II. Entropy of physical measures for $\mathcal{C}^\infty$ systems

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Lyapunov exponents

X a Borel space and $f : X \circlearrowleft$ measurable,
$\pi : V \to X$ a measurable fiber bundle over X equipped with a measurable Riemannian metric $\|\cdot\|_x$ on each fiber $V_x := \pi^{-1}x$,
$F : V \circlearrowleft$ a measurable bundle morphism with $\pi \circ F = f \circ \pi$.

Definition (Pointwise Lyapunov exponent)

For $x \in X$ and $v \in V_x$ we let

$$\chi(x, v) := \limsup_{n \to +\infty} \frac{1}{n} \log \|F^n(v)\|_{f^n x}.$$

For a fixed x, the set $\{v,\ \chi(x, v) \leq \lambda\}$ is a vector subspace nondecreasing in $\lambda \in \mathbb{R} \cup \{\pm\infty\}$.

Definition (Lyapunov flag)

There exist

- $r = r(x) \in \mathbb{N} \setminus \{0\}$,
- *vector spaces* $V_x = V_1(x) \supsetneq \cdots \supsetneq V_r(x) \neq 0$,
- $+\infty \geq \chi_1(x) > \cdots \chi_r(x) \geq -\infty$

$$\text{s.t. } \forall v \in V_i(x) \setminus V_{i+1}(x),\ \chi(x,v) = \chi_i(x).$$

The maps r, χ_i, V_i *are measurable. Moreover* r *and* χ_i *are* f*-invariant.*

Let denote by $(\chi_j(x))_{j=1,\cdots,\dim(V)}$ the Lyapunov exponents at x counted nonincreasingly with multiplicity, i.e.

$$\sharp\{j,\ \chi_j = \chi_i\} = \dim(V_i) - \dim(V_{i+1}).$$

χ^k the top Lyapunov exponent of $\Lambda^k F : \Lambda^k V \circlearrowleft$,

$\chi_\Lambda = \max_k \chi^k$ the top Lyapunov exponent of $\Lambda F : \Lambda V = \oplus_k \Lambda^k V \circlearrowleft$

Definition

x **Lyapunov regular** *when* $\chi^k(x) = \sum_{j=1}^k \chi_j(x)$ *for all* k.

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Theorem (Oseledets)

Lyapunov regular points have full μ*-measure for any* $\mu \in \mathcal{M}$*.*

Empirical measures

(X, d) compact metric, $f : X \circlearrowleft$ continuous,
$(\mathcal{M}, \mathfrak{d})$ compact set of f-invariant probas.

Definition

For $x \in X$, we let $p\omega(x)$ be the compact subset of $\mathcal{M}$ consisting of accumulation points of the sequence of empirical measures

$$(\mu_n^x)_n := \left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x} \right)_{n \in \mathbb{N}} .$$

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$(\mathcal{KM}, \mathfrak{d}^{Hau})$ compact set of nonempty closed subsets of $\mathcal{M}$.

Remark : $p\omega : x \mapsto p\omega(x)$ from X to $\mathcal{KM}$ is Borel measurable.

Statements

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Main Theorem

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$$\textit{for } \mathrm{Leb} \ \textit{a.e. } x,\ \max_{\mu \in p\omega(x)} h(\mu) \geq \chi_\Lambda^+(x).$$

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Theorem

There are $\mathcal{C}^r$ counterexamples for any finite $r \geq 1$.

$B_\mu := \{x,\ p\omega(x) = \{\mu\}\}$ basin of μ,
μ is physical when $\mathrm{Leb}(B_\mu) > 0$.

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Corollary

Let μ be a physical measure of a $\mathcal{C}^\infty$ system. Then

$$h(\mu) \geq \overline{\chi_\Lambda|_{B_\mu}},$$

with $\overline{\chi_\Lambda|_{B_\mu}}$ the essential supremum of χ_Λ on B_μ.

For the $\mathcal{C}^\infty$ Bowen eight's attractor, we get $\chi_1(x) = 0$ for Leb-a.e. x in the eyes.

Essential domain and image

X, Y metric spaces, Y separable,
$\phi : X \to Y$ Borel measurable,
m Borel measure on X.

Definition (Essential image)

$$\overline{\mathrm{Im}}_\phi(m) := \{y \in Y,\ \forall U \in \mathcal{V}(y)\ m(\phi^{-1}U) > 0\}.$$

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The essential image is the smallest closed subset K of Y for which $\phi(x) \in K$ for m-a.e. x. For $Y = \mathbb{R}$ the essential supremum is $\overline{\phi} = \sup\left(\overline{\mathrm{Im}}_\phi(m)\right)$.

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Definition (Essential domain)

$$\overline{\mathrm{Dom}}_\phi(m) := \left\{x \in X,\ \phi(x) \in \overline{\mathrm{Im}}_\phi(m)\right\}.$$

The essential domain has full m-measure.

Key Proposition

Fix $a < \overline{\chi^k}$ and let $\mathrm{Leb}_a = \mathrm{Leb}|_{\{\chi^k > a\}}$.

Proposition

For $x \in \overline{\mathrm{Dom}}_{p\omega}(\mathrm{Leb}_a)$, there exists $\mu \in p\omega(x)$ s.t.

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<u>Proof of Main Theorem :</u> For $(a_l)_{l \in \mathbb{N}}$ dense in $\mathbb{R}^+$ we let

$$F_l = \{\chi^k \leq a_l\} \cup \overline{\mathrm{Dom}}_{p\omega}(\mathrm{Leb}_{a_l}).$$

Then $\mathrm{Leb}(\bigcap_l F_l) = 1$. Moreover for $x \in \bigcap_l F_l$ and for a_l with $\chi^k(x) > a_l$, there is $\mu_l \in p\omega(x)$ with $h(\mu_l) \geq a_l$.

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By u.s.c. of the entropy, any accumulation point $\mu \in p\omega(x)$ of $(\mu_{l_n})_n$ with $a_{l_n} \xrightarrow{\nearrow_n} \chi^k(x)$ satisfies

$$h(\mu) \geq \limsup_n h(\mu_{l_n}) \geq \lim_n a_{l_n} = \chi^k(x).$$

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2. Bounded distortion : for a partition P with small diameter, we have for all x :
   - $P_x^n \cap D_u$ is contained in a u-disc with exp. small size,
   - $y \mapsto \mathrm{Jac}(df_y|_{E_u})$ is Hölder,

$$\text{therefore} \quad m(P_x^n) \leq C / \mathrm{Jac}(df_x^n|_{E_u}),$$

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3. Entropy computation : for $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f^k m$ and $\mu = \lim_k \mu_{n_k}$ with $\mu(\partial P) = 0$, we have

$$h(\mu, P) \geq \limsup_k -\frac{1}{n_k}\int \log m(P_x^{n_k})\, dm(x).$$

We get

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\begin{array}{rcl}
-\dfrac{1}{n_k}\displaystyle\int \log m(P_x^{n_k})\, d\mathrm{Leb}_{D_u}(x) & \geq & \dfrac{1}{n_k}\displaystyle\int \log \mathrm{Jac}(df^{n_k}|_{E_u})\, dm, \\
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then by letting $k \to +\infty$

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\begin{array}{ccc}
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h(\mu) & &
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Main lines of the proof

$x \in \overline{\mathrm{Dom}}_{p\omega}(\mathrm{Leb}_a)$, i.e. $p\omega(x) \in \overline{\mathrm{Im}}_{p\omega}(\mathrm{Leb}\,|_{\{\chi^k > a\}})$

1. NU Expanding disc : D a $\mathcal{C}^\infty$ embedded k-disc and $E \subset D$ with $\mathrm{Leb}_D(E) > 0$ s.t. for any $y \in E$
   - $\chi^k(y, \iota(T_y D)) > a$ with ι the Plücker embedding,
   - $\mathfrak{d}^{Hau}(p\omega(y), p\omega(x)) \ll 1$.

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2. <u>Borel-Cantelli argument</u> : for $n \in J \subset \mathbb{N}$ with $\sharp J = \infty$, subset E_n of E with $\mathrm{Leb}_D(E_n)$ not exp. small s.t. for any $y \in E_n$
   - $\|\Lambda^k df_y^n(\iota(T_y D))\| \geq e^{na}$,
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4. Entropy computation : for $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f^k m_n$ and $\mu = \lim_k \mu_{n_k}$ with $\mu(\partial P) = 0$, we have

$$\mathfrak{d}\,(\mu, p\omega(x)) \ll 1 \text{ and}$$

$$h(\mu, P) \gtrsim a.$$

We conclude by u.s.c. of the metric entropy on $\mathcal{M}$.

Step 1. Choice of D and E

$x \in \overline{\mathrm{Dom}}_{p\omega}(\mathrm{Leb}_a)$, thus $\forall \eta > 0$,

$$E' = \{y,\ \chi^k(y) > a \text{ and } \mathfrak{d}^{Hau}(p\omega(y), p\omega(x)) < \eta\}$$

satisfies $\mathrm{Leb}(E') > 0$. By reducing E' the Lyapunov space V_2 w.r.t. $\Lambda^k df$ is continuous on E'.

Let z be a Lebesgue density point of E' and $U \in \mathcal{V}(z)$ s.t. $\forall y \in U \cap E',\ \iota(H_y) \notin V_2(y)$ for some *constant* k-distribution H.

Since $\mathrm{Leb}(U \cap E') > 0$ one may take by Fubini $D \subset H_y$ with $\mathrm{Leb}_D(E') > 0$ for some $y \in U \cap E'$. Put $E = E' \cap D$.

Step 2 : Borel-Cantelli argument

Take

$$E_n := \{y \in E, \ \|\Lambda^k df_y(\iota(T_y D))\| \geq e^{na} \text{ and } \mathfrak{d}(\mu_n^y, p\omega(x)) \leq \eta\}.$$

For $\gamma > 0$ small error term, let us show that

$$\exists J \subset \mathbb{N} \text{ with } |J| = +\infty \text{ s.t.}$$

$$\mathrm{Leb}_D(E_n) > e^{-n\gamma}.$$

Argue by contradiction :

$$[\forall n \in \mathbb{N}\ \mathrm{Leb}_D(E_n) < e^{-n\gamma}]$$
$$\Rightarrow [\mathrm{Leb}_D(\limsup_n E_n) = 0.]$$

But $\limsup_n E_n \supset E$ and $\mathrm{Leb}_D(E) > 0$...

Step 3 : Bounded Distortion Lemma for maps

$f : M \circlearrowleft$ a $\mathcal{C}^\infty$ map,
$\sigma : (0,1)^k \to M$ a $\mathcal{C}^\infty$ immersed disc,
i.e. with $\inf_t \|\Lambda^k d_t \sigma\| > 0$,
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$\forall \gamma > 0 \, \forall \delta \in \mathbb{R}$, $\exists \epsilon = \epsilon(f, \gamma, \delta)$ and $C = C(f, \sigma, \gamma, \delta) > 0$ s.t.

Lemma

For any $n \in \mathbb{N}^*$ there exists $\Theta_n = \{\theta_n\}$ family of rep. of $(0,1)^k$ satisfying :

- $\bigcup_{\theta_n \in \Theta_n} \mathrm{Im}(\theta_n) \supset \{t \in \sigma^{-1} B_n(x, \epsilon), \ \|\Lambda^k d_t(f^n \circ \sigma)\| \geq e^{n\delta}\}$,
- $\forall \theta_n \in \Theta_n \, \forall 0 \leq l < n, \ \|d(f^l \circ \sigma \circ \theta_n)\| \leq 1$,
- $\forall \theta_n \in \Theta_n, \ \forall t, s \in \mathrm{Im}(\theta_n), \ \frac{\|\Lambda^k d_t(f^n \circ \sigma)\|}{\|\Lambda^k d_s(f^n \circ \sigma)\|} \leq e$,
- $\sharp \Theta_n \leq C e^{\gamma n}$.

Proof of Bounded Distortion Lemma for diffeomorphisms :

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DRL for $\mathcal{C}^\infty$ Cocycles

$\pi : V \to M$ a $\mathcal{C}^\infty$ Riemannian vector bundle over M,
$F : V \circlearrowleft$ a $\mathcal{C}^\infty$ semi-invertible bundle morphism with $\pi \circ F = f \circ \pi$,

$\mathbb{S}F : \mathbb{S}(V) \circlearrowleft$ associated sphere bundle automorphism,
$\phi : \mathbb{S}(V) \to \mathbb{R}$ a α-Hölder potential with $0 < \alpha \leq 1$,
$\Gamma : (0,1)^k \to \mathbb{S}(V)$ a $\mathcal{C}^\infty$ a disc, $\sigma = \pi \circ \Gamma$,
$x \in M$.

$\forall \gamma > 0$, $\exists \epsilon = \epsilon(F, \phi, \gamma)$ and $C = C(F, \phi, \sigma, \gamma) > 0$ s.t.

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- $\bigcup_{\theta_n \in \Theta_n} \mathrm{Im}(\theta_n) \supset \sigma^{-1} B_n^f(x, \epsilon)$,
- $\forall \theta_n \in \Theta_n \ \forall 0 \leq l < n, \ \|d\left(\mathbb{S}F^l \circ \Gamma \circ \theta_n\right)\| \leq 1$,
- $\forall \theta_n \in \Theta_n \ \forall t, s \in \mathrm{Im}(\theta_n), \ |S_n\phi(\Gamma(t)) - S_n\phi(\Gamma(s))| \leq 1$,
- $\sharp \Theta_n \leq C e^{\gamma n}$.

We consider

- the bundle morphism $F = \Lambda^k df$ on $\Lambda^k TM$,
- the k-disc Γ in $\mathbb{S}(\Lambda^k TM)$ induced by the immersed disc σ, i.e.

$$\Gamma(t) = \left(\sigma(t), \frac{\Lambda^k d\sigma(t)}{\|\Lambda^k d\sigma(t)\|}\right),$$

- the $\mathcal{C}^\infty$ potential $\phi : (x, v) \mapsto \log \|\Lambda^k df(x, v)\|$.

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Then we get

$$S_n\phi(\Gamma(t)) = \log \left(\frac{\|\Lambda^k d_t(f^n \circ \sigma)\|}{\|\Lambda^k d_t\sigma\|} \right)$$

... and BDL follows.

$\sigma : (0,1)^k \to M$ a smooth reparametrization of D,
m_n proba induced on E_n by Leb_D for $n \in J$,

$\epsilon > 0$ and $\Theta_n = \{\theta_n\}$ as in BDL at $x \in E_n$,
P partition with $\mathrm{diam}(P) < \epsilon$,
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$$
\begin{aligned}
m_n(P^n_x) &\leq \frac{1}{\mathrm{Leb}(E_n)} \sum_{\theta_n \in \Theta'_n} \mathrm{vol}_k(\sigma \circ \theta_n), \\
&\leq \frac{1}{\mathrm{Leb}(E_n)} \sum_{\theta_n \in \Theta'_n} e^{-na} \mathrm{vol}_k(f^n \circ \sigma \circ \theta_n), \\
m_n(P^n_x) &\leq \frac{e^{-na} \sharp \Theta_n}{\mathrm{Leb}(E_n)}.
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Step 4 : Entropy computation

$\sigma : (0,1)^k \to M$ a smooth reparametrization of D,
m_n proba induced on E_n by Leb_D for $n \in J$,

$\epsilon > 0$ and $\Theta_n = \{\theta_n\}$ as in BDL at $x \in E_n$,
P partition with $\mathrm{diam}(P) < \epsilon$,
$\Theta'_n = \{\theta_n \in \Theta_n,\ \mathrm{Im}(\theta_n) \cap E_n \neq \emptyset\}$.

$$\begin{aligned} m_n(P_x^n) &\leq \frac{1}{\mathrm{Leb}(E_n)} \sum_{\theta_n \in \Theta'_n} \mathrm{vol}_k(\sigma \circ \theta_n), \\ &\leq \frac{1}{\mathrm{Leb}(E_n)} \sum_{\theta_n \in \Theta'_n} e^{-na} \mathrm{vol}_k(f^n \circ \sigma \circ \theta_n), \\ m_n(P_x^n) &\leq \frac{e^{-na} \sharp \Theta_n}{\mathrm{Leb}(E_n)}. \end{aligned}$$

Therefore for $\mu = \lim_k \mu_{n_k}$ with $\mu(\partial P) = 0$:

$$h(\mu, P) \geq \liminf_{n \in J} -\frac{1}{n} \int \log m_n(P_x^n)\, dm_n(x) \gtrsim a.$$