{\epsilon,\gamma}^{a,k}$ of $(\nu_n)_{n\in I_{\epsilon,\gamma}}$, when $n \in I_{\epsilon,\gamma}$ goes to infinity, is ϵ-close to $p\omega(x)$ and satisfies*

$$h(\nu) \geq a - 2\gamma.$$

We postpone the proof of Lemma 7. To conclude the proof of Proposition 9 (admitting Lemma 7) we consider a weak-limit μ of $\nu_{\epsilon,\gamma}^{a,k}$ when ϵ and γ both go to zero. Clearly $\mu \in p\omega(x)$ and by upper semicontinuity of the metric entropy we get $h(\mu) \geq a$. □

Proof of Lemma 7. Let α be the scale given by the Reparametrization Lemma with respect to γ, k and a. We consider a partition P of M with diameter less than α. By standard arguments we may assume the boundary of P has zero ν-measure ; in particular the static entropy $\mu \mapsto H_\mu(P^m)$ is a continuous function for any m at ν. By Lemma 4

$$\forall m,\ \frac{1}{m}H_{\nu_n}(P^m) \geq \frac{1}{n}\left(H_{\mu_n}(P^n) - 3m\log\sharp P\right).$$

By taking the limit when $n \in I_{\epsilon,\gamma}$ goes to infinity we get

$$\frac{1}{m}H_\nu(P^m) \geq \liminf_{n\in I_{\epsilon,\gamma}} \frac{1}{n}H_{\mu_n}(P^n).$$

Let P_y^n being the element of the partition P^n containing $y \in M$. Then we have

$$H_{\mu_n}(P^n) = \int -\log\mu_n(P_y^n)d\mu_n(y).$$

We apply the Reparametrization Lemma at a given point y to a C^∞ map $\sigma : [0,1]^k \to M$ parametrizing the leaf L. By taking the foliation box U small enough we can assume $\|d\sigma\| \leq 1$ and $\Lambda^k d_t\sigma \neq 0$ for all $t \in [0,1]^k$. For n large enough we let θ be the resulting reparametrizations. The set $P_y^n \cap B_n^{V'}(x) \subset B(y, n, \alpha) \cap B_n^{V'}(x)$ is covered by the images of the θ's. The Lebesgue measure of each $f^n \circ \sigma \circ \theta$ is bounded from above by a universal constant C according to the second item of the Reparametrization Lemma. From the first item and the third item we get $\|\Lambda^k d_{\theta(t)}(f^n \circ \sigma)\| \geq \|\Lambda^k d_{\theta(t)}\sigma\| e^{na}/2$ for any $t \in [0,1]^k$. Together with the upperbound on the number of reparametrizations given in the last item we have for n large enough (independently of $y \in M$) :

$$\begin{aligned}
\underset{L}{\mathrm{Leb}}(P_y^n \cap B_n^{V'}(x)) &\leq \sum_\theta \mathrm{Leb}((\sigma \circ \theta)([0,1^k])),\\
&\leq \sum_\theta \int_{[0,1]^k} \|\Lambda^k d_{\theta(t)}\sigma\| \|\Lambda^k d_t\theta\|\, dt,\\
&\leq \sum_\theta 2e^{-na} \int_{[0,1]^k} \|\Lambda^k d_{\theta(t)}(f^n \circ \sigma)\| \|\Lambda^k d_t\theta\|\, dt,\\
&\leq \sum_\theta 2e^{-na}\, \mathrm{Leb}((f^n \circ \sigma \circ \theta)([0,1^k])),\\
&\leq 2Ce^{-na} \sharp\{\theta\},\\
\underset{L}{\mathrm{Leb}}(P_y^n \cap B_n^{V'}(x)) &\leq 2Ce^{-na} \times e^{\gamma n}.
\end{aligned}$$

But for $n \in I_{\epsilon,\gamma}$ we have also $\mathrm{Leb}_L(B_n^{V'}(x)) \geq e^{-n\gamma}$ so that we finally get for large enough $n \in I_{\epsilon,\gamma}$ and for all $y \in M$

$$\mu_n(P_y^n) \leq 2Ce^{-na} \times e^{2\gamma n},$$

$$H_{\mu_n}(P^n) \geq (a - 2\gamma)n - \log(2C)$$

and for all m

$$\frac{1}{m} H_\nu(P^m) \geq \liminf_{n \in I_{\epsilon,\gamma}} \frac{1}{n} H_{\mu_n}(P^n) \geq a - 2\gamma.$$

By taking the limit in m we conclude

$$h(\nu) \geq a - 2\gamma.$$

□

Appendix A. Counter-example for C^r interval maps for any finite r

For any positive integer r we give an example of a C^r (but not C^{r+1}) interval map $h : [0, 3/2] \circlearrowleft$ such that for x in a positive Lebesgue measure set the following properties hold:

(1) the empirical measures $(\mu_n^x)_n$ are converging to the Dirac measure at a fixed point (therefore with zero entropy),
(2) the Lyapunov exponent at x satisfies $\chi(x) = \frac{\log \|h'\|_\infty}{r} > 0$.

Consequently the Main Theorem does not hold true in finite smoothness.

Step 1: Let $\lambda > 1$. We first consider a C^r (even C^∞) interval map $f : [0, 3/2] \circlearrowleft$ with the following properties

- $f(0) = f(1) = 0$,
- f has a tangency of order r at 1, i.e. $f^{(k)}(1) = 0$ for $k = 1, ..., r$,
- f is affine with a slope equal to $\lambda = \|f'\|_\infty$ on the interval $[0, 1/\lambda]$.

Step 2: After a small C^∞ perturbation of f around 1 we may build a new map g such that for some n_0 and $n \geq n_0$, $g^k(1 - 1/n)$ lies in $[0, 1/\lambda]$ for $k = 1, ..., r^n - 1$ and $g^{r^n}(1 - 1/n) = 1 - 1/n + 1$. Indeed these conditions require $g(1-1/n) = (1-1/n+1)\lambda^{-r^n+1} = o(1/n^r)$, so that one can choose g arbitrarily C^∞ closed to f by taking n_0 large enough. For the interval map g, the empirical measures at $1 - 1/n$ are converging to the Dirac measure at the fixed point 0. We may also assume g is constant on $J_n := [1 - 1/n, 1 - 1/n - 1/2n^2]$ for $n \geq n_0$.

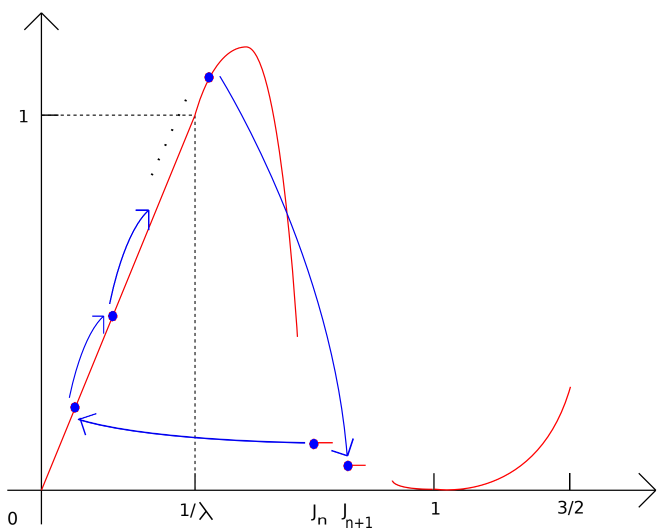

Figure 1: ***The graph of*** g ***in red.*** *The arrows and points in blue represent the orbit of* $1 - 1/n \in J_n$.

Step 3: We lastly modify g on J_n, $n \geq n_0$ such that the resulting map h satisfies the desired properties. Let us first introduce an auxiliary family of functions $(f_p)_{p \in \mathbb{N}}$. For any p we define f_p as the tent map $x \mapsto \max(x, 1-x)$ on $[1/p, 1/2 - 1/p] \cup [1/2 + 1/p, 1 - 1/p]$. We extend it into a C^r smooth interval map in such a way f_p vanishes and admits a tangency of order r at the points $0, 1/2$ and 1. Finally we extend f_p periodically on the whole real axis. The intervals $[1/p, 1/2 - 1/p] + k$ and $[1/2 + 1/p, 1 - 1/p] + k$ for $k \in \mathbb{Z}$

are called the affine branches of f_p. Observe that the C^r norm[§] of f_p may be chosen of order p^r. Then we let h be $x \mapsto \alpha_n f_{n^2}\left((x-1+1/n)2n^2N_n\right)+g(1-1/n)$ on J_n where $\alpha_n \in \mathbb{R}^+$ and $N_n \in \mathbb{N}$ are chosen such that

- for each affine branch I_n in J_n,
$$h^k(I_n) \subset [0,1/\lambda] \text{ for } k=1,...,r^n-1$$
and
$$h^{r^n}(I_n) = J_{n+1},$$
- the C^r norm of h on J_n goes to zero with n.

The first and second conditions are respectively fulfilled whenever

$$\lambda^{r^n-1} \times \alpha_n(1/2-2/n^2) = 1/2(n+1)^2$$

and

$$\max_{k=1,\dots,r} \|f_{n^2}^{(k)}\|_\infty \times \alpha_n \times (2n^2N_n)^r \sim n^{2r} \times \alpha_n \times (2n^2N_n)^r = 1/n.$$

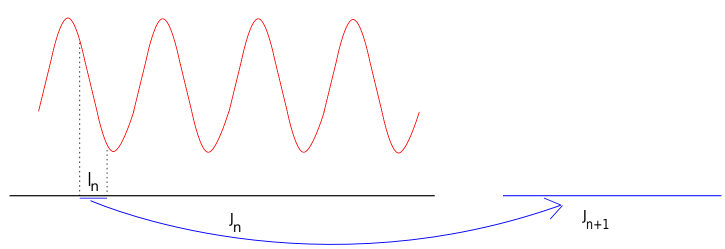

Figure 2: ***The graph of*** h ***on*** J_n ***in red.*** *The arrows and intervals in blue represent the image* J_{n+1} *of an affine branch* I_n *under* h^{r^n}.

Conclusion: Let $E_n = \bigcup_{I_n} I_n$ be the union of affine branches in J_n and let $E = E_{n_0} \cap h^{-r^{n_0}} E_{n_0+1} \cap h^{-r^{n_0}-r^{n_0+1}} E_{n_0+2} \cap \dots$ be the subset of points in J_{n_0} visiting successively the sets E_n, $n \geq n_0$. Clearly E is contained in the basin of the Dirac measure at 0. To conclude it remains to see that E has positive Lebesgue measure and that $\chi(x) \geq \frac{\log\lambda}{r}$ for any x in E. The set E is an affine dynamically defined Cantor set where we remove a proportion of $4/n^2$ at the n^{th} step. Therefore $\mathrm{Leb}(E) = \mathrm{Leb}(E_{n_0}) \prod_{n>n_0}(1-4/n^2) > 0$. Finally as $\log|h'|$ is equal on I_n to $\log(\alpha_n 4n^2N_n) \sim \frac{r-1}{r}\log\alpha_n \sim -r^{n-1}(r-1)\log\lambda$, the Lyapunov exponent at any $x \in E$ is given by

$$\begin{aligned}
\chi(x) &= \limsup_p \frac{1}{p}\log|(h^p)'(x)|, \\
&= \log\lambda \lim_q \frac{\sum_{q\geq n\geq n_0}\left(r^n - r^{n-1}(r-1)\right)}{\sum_{n\geq n_0} r^n}, \\
&= \frac{\log\lambda}{r}.
\end{aligned}$$

[§]The C^r norm of a C^r smooth interval map f is the maximum over $k=0,...,r$ of the supremum norms $\|f^{(k)}\|_\infty$.

Observe that any point in E is not recurrent.

Appendix B. Essential range of $x \mapsto p\omega(x)$

We recall here the definition of the essential range of a Borel map with respect to a Borel measure. Finally we relate the set of physical-like measures of a topological system (M, f) with the essential range of $M \ni x \mapsto p\omega(x)$.

We consider two metric spaces X and Y with Y separable. Let m be a Borel measure on X and $\phi : X \to Y$ be a Borel map.

Definition 1. With the above notations the essential range $\overline{\mathrm{Im}}_m(\phi)$ of ϕ with respect to m is the complement of $\{y \in Y,\ \exists U \text{ open with } y \in U \text{ and } m(\phi^{-1}U) = 0\}$.

The set $\overline{\mathrm{Im}}_m(\phi)$ is a closed subset of Y and for m-almost every x the point $\phi(x)$ belongs to $\overline{\mathrm{Im}}_m(\phi)$. Moreover it is the smallest set satisfying these properties.

Lemma 8. *Let (M, f) be a topological system. The map $p\omega : x \mapsto p\omega(x)$ from M to $\mathcal{KM}(M)$ is Borel.*

Proof. As the set $\mathcal{KM}(M)$ is separable, it is enough to show $p\omega^{-1}(B)$ is a Borel subset of M for any closed ball B of $\mathcal{KM}(M)$. Let B be the closed ball of radius ϵ centered at $K \in \mathcal{KM}(M)$, i.e. the set of compact subsets K' of M with $K' \subset K_\epsilon$ and $K \subset K'_\epsilon$ where K_ϵ and K'_ϵ denote respectively the closed ϵ-neighborhoods of K and K'. Firstly observe that $\{x \in M,\ p\omega(x) \subset K_\epsilon\}$ is closed. Then for a fixed sequence $(k_n)_{n\in\mathbb{N}}$ dense in K the following properties are equivalent :

$$\begin{array}{rll} & K \subset (p\omega(x))_\epsilon, & \\ \Leftrightarrow & \mathfrak{d}(k_n, p\omega(x)) \leq \epsilon & \text{for all } n, \\ \Leftrightarrow & \liminf_p \mathfrak{d}(k_n, \mu_x^p) < \epsilon' & \text{for all } n \text{ and } \mathbb{Q} \ni \epsilon' > \epsilon. \end{array}$$

The fonctions $x \mapsto \mathfrak{d}(k_n, \mu_x^p)$ being continous we conclude that $p\omega^{-1}(B)$ is a Borel set. □

Lemma 9. *The set $\mathcal{PL}(m)$ of physical-like measures is the union of all $K \in \overline{\mathrm{Im}}_m(p\omega)$.*

Proof. Firstly, the set $\overline{\mathrm{Im}}_m(p\omega)$ being a compact subset of $\mathcal{KM}(M)$, the set $\bigcup_{K\in\overline{\mathrm{Im}}_m(p\omega)} K$ is a compact subset of M. Therefore, from the definitions we get $\mathcal{PL}(m) \subset \bigcup_{K\in\overline{\mathrm{Im}}_m(p\omega)} K$. We argue by contradiction to prove the converse inclusion. Assume there is $K \in \overline{\mathrm{Im}}_m(p\omega)$ such that K is not contained in $\mathcal{PL}(m)$. Then this also holds for any K' close enough to K. Therefore there exists an open neighborhood U of K such that $p\omega^{-1}(U)$ has positive m-measure and for all x in this set $p\omega(x)$ is not contained in $\mathcal{PL}(m)$. It is impossible by definition of $\mathcal{PL}(m)$. □

References

1. L. Bareira and Y. Pesin, *Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents*,Encyclopedia of Mathematics and its Applications vol 115 (2007)
2. M.Boyle, T.Downarowicz, *The entropy theory of symbolic extension*, Invent. Math. 156 (2004), no.1, 119-161 .
3. Bonatti, C., Diaz, L., Viana, M., *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective*, Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, 2005. xviii+384 pp.
4. R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer Lecture Notes in Math. 470 (1975).
5. D. Burguet, *Symbolic extensions in intermediate smoothness on surfaces*, Ann. Sci. c. Norm. Supr. (4) 45 (2012), no. 2, 337362.
6. D. Burguet, *Periodic expansiveness of surface diffeomorphisms and applications*, J.E.M.S. (to appear).
7. D. Burguet, *A C^r surface diffeomorphism with a physical atomic measure whose basin contains a positive Lebesgue subset of points with positive Lyapunov exponent*, in preparation.
8. E. Catsigeras and H. Enrich, *SRB-like measures for C0 dynamics*, Bull. Pol. Acad. Sci. Math. 59 (2011), no. 2, 151164.
9. E. Catsigeras, M.Cerminara and H. Enrich, *The Pesin entropy formula for diffeomorphisms with dominated splitting*, Ergodic Theory and Dynamical Systems, 35(3), 737-761 (2015).
10. M. Gromov, *Entropy, homology and semi-algebraic geometry*, Seminaire Bourbaki 663 (1986).
11. O. Kozlovski, *An integral formula for topological entropy of C^{infty} maps*, Ergodic Theory and Dynamical Systems, 18(2), 405-424 (1998).
12. F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesins entropy formula.*, Ann. of Math. (2) 122 (1985), no. 3, 509-539.
13. L.-S. Young, *What are SRB measures, and which dynamical systems have them?*, J. Statist. Phys. 108 (2002), no. 5-6, 733754.
14. Misiurewicz, M.., *A short proof of the variational principle for $\mathbb{Z}^d$ actions*, Asterisque 40 (1976), 147158.
15. S.Newhouse, *Continuity properties of entropy*, Annals of Math., 129 (1989), 215-235.
16. V. I. Oseledets, *A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. 19 (1968), 197231.
17. Ott, William; Yorke, James A., *When Lyapunov exponents fail to exist*, Phys. Rev. E (3) 78 (2008), no. 5
18. Y. Pesin, *Lyapunov characteristic exponent and smooth ergodic theory*, Russian Math. Surveys 32(1977), no. 4, 55-114.
19. M. Qian and S. Zhu, *SRB measures and Pesins entropy formula for endomorphisms*, Trans. of the Amer. Math. Soc., 354(4):14531471, 2002.
20. D. Ruelle, *A measure associated with Axiom A attractors*, Amer. J. Math. 98: 619654 (1976).
21. David Ruelle, *An inequality for the entropy of differentiable maps*, Bol. Soc. Brasil. Mat. 9 (1978), no. 1, 8387.
22. Ya. G. Sinai, *Gibbs measure in ergodic theory*, Russian Math. Surveys 27:2169
23. M. Tsujii, *Regular point for ergodic Sinai measures* I, Transactions of the American Mathematical Society Volume 328, Number 2, December 1991.
24. L. S. Young, *Entropy of continuous flows on compact 2-manifolds*, Topology 16 (4) (1977), 469471.
25. Y. Yomdin, *C^k resolution of semialgebraic mappings*, Israel J. Math., 57, 3 (1987), 301-317.

Sorbonne Universite, LPSM, 75005 Paris, France
E-mail address: `david.burguet@upmc.fr`