III. Equidistribution of periodic points for $\mathcal{C}^\infty$ surface diffeomorphisms

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(X, f) a topological system,

$$h_{top} \text{ topological entropy,}$$

$$\forall n \in \mathbb{N},\ \mathrm{Per}_n := \{x \in X,\ f^n x = x\}.$$

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$$\forall n \in \mathbb{N},\ \mathrm{Per}_n := \{x \in X,\ f^n x = x\}.$$

Fact

(X, f) expansive, then

$$h_{top} \geq \limsup_n \frac{1}{n} \log \sharp \mathrm{Per}_n.$$

Theorem (Bowen)

(X, f) expansive with specification,
$\phi : X \to \mathbb{R}$ a continuous potential with the Bowen property,

$$\text{then} \qquad P_{top}(\phi) = \lim_n \frac{1}{n} \log \sum_{x \in \mathrm{Per}_n} e^{S_n\phi(x)}.$$

Moreover there is a unique equilibrium measure μ_ϕ satisfying

$$\mu_\phi = \lim_n \frac{\sum_{x \in \mathrm{Per}_n} e^{S_n\phi(x)}\delta_x}{\sum_{x \in \mathrm{Per}_n} e^{S_n\phi(x)}}.$$

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A topologically mixing uniformly hyperbolic system with a Hölder potential satisfies the above assumptions. In particular periodic points are equidistributed along the unique maximal measure.

Periodic growth for surface diffeos

M a compact surface, $f : M \circlearrowleft$ diffeo $\mathcal{C}^1$,
any $\mu \in \mathcal{M}$ ergodic with $h(\mu) > 0$ is hyperbolic,

Theorem (Katok)

$f : M \circlearrowleft \mathcal{C}^{1+}$,

$$h_{top} \leq \limsup_n \frac{1}{n} \sharp \log \mathrm{Per}_n .$$

Does the equality hold ?
Is the periodic growth $\limsup_n \frac{1}{n} \sharp \log \mathrm{Per}_n$ even finite ?

Densely...

Theorem (Artin-Mazur, Kaloshin)

For any $1 \leq r \leq +\infty$, there is a $\mathcal{C}^r$ dense set of diffeos s.t.

$$\limsup_n \frac{1}{n} \sharp \log \mathrm{Per}_n < +\infty$$

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$$\limsup_n \frac{1}{n} \sharp \log \mathrm{Per}_n < +\infty$$

but not generically.

Theorem (Gonchenko-Turaev-Shilnikov, Kaloshin)

For any $2 \leq r \leq +\infty$, there is a $\mathcal{C}^r$ open set $\mathcal{N}$ of diffeos s.t. generically in $\mathcal{N}$

$$\limsup_n \frac{1}{n} \sharp \log \mathrm{Per}_n = +\infty$$

$x \in \mathrm{Per}_n$, $|\chi|(x) = \max_{i=1,2}(|\chi_1(x)|, |\chi_2(x)|)$
with $\chi_i(x)$ $i = 1, 2$ the Lyapunov exponents at x,

$$\forall \delta > 0,\ \mathrm{Per}_n^\delta := \{x \in M,\ f^n x = x \text{ saddle and } |\chi|(x) > \delta\},$$

$0 < \delta < h_{top}$.

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ϕ Hölder and *small*, i.e. $P_{top}(\phi) > \|\phi\|_\infty$,
$S_n\phi = \sum_{l=0}^{n-1} \phi \circ f^l$,
$0 < \delta < P_{top}(\phi) - \|\phi\|_\infty$.

Theorem (Gelfert-Wolf)

f $\mathcal{C}^{1+}$,

$$P_{top}(\phi) \leq \limsup_n \frac{1}{n} \log \sum_{x \in \mathrm{Per}_n^\delta} e^{S_n\phi(x)}.$$

Theorem

f $\mathcal{C}^\infty$,

$$P_{top}(\phi) = \limsup_n \frac{1}{n} \log \sum_{x \in \mathrm{Per}_n^\delta} e^{S_n\phi(x)}.$$

$$\textit{Moreover any } \mu = \lim_k \frac{\sum_{x \in \mathrm{Per}_{n_k}^\delta} e^{S_{n_k}\phi(x)} \delta_x}{\sum_{x \in \mathrm{Per}_{n_k}^\delta} e^{S_{n_k}\phi(x)}}$$

$$\textit{with } \lim_k \frac{1}{n_k} \log \sum_{x \in \mathrm{Per}_{n_k}^\delta} e^{S_{n_k}\phi(x)} = P_{top}(\phi),$$

is an equilibrium measure w.r.t. ϕ.

Theorem (Sarig)

f $\mathcal{C}^{1+}$, $\exists p \in \mathbb{N}^* \, \exists C > 0$,

$$\liminf_{p|n} e^{-nP_{top}(\phi)} \sum_{x \in \mathrm{Per}_n^\delta} e^{S_n\phi(x)} > C.$$

Theorem (Buzzi, Crovisier, Sarig)

f $\mathcal{C}^\infty$,

- *topologically transitive, then*
  there exists a unique equilibrium measure μ_ϕ *w.r.t.* ϕ,
- *topologically mixing, then*
  $\exists C > 0$, $\liminf_n e^{-nP_{top}(\phi)} \sum_{x \in \mathrm{Per}_n^\delta} e^{S_n\phi(x)} > C.$

Theorem

f $\mathcal{C}^\infty$ topologically mixing,

$$\textit{then} \qquad \lim_n \frac{1}{n} \log \sum_{x \in \mathrm{Per}_n^\delta} e^{S_n\phi(x)} = P_{top}(\phi) \textit{ and}$$

$$\mu_\phi = \lim_n \frac{\sum_{x\in\mathrm{Per}_n^\delta} e^{S_n\phi(x)}\delta_x}{\sum_{x\in\mathrm{Per}_n^\delta} e^{S_n\phi(x)}}.$$

Tail periodic growth

(X, T) topological system,
$\mathcal{P} = (\mathcal{P}_n)_{n\in\mathbb{N}}$ with $\mathcal{P}_n \subset \mathrm{Per}_n$,

Definition

- *$\mathcal{P}$-growth*

$$g_{\mathcal{P}} := \limsup_n \frac{1}{n} \log \sharp\mathcal{P}_n,$$

- *tail $\mathcal{P}$-growth*

$$g^*_{\mathcal{P}} := \lim_{\epsilon\to 0} g^*_{\mathcal{P}}(\epsilon)$$

with

$$g^*_{\mathcal{P}}(\epsilon) := \limsup_n \frac{1}{n} \sup_{x\in X} \log \sharp B_n(x,\epsilon) \cap \mathcal{P}_n.$$

Lemma

Any $\mu = \lim_k \frac{1}{\sharp\mathcal{P}_{n_k}} \sum_{x\in\mathcal{P}_{n_k}} \delta_x$ *with* $\lim_k \frac{1}{n_k} \log \sharp\mathcal{P}_{n_k} = g_{\mathcal{P}}$ *satisfies*

$$h(\mu) \geq g_{\mathcal{P}} - g^*_{\mathcal{P}}.$$

In particular $h_{top} \geq g_{\mathcal{P}} - g^*_{\mathcal{P}}$.

Lemma

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$$h(\mu) \geq g_{\mathcal{P}} - g^*_{\mathcal{P}}.$$

In particular $h_{top} \geq g_{\mathcal{P}} - g^*_{\mathcal{P}}$.

Proof : Denote $\mu_n := \frac{1}{\sharp \mathcal{P}_n} \sum_{x \in \mathcal{P}_n} \delta_x$ for all n.
Let P partition with diameter $< \epsilon$, $Q_n \succ P^n$ s.t.
$\forall n \forall A \in Q_n,\ \sharp A \cap \mathcal{P}_n \leq 1$.

$$\begin{aligned} \forall l \leq n, \qquad \frac{1}{l} H_{\mu_n}(P^l) &\geq \frac{1}{n} H_{\mu_n}(P^n), \\ &\geq \frac{1}{n} \left(H_{\mu_n}(Q_n) - H_{\mu_n}(Q_n | P^n) \right), \\ &\geq \frac{1}{n} \log \sharp \mathcal{P}_n - \sup_{x \in X} \log \sharp \mathcal{P}_n \cap B_n(x, \epsilon), \end{aligned}$$

$$h(\mu, P) \leftarrow_l \frac{1}{l} H_\mu(P^l) \geq g_{\mathcal{P}} - g^*_{\mathcal{P}}(\epsilon).$$

$f : M \circlearrowleft$ a $\mathcal{C}^1$ surface diffeo,
$g^*_\delta = g^*_{\mathrm{Per}^\delta}$ with $\mathrm{Per}^\delta = (\mathrm{Per}^\delta_n)_n$.

Theorem (Kaloshin)

For f $\mathcal{C}^2$,

$$\forall \delta > 0,\ g^*_\delta < +\infty.$$

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Main Theorem

For f $\mathcal{C}^\infty$,

$$\forall \delta > 0, \ g^*_\delta = 0.$$

Explicit (sharp ?) upperbound of g^*_δ for f $\mathcal{C}^r$ with $r < +\infty$ in terms of r, δ and $\|df\|$.

Bounded distorsion lemma for interval maps

f a $\mathcal{C}^\infty$ interval map,
$\delta > 0$ lower bound of the Lyapunov exponent,
$x \in M$ center of the dynamical ball.

Lemma

For $\gamma > 0$ and $x \in M$ there exist

- $\epsilon = \epsilon(\gamma)$ *radius of the dynamical ball,*
- $C = C(\gamma)$ *constant,*
- $\Theta_n = \Theta_n(\gamma, x) = \{\theta_n\}$ *families of rep. of* $(0,1)$,

s.t. we have for all $n \in \mathbb{N}$

- $\bigcup_{\theta_n \in \Theta_n} \mathrm{Im}(\theta_n) \supset \{t \in B_n(x,\epsilon),\ |(f^n)'(t)| \geq e^{n\delta}\}$,
- $\forall \theta_n \in \Theta_n,\ \forall t, s \in \mathrm{Im}(\theta_n),\ |\log|(f^n)'(t)| - \log|(f^n)'(s)|| \leq 1$,
- $\sharp\Theta_n \leq Ce^{\gamma n}$.

On $\mathrm{Im}(\theta_n)$ the map f^n is expanding, so that there exists at most one point in $\mathrm{Im}(\theta_n) \cap \mathrm{Per}_n$ for each $\theta_n \in \mathcal{F}_n$, therefore

$$g^*_\delta(\epsilon) \leq \gamma, \text{ then } g^*_\delta = 0.$$

Theorem

Main Theorem and its corollaries also hold for $\mathcal{C}^\infty$ interval maps.

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Strategy of the proof for surface diffeos :

- define elementary f^n-hyperbolic pieces as an analogous of f^n-expanding intervals,
- cover a given dynamical ball of length n by a exponentially small number of such pieces.

Reduction

$M = \mathbb{R}^2/\mathbb{Z}^2$,
$e_u(x), e_s(x) \in \mathbb{S}(\mathbb{R}^2)$ generating the u/s space at $x \in \mathrm{Per}_n^\delta$,

Fact

There exists $\alpha = \alpha(\delta) \in]0, \pi/2[$ *s.t.*

$$\forall x \in \mathrm{Per}_n^\delta, \ \max_{0 \le k < n} \angle e_u(x), e_s(x) > \alpha.$$

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$\mathfrak{C} = (\mathfrak{C}_s, \mathfrak{C}_u)$ is called a α-bicone if $\mathfrak{C}_s$ and $\mathfrak{C}_u$ cones with

$$\angle \mathfrak{C}_u, \angle \mathfrak{C}_s < \alpha/2 \text{ and } \angle(\mathfrak{C}_s, \mathfrak{C}_u) > \alpha/2,$$

Let $\mathfrak{F} = \{\mathfrak{C}\}$ be a family of α-bicones with $\sharp\mathfrak{F} \le C/\alpha^2$ s.t.

$$\forall v, w \in \mathbb{S}(\mathbb{R}^2) \text{ with } \angle v, w > \alpha,$$

$$\exists \mathfrak{C} \in \mathfrak{F} \text{ with } \ v \in \mathfrak{C}_s \text{ and } w \in \mathfrak{C}_u.$$

For any α-bicone $\mathfrak{C}$ we let

$$\mathrm{Per}_n^\delta(\mathfrak{C}) = \{x \in \mathrm{Per}_n^\delta \text{ with } e_{u/s}(x) \in \mathfrak{C}_{u/s}\}.$$

Then we have

$$\forall \epsilon > 0,\ \sup_{x \in M} \sharp \mathrm{Per}_n^\delta \cap B_n(x,\epsilon) \leq \tfrac{nC}{\alpha(\delta)^2} \sup_{\mathfrak{C}, x \in \mathrm{Per}_n^\delta(\mathfrak{C})} \sharp \mathrm{Per}_n^\delta(\mathfrak{C}) \cap B_n(x, 2\epsilon).$$

Therefore it is enough to estimate $\sharp \mathrm{Per}_n^\delta(\mathfrak{C}) \cap B_n(x, 2\epsilon)$ for a fixed α-bicone $\mathfrak{C}$ and for $x \in \mathrm{Per}_n^\delta(\mathfrak{C})$.

1. Hyperbolic structure : n-hyperbolic sets.

$\mathcal{F}_n = (\mathrm{f}_0, \cdots, \mathrm{f}_{n-1})$ with $\mathrm{f}_l : B \to \mathbb{R}^2$ $\mathcal{C}^\infty$ (applied to local dynamics at $x \in \mathrm{Per}_n^\delta(\mathfrak{C})$),

$\mathfrak{C}_{s/u}$ constant cone fields on $B \subset \mathbb{R}^2$ centered at $(1,0)$ and $(0,1)$,

$U_n \subset B_n$ is said (δ, n)-*hyperbolic* when there are two $\mathcal{C}^\infty$ unit vector fields e_n^s and e_n^u on U_n contracted and expanded by df^n s.t.

- $e_n^{s/u}, d\mathrm{f}^n(e_{s/u}) \in \mathfrak{C}_{e_n^{s/u}}$,
- $\left|\log \|d\mathrm{f}^n\left(e_n^{s/u}\right)\|\right| > \frac{n\delta}{2}$.

We can assume $e_n^u = (0,1)$.

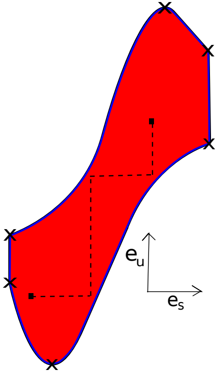
2. Accessibility condition : hexagons.

$H_n \subset U_n$ is a *n-hyperbolic hexagon* when any two points in U_n may be joined by $^s/_u$-paths with the same orientations, i.e.

$$\forall x, y \in H_n,\ \exists \alpha = \alpha_{x,y} : [0,1] \to H_n \text{ continuous s.t.}$$

$$\exists \epsilon_{s/u} \in \{\pm 1\} \text{ and } P \text{ finite partition into intervals of } [0,1] \textit{with}$$

$$\forall I \in P,\ \alpha \in \mathcal{C}^1(I) \text{ and } \forall t \in I, \alpha'(t) \in \epsilon_{s/u} \mathbb{R}^+ e_n^{s/u}.$$



Lemma

Let H_n be a n-hyperbolic hexagon, then for large n,

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Proof : For $x, y \in H_n$, let

$$v_{s/u} = \sum_{\substack{I \in P, \\ \alpha'_{x,y}|_I \in \mathbb{R} e_{s/u}}} \int_I \alpha'_{x,y}(t)\, dt \in \mathfrak{C}_{s/u}$$

and

$$w_{s/u} = \sum_{\substack{I \in P, \\ \alpha'_{x,y}|_I \in \mathbb{R} e_{s/u}}} \int_I d_{\alpha_{x,y}(t)} \mathrm{f}^n(\alpha'_{x,y}(t))\, dt \in \mathfrak{C}_{s/u}$$

But $\|w_s\| \ll \|v_s\|$, $\|v_u\| \ll \|w_u\|$ for large n and $\angle(\mathfrak{C}_s, \mathfrak{C}_u) > \alpha/2$ so that

$$\begin{array}{ccc} w_u - w_s & \neq & v_u - v_s, \\ \| & & \| \\ \mathrm{f}^n x - \mathrm{f}^n y & \neq & x - y \end{array}$$

Algebraic properties of reparametrizations

A map $\phi = (\phi^1, \cdots, \phi^k) : (0,1)^k \circlearrowleft$ is *semi-algebraic* :

$$\forall i \, \exists Q_i \in \mathbb{R}[X_1, \cdots, X_{k+1}] \setminus \{0\} \text{ s.t.}$$

$$\forall x \in (0,1)^k, \;\; Q_i(x, \phi^i(x)) = 0.$$

The degree of ϕ is $\deg(\phi) = \max_i \min_{Q_i} \deg(Q_i)$.

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Lemma (Algebraic RL)

B the unit euclidean ball in $\mathbb{R}^d$,
$P : (0,1)^k \to \mathbb{R}^d$ with $P = (P_1, \cdots, P_d) \in \mathbb{R}^d[X_1, \cdots, X_k]$,
$s = \max_i \deg P_i$, $r \in \mathbb{N}$,

There exists a family $\Theta = \{\theta\}$ of semi-algebraic rep. of $(0,1)^k$ s.t.

- $\bigcup_{\theta \in \Theta} \mathrm{Im}(\theta) = P^{-1}(B)$,
- $\forall \theta \in \Theta, \ \ \|\theta\|_r, \|P \circ \theta\|_r, \leq 1$,
- $\sharp\Theta, \max_{\theta \in \Theta} \deg(\theta) \leq \mathfrak{C} = \mathfrak{C}(k, d, r, s)$.

Lemma

$\theta_1, \cdots, \theta_m : (0,1)^k \to (0,1)^k$ *semi-algebraic maps, then*

$$\deg(\theta_m \circ \cdots \theta_1) \leq \prod_{i=1}^{m} \deg(\theta_i)^k.$$

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<u>Proof :</u> $k = 1$, $m = 2$, $(Q_i)_{i=1,2}$ vanishing polynomials of $(\theta_i)_{i=1,2}$. Eliminate the variable Y in

$$\begin{cases} Q_1(X, Y) = 0, \\ Q_2(Y, Z) = 0. \end{cases}$$

For $x \in (0,1)^k$ this system has $(x, \theta_1(x), \theta_2 \circ \theta_1(x))$ as a solution, therefore $\mathrm{Res}_Y(Q_1, Q_2)$ vanishes at $(x, \theta_2 \circ \theta_1(x)) = 0$. Finally $\deg(\mathrm{Res}_Y(Q_1, Q_2)) \leq \deg(Q_1) \times \deg(Q_2)$.

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In the conclusion of DRL, we also have

- $\max_{\theta_n \in \Theta_n} \deg(\theta_n) \leq Ce^{\gamma n}$.

We apply DRL for the derivative cocycle $\mathbb{S}_2(df)$ acting on $\mathbb{S}_2(\mathbb{R}^2) = \{(z, v, w),\ z \in \mathbb{R}^2, v, w \in \mathbb{S}(T_z\mathbb{R}^2) \simeq \mathbb{S}^1\}$.

Lemma

For $\gamma > 0$ there exist $\epsilon = \epsilon(\gamma)$, $C = C(\gamma, \delta)$ and $\Theta_n = \Theta_n(\gamma) = \{\theta_n\}$ families of s.a. maps from $[0,1]^2 \times [0,1] \times [0,1]$ to $\mathbb{S}_2(\mathbb{R}^2)$, s.t. we have for all $n \in \mathbb{N}$:

- $\forall \theta_n \in \Theta_n,\ \theta_n(t, s_1, s_2) = \left(\theta_n^1(t), \theta_n^2(t, s_1), \theta_n^3(t, s_2)\right)$,
- $\bigcup_{\theta_n \in \Theta_n} \mathrm{Im}(\theta_n^1) \supset B_n(0, \epsilon)$,
- $\forall \theta_n \in \Theta_n,\ \ \|d\left(\mathbb{S}_2(df^n \circ \theta_n\right)\| \leq \alpha/2$,
- $\forall \theta_n \in \Theta_n\ \forall t, t' \in [0,1]^2, s_i, s_i' \in [0,1], i = 2, 3$
$$\left|\log \|d_{\theta_n^1(t)} f^n(\theta_n^i(t, s_i))\| - \log \|d_{\theta_n^1(t)} f^n(\theta_n^i(t', s_i'))\|\right| \leq 1,$$
- $\sharp\Theta_n, \max_{\theta_n \in \Theta_n} \deg(\theta_n) \leq Ce^{\gamma n}$.

(n, δ)-hyperbolic s.a. set

$$\Theta'_n := \{\theta_n \in \Theta_n,\ \exists x \in \mathrm{Per}^{\delta}_{\mathcal{F}_n}(\mathfrak{C}) \text{ s.t. } (x, e_s(x), e_u(x)) \in \mathrm{Im}(\theta_n)\},$$

For $\theta_n \in \Theta'_n$ the set $\theta^1_n([0,1]^2)$ is endowed with a (n, δ)-hyperbolic structure by letting

$$\begin{aligned} e_s(\theta^1_n(t)) &= \theta^2_n(t, 0), \\ e_u(\theta^1_n(t)) &= \theta^3_n(t, 0). \end{aligned}$$

Covering (n, δ)-hyperbolic s.a. set by hexagons

Lemma

There is a universal polynomial P, such that any $\theta_n^1([0,1])^2$ may be partitioned into $P(\deg(\theta_n))$ hexagons.

Proof : Elimination. □