I. Smooth Parametrizations in dynamics

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Luminy, May 20, 2019

Shub's entropy conjecture

M a compact manifold,
$f : M \circlearrowleft$ a $\mathcal{C}^0$ map,
h_{top} topological entropy of f,

$f_* : H_*(M) \circlearrowleft$,
ρ spectral radius of f_*.

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f $\mathcal{C}^1$,

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Theorem (Yomdin)

The conjecture holds true for any $\mathcal{C}^\infty$ map.

Volume growth

$(M, \|\cdot\|)$ compact $\mathcal{C}^\infty$ Riemannian manifold,
$\mathcal{C}^\infty$ disc $\sigma : (0,1)^k \to M$, i.e. a $\mathcal{C}^\infty$ map
with $\|d^r\sigma\| < +\infty$ for all $r \in \mathbb{N}$.

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Volume growth of σ :

$$\begin{aligned} v(\sigma) &= \limsup_n \frac{1}{n} \log \mathrm{vol}_k(f^n \circ \sigma), \\ &= \limsup_n \frac{1}{n} \log \int_{(0,1)^k} \|\Lambda^k d_t(f^n \circ \sigma)\| \, dt, \end{aligned}$$

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Fact

f $\mathcal{C}^1$,

$$\log \rho \leq v.$$

Volume growth of σ at scale $\epsilon > 0$:

$$v^*(\sigma,\epsilon) = \limsup_n \frac{1}{n} \log \sup_{x\in M} \mathrm{vol}_k(f^n \circ \sigma|_{\sigma^{-1}B_n(x,\epsilon)}),$$
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Local volume growth :

$$v^* = \overset{\searrow}{\lim_{\epsilon \to 0}} \sup_{\sigma} v^*(\sigma, \epsilon)$$

Fact

$$v \leq h_{top} + v^*.$$

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$\mathcal{M}$ compact set of f-invariant probas,
$h(\nu)$ metric entropy of $\nu \in \mathcal{M}$

Theorem (Newhouse)

f $\mathcal{C}^{1+}$,

- $h_{top} \leq v$,
- $\forall \mu \in \mathcal{M},\ \limsup_{\nu \to \mu} h(\nu) \leq h(\mu) + v^*$.

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Proof of Entropy Conjecture for $\mathcal{C}^\infty$ systems :

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Corollary

f $\mathcal{C}^\infty$,

- $h_{top} = v$,
- $\forall \mu \in \mathcal{M}$, $\limsup_{\nu \to \mu} h(\nu) \leq h(\mu)$. *In particular there exists an equilibrium measure w.r.t. any* $\mathcal{C}^0$ *potential.*

Finite smoothness

$f : M \circlearrowleft$ a $\mathcal{C}^r$ map with $+\infty > r \geq 1$, $d = \dim(M)$,
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$\mu \in \mathcal{M}$, $\chi_1^+(\mu) = \lim_n \frac{1}{n} \int \log^+ \|d_x f^n\| \, d\mu(x)$,
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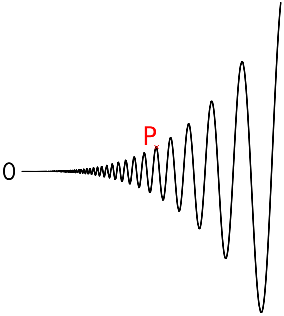
Theorem

- $v^* \leq \frac{kR(f)}{r}$,
- $\limsup_{\nu \to \mu} h(\nu) \leq h(\mu) + \frac{d\chi_1^+(\mu)}{r}$.

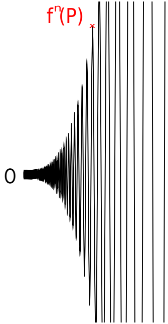
These upperbounds are essentially sharp. Moreover there are $\mathcal{C}^r$ examples without maximal measures (Misiurewicz, Buzzi).

$\underline{\mathcal{C}^r \text{ example with } v^* \neq 0 :}$ $f : \mathbb{R}^2 \circlearrowleft$ given by $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ with $\lambda > 1$

$$\begin{array}{rcl} \sigma : (0,1) & \to & \mathbb{R}^2, \\ t & \mapsto & (t, t^{2r+1}\sin(1/t)) \end{array}, \qquad x = 0.$$



$\sigma \cap [0,1]^2$



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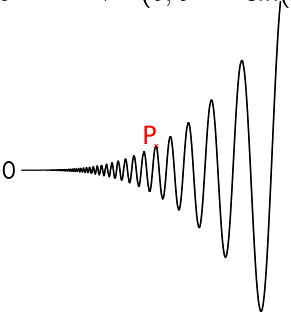

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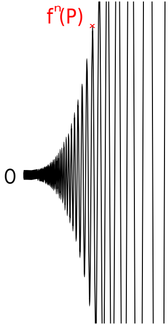

$f^n(\sigma \cap B_n(0,1))$

$y_{f^n(P)} = x_P^{2r+1} \times \lambda^n \simeq 1$ and
$\simeq 1/x_P$ disc. branches in $f^n(\sigma \cap B_n(0,1))$

$$v^*(\sigma, 1) \geq \lim_n \frac{\log(1/x_P)}{n} = \frac{\log \lambda}{2r+1}.$$

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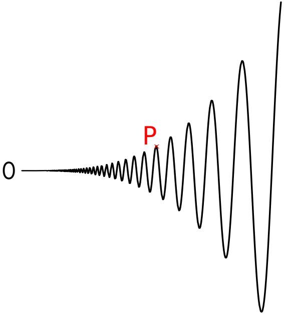

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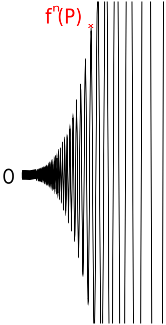

$f^n(\sigma \cap B_n(0,1))$

$$v^*(\sigma) \geq \frac{\log \lambda}{2r+1}.$$

Algebraic RL

B the unit euclidean ball in $\mathbb{R}^d$,
$P : (0,1)^k \to \mathbb{R}^d$ with $P = (P_1, \cdots, P_d) \in \mathbb{R}^d[X_1, \cdots, X_k]$,
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$r \in \mathbb{N}$, for a $\mathcal{C}^r$ disc $\varphi : (0,1)^k \to \mathbb{R}^d$,
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Lemma (Gromov)

There exists a family $\Theta = \{\theta\}$ of rep. of $(0,1)^k$ s.t.

1. $\bigcup_{\theta \in \Theta} \mathrm{Im}(\theta) = P^{-1}(B)$,
2. $\forall \theta \in \Theta$, $\|\theta\|_r \leq 1$ *and* $\|P \circ \theta\|_r \leq 1$,
3. $\sharp\Theta \leq \mathfrak{C}$ *with* $\mathfrak{C} = \mathfrak{C}(k, d, r, s)$.

Polynomial Estimate of $\mathfrak{C}$

Lemma (B.-Liao-Yang, Binyamini-Novikov)

There exists $R_k \in \mathbb{R}[X, Y]$, s.t.

$$\mathfrak{C}(k, d, r, s) = R_{k,d}(r, s).$$

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Theorem (Yomdin, B.-Liao-Yang)

There is an explicit (essentially sharp) rate of convergence of $\lim_{\epsilon \to 0} \sup_\sigma v^*(\sigma, \epsilon) = 0$ *for $\mathcal{C}^\infty$ maps f and σ in a given ultradifferentiable class, e.g. in the analytic case*

$$\forall \epsilon > 0, \quad \sup_\sigma v^*(\sigma, \epsilon) \leq O\left(\|df\|\right) \frac{\log(|\log \epsilon|)}{|\log \epsilon|}.$$

$\mathfrak{s} : (0,1)^k \to \mathbb{R}^d$ a $\mathcal{C}^r$ disc with $r \in \mathbb{N}^*$.

Lemma

There exists a family $\Theta = \{\theta\}$ of rep. of $(0,1)^k$ s.t.

- $\bigcup_{\theta\in\Theta} \mathrm{Im}(\theta) \supset \mathfrak{s}^{-1}(B)$,
- $\forall\theta \in \Theta,\ \|\theta\|_r \leq 1$ *and* $\|\mathfrak{s}\circ\theta\|_r \leq 1$,
- $\sharp\Theta \leq \mathfrak{D} \max\left(\|d^r\mathfrak{s}\|, 1\right)^{\frac{k}{r}}$ *with* $\mathfrak{D} = \mathfrak{D}(k,d,r)$.

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Proof : We may assume $\|d^r\mathfrak{s}\| \leq 1$: consider k-subcubes C of $(0,1)^k$ of size $|C| = \max\left(\|d^r\mathfrak{s}\|, 1\right)^{-1/r}$ covering $(0,1)^k$ and $\psi_C : (0,1)^k \to$ C affine parametrization of C, then $\|d^r(\mathfrak{s} \circ \psi_C)\| = |C|^r \|d^r\mathfrak{s}\| \leq 1$...

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If $\|d^r\mathfrak{s}\| \leq 1$, let P be the $(r-1)$-Lagrange polynomial of $\mathfrak{s}$ at $x_0 \in (0,1)^k$ and $\Theta = \{\theta\}$ as in the Algebraic RL for $\frac{P}{2}$, then

- $\mathfrak{s}^{-1}(B) \subset P^{-1}(2B) = \bigcup_{\theta\in\Theta} \mathrm{Im}(\theta)$,
- $\|\mathfrak{s}\circ\theta\|_r \leq \|P\circ\theta\|_r + \|(\mathfrak{s}-P)\circ\theta\|_r \leq \mathfrak{E} = \mathfrak{E}(k,d,r)$.

DRL for non autonomous $\mathcal{C}^r$ dynamical systems

$\mathcal{F} = (\mathrm{f}_m)_{m\in\mathbb{N}^*}$ family of $\mathcal{C}^r$ maps from B to $\mathbb{R}^d$ with $\mathrm{f}_0 = \mathrm{Id}_B$,
$\mathrm{f}^m = \mathrm{f}_m \circ \cdots \circ \mathrm{f}_0$ from B_m to $\mathbb{R}^d$ with
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Proof : By induction. For each $\theta_m \in \Theta_m$, let $\Theta(\theta_m)$ be the family of rep. obtained when applying the above Lemma to $\mathrm{f}^m \circ \mathfrak{s} \circ \theta_m$.
Take $\Theta_{m+1} = \{\theta_m \circ \theta \mid \theta_m \in \Theta_m, \ \theta \in \Theta(\theta_m)\}$.

$\Phi = (\phi_m)_{m\in\mathbb{N}}$ family of α-Hölder maps from B to $\mathbb{R}$
with $\sup_m |\phi_m|_\alpha \leq 1$ for some $0 < \alpha \leq 1$,
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- $\forall \theta_m \in \Theta_m,\ |S_m\Phi \circ \mathfrak{s} \circ \theta_m|_\alpha \leq 1$,
- $\sharp\Theta_m \leq m^{1/\alpha}\mathfrak{D}^m \max\left(\|d^r\mathfrak{s}\|, 1\right)^{\frac{k}{r}} \prod_{l=0}^{m-1} \max\left(\|\mathsf{f}_l\|_r, 1\right)^{\frac{k}{r}}$.

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Proof : Let θ'_m be the rep. of the previous Lemma, then
$|\phi_l \circ \mathsf{f}^l \circ \mathfrak{s} \circ \theta'_m|_\alpha \le |\phi_l|_\alpha \|\mathsf{f}^l \circ \mathfrak{s} \circ \theta'_m\|_1^\alpha \le 1$, thus
$|S_m\Phi \circ \mathfrak{s} \circ \theta'_m|_\alpha \le m$. Take finally $\theta_m = \theta'_m \circ \psi_C$ with $|C| = m^{-1/\alpha}$.

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Proof of Yomdin's theorem :

$$\sup_\sigma v^*(\sigma, \epsilon) \leq \gamma \text{ and then } v^* = 0.$$

$\phi : M \to \mathbb{R}$ a α-Hölder potential with $0 < \alpha \leq 1$,
$S_n\phi = \sum_{l=0}^{n-1} \phi \circ f^l$,

$\forall \gamma > 0$, $\exists \epsilon = \epsilon(f, \phi, \gamma)$ and $C = C(f, \phi, \sigma, \gamma) > 0$ s.t.

Lemma

For any $n \in \mathbb{N}^*$ there exists $\Theta_n = \{\theta_n\}$ family of rep. of $(0,1)^k$ satisfying :

- $\bigcup_{\theta_n \in \Theta_n} \mathrm{Im}(\theta_n) \supset \sigma^{-1} B_n(x, \epsilon)$,
- $\forall \theta_n \in \Theta_n\ \forall 0 \leq l < n,\ \|d(f^l \circ \sigma \circ \theta_n)\| \leq 1$,
- $\forall \theta_n \in \Theta_n,\ \forall t, s \in \mathrm{Im}(\theta_n),\ |S_n\phi \circ \sigma(t) - S_n\phi \circ \sigma(s)| \leq 1$,
- $\sharp\Theta_n \leq Ce^{\gamma n}$.

Remark : Third item may be seen as a weak Bowen property for ϕ :

$$\exists \epsilon > 0\, \exists C > 0 \text{ s.t.}$$
$$\forall n \in \mathbb{N}\, \forall y \in B_n(x, \epsilon),\ |S_n\phi(x) - S_n\phi(y)| < C.$$

Local dynamics of a $\mathcal{C}^\infty$ system (f, M) with $\phi : M \to \mathbb{R}$ a α-Hölder potential

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$\Phi = (\phi_m)_m$ with $\phi_m = \sum_{k=0}^{p-1} \phi \circ f^k \circ \psi^\epsilon_{f^{pm}\overline{x}}$,

$$\forall s \geq 1,\ \sup_m \|d^s \mathrm{f}_m\| = O(\epsilon^{s-1})$$
$$\text{and } \sup_m |\phi_m|_\alpha = O(\epsilon^\alpha) \text{ uniformly in } x.$$

Proof of DRL for $f : M \circlearrowleft \mathcal{C}^\infty$ and $\phi : M \to \mathbb{R}$ α-Hölder :

Choose r, p, ϵ w.r.t. small error term $\gamma > 0$ s.t.

- $r \in \mathbb{N}$ with $\|df\|^{k/r} < e^{\gamma/2}$,
- $p \in \mathbb{N}$ with $\mathfrak{D}^{1/p} < e^{\gamma/2}$,
- $\epsilon > 0$ with $2\epsilon \max(\|df^p\|, 1) < 1$, $\|\mathrm{f}_m\|_r \leq \|df^p\|$ and $|\phi_m|_\alpha \leq 1$ for all $m \in \mathbb{N}$ with $\mathcal{F} = (\mathrm{f}_m)_m$ and $(\phi_m)_m$ as above.

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For $n = pm$, we have

$$B_n^f(\overline{x}, \epsilon) \subset \psi_{\overline{x}}^{\epsilon}(B_m(\mathcal{F})) \text{ and}$$

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Let $(\Theta_m(\mathcal{F}))_m$ be the families of rep. given by DRL for n.a. $\mathcal{C}^r$ systems applied to $\mathcal{F}$. The family $\Theta'_n = \{\theta'_n = \theta_m,\ \theta_m \in \Theta_m(\mathcal{F})\}$ satisfies DRL for f :

$$\begin{aligned}
\sharp\Theta'_n &\leq m^{1/\alpha}\mathfrak{D}^m \prod_{l=0}^{m-1} \max\left(\|\mathrm{f}_l\|_r, 1\right)^{\frac{k}{r}}, \\
&\leq m^{1/\alpha}\mathfrak{D}^m \max\left(\|df\|, 1\right)^{n\frac{k}{r}}, \\
&\leq Ce^{\gamma n}.
\end{aligned}$$

Asymptotic h-expansiveness of $\mathcal{C}^\infty$ systems

(X, T) top. system, i.e. (X, d) compact metric space and $T : X \circlearrowleft$ continuous,

$n \in \mathbb{N}$, $\delta > 0$, $K \subset X$,

$$r_n(\delta, K) = \min \left\{ \sharp E_\delta,\ \bigcup_{x \in E_\delta} B_n(x, \delta) \supset K \right\}.$$

Tail entropy of T :

$$h^* = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \limsup_n \frac{1}{n} \log \sup_{x \in X} r_n\left(\delta, B_n(x, \epsilon)\right).$$

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Theorem (Misiurewicz)

$$\forall \mu \in \mathcal{M}, \ \limsup_{\nu \to \mu} h(\nu) \leq h(\mu) + h^*.$$

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Proof : With the notations of DRL, take $\sigma = \psi_x^\epsilon$. If F_δ is δ dense in $(0,1)^k$ then $E_\delta = \bigcup_{\theta_n \in \Theta_n} \theta_n(F_\delta)$ is δ-dense for the distance d_n in $B_n(x,\epsilon)$ with $\forall x, y \in M,\ d_n(x,y) = \max_{0 \leq k < n} d(f^k x, f^k y)$.

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$$\forall \delta > 0, \quad r_n(\delta, B_n(x, \epsilon)) \le \sharp\Theta_n \times \sharp F_\delta, \\ \le C e^{\gamma n} \sharp F_\delta$$

and

$$h^* \le \gamma.$$

DRL for $\mathcal{C}^\infty$ Cocycles

$f : M \circlearrowleft$ a $\mathcal{C}^\infty$ map,
$\pi : V \to M$ a $\mathcal{C}^\infty$ Riemannian vector bundle over M,
$F : V \circlearrowleft$ a $\mathcal{C}^\infty$ *semi-invertible* bundle morphism with $\pi \circ F = f \circ \pi$,

$\mathbb{S}F : \mathbb{S}(V) \circlearrowleft$ associated sphere bundle morphism,
$\phi : \mathbb{S}(V) \to \mathbb{R}$ a α-Hölder potential with $0 < \alpha \leq 1$,
$\Gamma : (0,1)^k \to \mathbb{S}(V)$ a $\mathcal{C}^\infty$ disc, $\sigma = \pi \circ \Gamma$,
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Lemma

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- $\forall \theta_n \in \Theta_n\ \forall t, s \in \mathrm{Im}(\theta_n),\ |S_n\phi(\Gamma(t)) - S_n\phi(\Gamma(s))| \leq 1$,
- $\sharp \Theta_n \leq C e^{\gamma n}$.

Lemma

For $\gamma > 0$ *and* $x \in M$ *there exist* $\tilde{\epsilon} = \tilde{\epsilon}(\gamma)$ *and* $C = C(\gamma)$ *constant s.t. we have for all* $n \in \mathbb{N}$ *and for all* $0 < \epsilon < \tilde{\epsilon}$ *:*

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- Buzzi : Take $\epsilon' = \epsilon'(\gamma) > 0$ s.t. $\forall \delta > 0, \; \limsup_n \frac{1}{n} \log \sup_{y \in \mathbb{S}(V)} r_n^{\mathbb{S}(F)}\left(\delta, B_n^{\mathbb{S}(F)}(y, \epsilon')\right) < \gamma/2.$