Examples of $\mathcal{C}^r$ interval map with large symbolic extension entropy

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Abstract : For any integer $r \geq 2$ and any real $\epsilon > 0$, we construct an explicit example of $\mathcal{C}^r$ interval map f with symbolic extension entropy $h_{sex}(f) \geq \frac{r}{r-1}\log\|f'\|_\infty - \epsilon$ and $\|f'\|_\infty \geq 2$. T.Downarawicz and A.Maass [11] proved that for $\mathcal{C}^r$ interval maps with $r > 1$, the symbolic extension entropy was bounded above by $\frac{r}{r-1}\log\|f'\|_\infty$. So our example prove this bound is sharp. Similar examples had been already built by T.Downarowicz and S.Newhouse for diffeomorphisms in higher dimension by using generic arguments on homoclinic tangencies.

1 Introduction

1.1 Entropy of symbolic extensions

Let T be a dynamical system defined on a compact metrizable space X. We denote $\mathcal{M}(X,T)$ the set of invariant probability measures of (X,T) endowed with the weak star topology and $\mathcal{M}_e(X,T) \subset \mathcal{M}(X,T)$ the subset of ergodic measures. A symbolic extension (Y,S) of (X,T) is an extension which is a subshift of a full shift over a finite alphabet. Given a dynamical sytem one can wonder if it admits a symbolic extension and how far this extension is from the initial system in the point of view of entropy. The symbolic extension entropy function estimates this defect. Let $\pi : (Y,S) \to (X,T)$ be a symbolic extension. We consider the fonction $h^\pi_{ext} : \mathcal{M}(X,T) \to \mathbb{R}$ defined by $h^\pi_{ext}(\mu) := \sup_{\pi^*\nu=\mu} h(\nu)$, where h denotes the usual Kolmogorov-Sinai entropy. Then the symbolic extension entropy function $h_{sex} : \mathcal{M}(X,T) \to \mathbb{R}$ is defined as follows :

$$h_{sex}(\mu) := \inf_{\pi:(Y,S)\to(X,T)} h^\pi_{ext}(\mu)$$

where the infinimum is taken over all symbolic extensions (Y,S) of (X,T) (when there is no symbolic extension, we put $h_{sex} \equiv +\infty$).

Finally the topological symbolic extension entropy $h_{sex}(T)$ is the infimum of the topological entropy of the symbolic extensions of (X,T) :

$$h_{sex}(T) = \inf_{\pi:(Y,S)\to(X,T)} h_{top}(T)$$

In fact the topological symbolic extension entropy $h_{sex}(T)$ is equal to the supremum of the symbolic extension entropy function h_{sex} [1].

M.Boyle and T.Downarowicz [1] reduce the problem of existence of symbolic extensions to the study of the convergence of the entropy computed at finer and finer scale. Let us explain more precisely their main result.

Let $h_k : \mathcal{M}(X,T) \to \mathbb{R}$ be the Katok entropy (cf Appendix) computed with precision ϵ_k where $(\epsilon_k)_k$ is a decreasing sequence converging to zero. One can define by induction the following transfinite sequence (for a real map f on $\mathcal{M}(X,T)$ we denote $\widetilde{f}$ the smallest upper-semicontinuous function larger than f [1]) :

- $u_0 := 0$;
- if α is a successor ordinal [2]:

$$u_\alpha := \lim_{k\to+\infty} \widetilde{u_{\alpha-1} + h} - h_k$$

- if α is a limit ordinal :

$$u_\alpha := \widetilde{\sup_{\beta<\alpha} u_\beta}$$

The sequence $(u_\alpha)_\alpha$ is stationary at some countable step α^* (the ordinal α^* is called the order of accumulation of (X,T)). The main result of [1] can then be stated in the following way :

$$h_{sex} = h + u_{\alpha^*} \tag{1}$$

1.2 Tail entropy

The tail entropy [15] of a dynamical system estimates the entropy appearing at arbitrarily small scales :

$$h^*(T) = \lim_{\epsilon\to 0}\lim_{\delta\to 0}\limsup_{n\to+\infty}\frac{1}{n}\log\sup_{x\in X} r(n,\delta,B(x,n,\epsilon))$$

where $B(x,n,\epsilon) := \{y\in X,\ \forall k=0,...,n-1,\ d(T^kx,T^ky)<\epsilon\}$ is the usual Bowen ball. In fact the tail entropy satisfies also a variational principle [9, 4] and can be also written in terms of u_1 and the sequence h_k :

$$h^*(T) = \lim_{k\to+\infty}\|h-h_k\|_\infty = \sup_{\mu\in\mathcal{M}(X,T)} u_1(\mu) \tag{2}$$

A system is said to be asymptotically h-expansive if $h^*(T)=0$. It was proved by M. Boyle, D. Fiebig, U. Fiebig [3] that any asymptotically h-expansive system satisfies $h_{sex}=h$. Moreover it follows from Yomdin's theory [7] that $h^*(T)=0$ for $\mathcal{C}^\infty$ dynamical systems defined on a compact smooth manifold M and therefore $h_{sex}=h$ for such systems. In fact Yomdin's theory provides us the following upper bound on the tail entropy for $\mathcal{C}^r$ systems, which is due to J.Buzzi [7] :

$$h^*(T) \le \frac{\dim(M)R(T)}{r} \tag{3}$$

where $R(T) := \lim_n \frac{1}{n}\log\|(T^n)'\|_\infty$ for any riemmanian metric $\|\|$ on M. This upper bound (3) is known to be sharp [14],[7].

1.3 Existence of symbolic extensions for $\mathcal{C}^r$ maps

It is still unknown if general $\mathcal{C}^r$ dynamical systems admit symbolic extensions. But it was recently proved by A.Maass and T.Downarowicz in the case of interval maps [11]. If ν is an ergodic measure of a $\mathcal{C}^1$ interval map f, one can define its Lyapounov exponent $\chi(\nu) := \int \log|f'|d\nu$. We consider $\chi^+ = \max(\chi,0)$ and we denote $\overline{\chi^+}$ its harmonic extension on $\mathcal{M}([0,1],f)$ (the function $\overline{\chi^+}$ is given by the formula $\overline{\chi^+}(\mu) = \lim_{n\to+\infty}\frac{1}{n}\int\max(\log|f^n|'(x),0)d\mu(x)$ for all $\mu\in\mathcal{M}([0,1],f)$). Observe that $\overline{\chi^+}(\mu)\le\log\|f'\|_\infty$.

[1] if f is bounded, then the function $\widetilde{f}$ can be written in the following form : $\widetilde{f}(\mu) = \limsup_{\nu\to\mu} f(\nu)$; if f is unbounded, then we put $\widetilde{f}\equiv+\infty$

[2] in this case we denote $\alpha-1$ the ordinal preceding α

Theorem 1 *[11] Let $r > 1$. Let $f : [0,1] \to [0,1]$ be a $\mathcal{C}^r$ map, then for all ordinal α and for all $\mu \in \mathcal{M}([0,1], f)$,*

$$u_\alpha(\mu) \leq \frac{\overline{\chi^+}(\mu)}{r-1} \tag{4}$$

Moreover for all $n \in \mathbb{N}$,

$$u_n(\mu) \leq \sum_{k=1}^{n} \frac{\overline{\chi^+}(\mu)}{r^k} \tag{5}$$

In particular (according to (1)),

$$h_{sex}(f) \leq h_{top}(f) + \frac{\log \|f'\|_\infty}{r-1}$$

In higher dimension we conjecture :

Conjecture 1 *Let $r > 1$. Let $T : M \to M$ is a $\mathcal{C}^r$ map, then for all ordinal α and for all $\mu \in \mathcal{M}(X,T)$,*

$$u_\alpha(\mu) \leq \frac{\overline{\sum_{i=1}^{d} \chi_i^+}(\mu)}{r-1}$$

where $(\chi_i)_{i=1,\ldots,d}$ denote the d Lyapounov exponents.
Moreover forall $n \in \mathbb{N}$,

$$u_n(\mu) \leq \left(\sum_{k=1}^{n} \frac{1}{r^k}\right) \left(\overline{\sum_{i=1}^{d} \chi_i^+}(\mu)\right)$$

In particular,

$$h_{sex}(T) \leq h_{top}(T) + \frac{\log R(T)}{r-1} \tag{6}$$

1.4 Previous examples of higher dimensional diffeomorphisms with large symbolic extension entropy

S.Newhouse and T.Downarowicz [10] built examples of $\mathcal{C}^r$ $(r > 1)$ diffeomorphism on any manifold of dimension ≥ 2 such that $\sup_{\mu \in \mathcal{M}(X,T)} h_{sex}(\mu)$ is equal to $\frac{\dim(M)R(T)r}{r-1}$. Therefore their examples would prove the upper bound (6) is sharp. They also gave $\mathcal{C}^1$ examples without symbolic extensions. Their examples are generic and the construction use homoclonic tangencies.

M.Boyle and T.Downarowicz [2] built explicitly a $\mathcal{C}^r$ example on a manifold of dimension 4 with $h_{sex}(T) > h_{top}(T)$ by adapting an example of $\mathcal{C}^r$ diffeomorphism without measure of maximal entropy due to M.Misiurewicz [14].

1.5 Main statements

In the following paper we prove that Theorem 1 is sharp.

Theorem 2 *Let $r \in \mathbb{N}^*$. There exists a $\mathcal{C}^r$ interval map $f_r : [0,1] \to [0,1]$ fixing 0, such that for all integers $n \geq 1$:*

$$u_n(\delta_0) = \left(\sum_{k=1}^{n} \frac{1}{r^k}\right) \log \|f_r'\|_\infty > 0$$

where $\delta_0 \in \mathcal{M}([0,1], f_r)$ denotes the dirac measure at the point 0.
In particular, if ω is the first ordinal with infinite cardinal, we have :

- *if* $r > 1$, *then* $u_\omega(\delta_0) = \frac{\log \|f_r'\|_\infty}{r-1}$;
- *if* $r = 1$, *then* $u_\omega \equiv +\infty$ *and therefore* f_1 *does not admit symbolic extensions.*

Recall that $\overline{\chi^+}(\mu) \leq \log \|f_r'\|_\infty$ for all invariant measure μ. Therefore the inequalities (4) and (5) of Theorem 1 are sharp for $\mathcal{C}^r$ interval maps.

Remark 1 *One could also wonder if for all* $\mathcal{C}^r$ $(r > 1)$ *interval maps* f *and for all* $n \in \mathbb{N}$ *we have* $\sup_{\mu \in \mathcal{M}([0,1],f)}(u_{n+1} - u_n)(\mu) \leq \frac{\log \|f'\|_\infty}{r^{n+1}}$. *In fact it is false : we explain in Section 3.5 how to modify the previous example to get a counter-example.*

Recall that T.Downarowicz and A.Maass (Theorem 1) obtain the following upper bound on the topological symbolic extension entropy of a $\mathcal{C}^r$ $(r > 1)$ interval map f :

$$h_{sex}(f) \leq h_{top}(f) + \frac{\log \|f'\|_\infty}{r-1} \leq \frac{r \log \|f'\|_\infty}{r-1}$$

By using the construction of the previous example we prove this upper bound is sharp in the following sense :

Theorem 3 *Let* $r \geq 2$ *be an integer. For any* $\epsilon > 0$, *there exists a* $\mathcal{C}^r$ *interval map* $f_{r,\epsilon}$ *with* $\|f'_{r,\epsilon}\|_\infty \geq 2$ *such that :*

$$h_{sex}(f_{r,\epsilon}) \geq \frac{r \log \|f'_{r,\epsilon}\|_\infty}{r-1} - \epsilon$$

But we do not know if our example can provide a new one satisfying $h_{sex}(f_r) = h_{top}(f_r) + \frac{\log \|f_r'\|_\infty}{r-1}$.

Our examples are in the spirit of those of T.Downarowicz and S.Newhouse : we accumulate horseshoes at different small scales. The construction of such horseshoes is similar of examples due to J.Buzzi of $\mathcal{C}^r$ interval maps without measures of maximal entropy [7],[17].

2 Sex entropy by accumulating small horseshoes

We recall first the main idea used by S.Newhouse and T.Downarowicz [10] to get a lower bound of the symbolic extension entropy by accumulating entropy at small scales. The following lemma is valid for general dynamical systems. Recall that h_k denotes the Katok entropy at some scale ϵ_k where $(\epsilon_k)_k$ is a decreasing sequence converging to zero. Also if p is a periodic point we denote $\mathcal{O}(p)$ the orbit of p and $\gamma_p := \frac{1}{\sharp \mathcal{O}(p)} \sum_{q \in \mathcal{O}(p)} \delta_q$ the periodic measure associated to p.

Lemma 1 *Let* $T : X \to X$ *be a continuous map defined on a compact metrizable space* X. *Let* μ *be an invariant probability measure.*

We assume that for all $k \in \mathbb{N}$, *there exists periodic points* $(p_{(i_1,\ldots,i_{2k+1})})_{(i_1,\ldots,i_{2k+1}) \in \mathbb{N}^{2k+1}}$ *and invariant probability measures* $(\mu_{(i_1,\ldots,i_{2k})})_{(i_1,\ldots,i_{2k}) \in \mathbb{N}^{2k}}$ *(we put* $\mathbb{N}^0 = \{\emptyset\}$ *and* $\mu_\emptyset = \mu$*) such that :*

1. *for all* $(i_1, \ldots, i_{2k}) \in \mathbb{N}^{2k}$, *the periodic measures* $\gamma_{p_{(i_1,\ldots,i_{2k+1})}}$ *are converging to* $\mu_{(i_1,\ldots,i_{2k})}$ *when* i_{2k+1} *goes to* $+\infty$;
2. *for all* $(i_1, \ldots, i_{2k+1}) \in \mathbb{N}^{2k+1}$, *the measures* $\mu_{(i_1,\ldots,i_{2k+2})}$ *are converging to* $\gamma_{p_{(i_1,\ldots,i_{2k+1})}}$ *when* i_{2k+2} *goes to* $+\infty$;

3. *for all* $q \in \mathbb{N}$,
$$\lim_{i_{2k}\to+\infty} h_q(\mu_{(i_1,\dots,i_{2k})}) = 0$$

4. *the limits* $\lim_{i_1\to+\infty}\left(\lim_{i_2\to+\infty}\dots\left(\lim_{i_{2k}\to+\infty} h(\mu_{(i_1,\dots,i_{2k})})\right)\dots\right)$ *exist.*

Then for all $n \in \mathbb{N}$ *:*

$$u_n(\mu) \geq \sum_{l=1}^{n} \lim_{i_1\to+\infty}\left(\lim_{i_2\to+\infty}\dots\left(\lim_{i_{2l}\to+\infty} h(\mu_{(i_1,\dots,i_{2l})})\right)\dots\right) \tag{7}$$

PROOF : We prove (7) by induction on n. Assume the lemma for n and prove it for $n+1$. By definition, we have :

$$u_{n+1}(\mu) = \lim_q \widetilde{h - h_q + u_n}(\mu)$$

Then for all $q \in \mathbb{N}$ we get by using the first and the second hypothesis and by upper semi-continuity of $\widetilde{h - h_q + u_n}$:

$$\widetilde{h - h_q + u_n}(\mu) \geq \limsup_{i_1}\left(\limsup_{i_2}(h - h_q + u_n)(\mu_{(i_1,i_2)})\right)$$

Then according to the third hypothesis :

$$\widetilde{h - h_q + u_n}(\mu) \geq \limsup_{i_1}\left(\limsup_{i_2}(h + u_n)(\mu_{(i_1,i_2)})\right)$$

and as the limits $\lim_{i_1}\left(\lim_{i_2} h(\mu_{(i_1,i_2)})\right)$ exist, we obtain :

$$\widetilde{h - h_q + u_n}(\mu) \geq \lim_{i_1}\left(\lim_{i_2} h(\mu_{(i_1,i_2)})\right) + \limsup_{i_1}\left(\limsup_{i_2} u_n(\mu_{(i_1,i_2)})\right)$$

We apply finally the induction hypothesis to each measure $\mu_{(i_1,i_2)}$ to get :

$$u_n(\mu_{(i_1,i_2)}) \geq \sum_{l=2}^{n} \lim_{i_3\to+\infty}\left(\lim_{i_4\to+\infty}\dots\left(\lim_{i_{2l}\to+\infty} h(\mu_{(i_1,\dots,i_{2l})})\right)\dots\right)$$

We conclude that :

$$u_{n+1}(\mu) \geq \sum_{l=1}^{n} \lim_{i_1\to+\infty}\left(\lim_{i_2\to+\infty}\dots\left(\lim_{i_{2l}\to+\infty} h(\mu_{(i_1,\dots,i_{2l})})\right)\dots\right)$$

□

In the following we also use the following equivalent version of the previous lemma :

Lemma 2 *Let* $f : X \to X$ *be a continuous map defined on a compact metrizable space* X *with a fixed point* p.

We assume that for all $k \in \mathbb{N}$, *there exists periodic points* $(p_{(i_1,\dots,i_{2k+1})})_{(i_1,\dots,i_{2k})\in\mathbb{N}^{2k}}$ *and invariant measures* $(\mu_{(i_1,\dots,i_{2k+1})})_{(i_1,\dots,i_{2k+1})\in\mathbb{N}^{2k+1}}$ *such that :*

1. *for all* $(i_1,\dots,i_{2k-1}) \in \mathbb{N}^{2k-1}$, *the periodic measures* $\gamma_{p_{(i_1,\dots,i_{2k})}}$ *are converging to* $\mu_{(i_1,\dots,i_{2k-1})}$ *when* i_{2k} *goes to* $+\infty$ *;*

2. *for all* $(i_1,\dots,i_{2k}) \in \mathbb{N}^{2k}$, *the measures* $\mu_{(i_1,\dots,i_{2k+1})}$ *are converging to* $\gamma_{p_{(i_1,\dots,i_{2k})}}$ *when* i_{2k+1} *goes to* $+\infty$ *(*$\lim_n \mu_n = \delta_p$ *for* $k=0$*) ;*

3. *for all* $q \in \mathbb{N}$,
$$\lim_{i_{2k+1}\to+\infty} h_q(\mu_{(i_1,\dots,i_{2k+1})}) = 0$$

4. *the limits* $\lim_{i_1\to+\infty}\left(\lim_{i_2\to+\infty}\dots\left(\lim_{i_{2k+1}\to+\infty} h(\mu_{(i_1,\dots,i_{2k+1})})\right)\dots\right)$ *exist.*

Then for all $n \in \mathbb{N}$ *:*
$$u_n(\delta_p) \geq \sum_{l=0}^{n-1} \lim_{i_1\to+\infty}\left(\lim_{i_2\to+\infty}\dots\left(\lim_{i_{2l+1}\to+\infty} h(\mu_{(i_1,\dots,i_{2l+1})})\right)\dots\right)$$

3 Our construction on the interval

3.1 Horseshoe for interval maps

The following notion of horseshoe for interval maps is due to M.Misiurewicz.

Definition 1 *Let f be an interval map. A family $J = (J_1, ..., J_p)$ of closed disjoint intervals is called a p* ***horseshoe*** *if $J_k \subset f(J_i)$ for all j, k.*

To simplify the notations we mean sometimes by J the union of the intervals defining the horseshoe J. Remark that any subfamily K of J is itself a horseshoe. If $J = (J_1, ..., J_p)$ is a p horseshoe ordered increasingly, i.e. if $i < j$ then $x_i < x_k$ for all $(x_i, x_k) \in J_i \times J_k$, we denote by J' the $p-1$ horseshoe $(J_1, ..., J_{p-1})$.

Let us denote $H_J := \bigcap_{n\in\mathbb{Z}} T^n J$ and $(\{1, ..., p\}^{\mathbb{N}}, \sigma)$ the one sided shift with p symbols. The map $\pi : (H_J, T) \to (\Sigma_p^+, \sigma)$ defined by $(\pi(x))_k = q$ if $f^k(x) \in J_q$ is a semi-conjugacy. In particular $h_{top}(f) \geq \log p$. In fact horseshoes characterize entropy of continuous interval maps [13] : if f is a continuous interval map with entropy $h_{top}(f) > 0$ then for all $h < h_{top}(f)$ there exists a p horseshoe for f^N with $\log(p)/N > h$.

In our construction we consider horseshoes of the following simple form.

Definition 2 *Let $f : [0, 1] \to [0, 1]$ be a $\mathcal{C}^r$ interval map and let p and N be integers. A (p, N)* ***quasi linear horseshoe*** *(resp. a (p, N)* ***linear horseshoe****) for f is a p horseshoe ordered increasingly $J = (J_1, ..., J_p)$ for f^N such that :*

- $|J_1| = |J_2| = ... = |J_p|$ *;*
- $f(J_1) = f(J_2) = ... = f(J_p)$ *;*
- *f is increasing on J_i when i is odd and f is decreasing on J_i when i is even ;*
- *f is affine on J_i for all $i = 1, ..., p-1$ (resp. for all $i = 1, ..., p$) ;*
- *there exists $J_i < b_i < J_{i+1}$ such that $f^{(l)}(b_i) = 0$ for $l = 1, ..., r$ and $i = 1, ..., p-1$;*
- *$f^{N-1}_{/f(J_1)}$ is affine.*

The slope of a quasi linear horseshoe J is defined by $s(J) := \|(f^N_{/J_1})'\|_\infty$.

We will write H_J^N the compact f^N invariant set associated to a (p, N) (quasi-) linear horseshoe $J = (J_1, ..., J_p)$ for f, that is $H_J^N := \bigcap_{n\in\mathbb{N}} f^{-nN} J$, and we denote H_J the compact f invariant set associated, that is $H_J := \bigcup_{k=0,...,N-1} f^k(H_J^N)$.

We will use the following technical lemma :

Lemma 3 *Let $f : [0,1] \to [0,1]$ be a $\mathcal{C}^r$ interval map and $J = (J_1, ..., J_p)$ a (p, N) quasi linear horseshoe for f with slope $s(J) > 1$. Then there exists a sequence of periodic points $(p_n)_{n\in\mathbb{N}}$ in J' with periods $(NP_n)_{n\in\mathbb{N}}$ and a sequence of points $(p'_n)_{n\in\mathbb{N}}$ in J' such that :*

- *the periodic measures γ_{p_n} converge to the measure of maximal entropy of $H_{J'}$;*
- *the sequence $(f(p_n))_{n\in\mathbb{N}}$ is monotone ;*
- *f^{NP_n} is increasing and affine on $[p_n, p'_n]$;*
- *$J_p \subset f^{NP_n}([p_n, p'_n])$.*

PROOF : Put $K := f(J_1) = ... = f(J_p)$. We assume $f^{N-1}_{/K}$ is increasing (one can easily adapt the proof in the decreasing case).The map $\pi : (H^N_J, f^N) \to (\{1, ..., p\}^{\mathbb{N}}, \sigma)$ defined by $(\pi(x))_k = q$ if $f^{Nk}(x) \in J_q$ for all $k \in \mathbb{N}$ is a semi-conjugacy. As f^N is expanding on each element of J' (because $s(J) > 1$) the restriction of π on $H^N_{J'}$ is one-to-one and therefore $\pi : (H^N_{J'}, f^N) \to (\{1, ..., p-1\}^{\mathbb{N}}, \sigma)$ is a conjugacy. It is well-known that the periodic measures are dense in $\mathcal{M}((\{1, ..., p-1\}^{\mathbb{N}}, \sigma))$: in particular there exists a sequence $(q_n)_{n\in\mathbb{N}}$ of periodic points of $(\{1, ..., p-1\}^{\mathbb{N}}, \sigma)$ with periods $(P_n)_{n\in\mathbb{N}}$ such that the associated periodic measures converge to the measure of maximal entropy μ of $(\{1, ..., p-1\}^{\mathbb{N}}, \sigma)$. One can also clearly arrange this sequence such that for all $n \in \mathbb{N}$ the integer $\sharp\{k \in [0, P_n - 1],\ (\sigma^k q_n)_0 \text{ is even}\}$ is even. We put $p_n = \pi^{-1}(q_n)$ so that $p_n \in J'$ is a periodic point of f with period NP_n. Moreover f^{NP_n} is increasing near p_n because we assume $f^{N-1}_{/K}$ is increasing and that $\sharp\{k \in [0, P_n - 1],\ (\sigma^k q_n)_0 \text{ is even}\}$ is even. By extracting a subsequence one can also assume that $(f(p_n))_{n\in\mathbb{N}}$ is monotone.

The periodic measures γ_{p_n} converge to $\frac{1}{N}\sum_{k=0}^{N-1} f^{*k}\pi^{*-1}\mu \in \mathcal{M}([0,1], f)$ which is a measure of maximal entropy of $H_{J'}$. Indeed, as π is a conjugacy from $(H^N_{J'}, f^N)$ to $(\{1, ..., p-1\}^{\mathbb{N}}, \sigma)$ we have $h(\pi^{*-1}\mu, f^N) = h(\mu, \sigma) = h_{top}(\{1, ..., p-1\}^{\mathbb{N}}, \sigma) = h_{top}(f^N, H^N_{J'})$. Finally it is easily seen that $h(\frac{1}{N}\sum_{k=0}^{N-1} f^{*k}\pi^{*-1}\mu, f) = \frac{1}{N}h(\pi^{*-1}\mu, f^N)$ and $h_{top}(f^N, H^N_{J'}) = Nh_{top}(f, H_{J'})$ so that $h(\frac{1}{N}\sum_{k=0}^{N-1} f^{*k}\pi^{*-1}\mu, f) = h_{top}(f, H_{J'})$.

Observe now that f^N is affine on each interval which is a connected component of $\bigcap_{k=0}^{P_n-1} f^{-kN}J'$ because f is affine on each element of J'. Moreover the image by f^{NP_n} of any such interval contains J_i for all $i = 1, ..., p$ because J is a horseshoe for f^N. Let us denote $[p''_n, p'_n]$ the interval containing p_n. As f^{NP_n} is increasing near p_n and as J_p stands at the right of p_n we conclude that $J_p \subset f^{NP_n}([p_n, p'_n])$.

□

3.2 A model of $\mathcal{C}^r$ interval maps with entropy of first order

The question of continuity of the entropy for smooth dynamical systems was studied early on. M.Misiurewicz [14] gave the first examples of $\mathcal{C}^r$ diffeomorphisms defined on a compact manifold of dimension 4 without measures of maximal entropy. Then S.Newhouse [16] proved, using Yomdin's theory, that the entropy function was upper semi-continuous for $\mathcal{C}^\infty$ systems. Counterexamples for interval maps appear much later. In his thesis [8] J.Buzzi built an example of $\mathcal{C}^r$ maps without measure of maximal entropy (see also [17]).

In Misiurewicz's and Buzzi's examples the stategy is the same : you construct "smaller and smaller horseshoes" converging to a fixed point such that their entropies converge increasingly to the topological entropy. By a "small" horseshoe J we mean that the orbit of the associated compact invariant set H_J is contained in the ϵ-neighborhood of some periodic orbit for $\epsilon > 0$ small.

In this section we recall the main idea in the example of J.Buzzi, which will be a model of "first order" in our example. We first begin with the following easy lemma, which will be useful in the next constructions :

Lemma 4 *1. There exists a constant $1 \geq M_1 > 0$ with the following properties.*
Let $a, b \in [0,1]$. Let $\alpha \in \mathbb{R}^+$ and $c, d \in \mathbb{R}$ with $|c-d| \leq M_1\alpha|a-b|^r$. Then there exists a $\mathcal{C}^\infty$ monotone map $f : \mathbb{R} \to \mathbb{R}$ such that :

- $\|f\|_r := \max_{k=1,...,r} \|f^{(k)}\|_\infty \leq \alpha$;
- $f(a) = c$, $f(b) = d$;
- $f^{(k)}(a) = f^{(k)}(b) = 0$ *for* $k = 1, ..., r$.

2. There exists a constant $1 \geq M_2 > 0$ with the following properties. Let $a, b \in [0,1]$. Let $\alpha \in \mathbb{R}^+$, $c \in \mathbb{R}$ and $c' \in \mathbb{R}$ with $c'(b-a) \geq 0$ and $|c'| \leq M_2\alpha|a-b|^{r-1}$. Then there exists a $\mathcal{C}^\infty$ monotone map $f : \mathbb{R} \to \mathbb{R}$ such that :

- $\|f\|_r := \max_{k=1,...,r} \|f^{(k)}\|_\infty \leq \alpha$;
- $f(a) = c$, $|f(a) - f(b)| \leq \alpha|a-b|^r$;
- $f'(a) = c'$, $f'(b) = 0$;
- $f^{(k)}(a) = f^{(k)}(b) = 0$ *for* $k = 2, ..., r$.

PROOF : (1) We are easily reduce to the case $a < b$ and $c < d$. Let $F : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ non-decreasing map such that $F(0) = 0$ and $F(1) = 1$ and $F^{(k)}(0) = F^{(k)}(1) = 0$ for $k = 1, ..., r$. Put $M_1 := \min(\frac{1}{\|F\|_r}, 1)$. Fix a, b, c, d, α as in the statement (1) of the lemma. We define f as follows

$$f := |c-d|F(|a-b|^{-1}(.-a)) + c$$

Clearly $f^{(k)}(a) = f^{(k)}(b) = 0$ for $k = 1, ..., r$ and $f(a) = c$, $f(b) = d$. Moreover $\|f^{(k)}\|_\infty = |a-b|^{-k}|c-d|\|F^{(k)}\|_\infty \leq \alpha$ for all $k = 1, ..., r$.

(2) We are easily reduce to the case $a < b$ and $c' > 0$. Let $F : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ non-decreasing map such that $F(0) = 0$, $F(1) = 1$, $F'(0) = 1$, $F'(1) = 0$ and $F^{(k)}(0) = F^{(k)}(1) = 0$ for $k = 2, ..., r$. Put $M_2 := \min(\frac{1}{\|F\|_r}, 1)$. Fix a, b, c, c', α as in the statement (2) of the lemma. We define f as follows

$$f := c'|a-b|F(|a-b|^{-1}(.-a)) + c$$

Clearly $f^{(k)}(a) = f^{(k)}(b) = 0$ for $k = 2, ..., r$ and $f(a) = c$, $f'(a) = c'$. We put $d := f(b)$. Moreover $\|f^{(k)}\|_\infty = c'|a-b|^{-k+1}\|F^{(k)}\|_\infty \leq \alpha$ for all $k = 1, ..., r$ and $|f(a) - f(b)| = |\int_{[a,b]} f'(t)dt| \leq c'|a-b|\|F'\|_\infty \leq \alpha|a-b|^r$.

□

We can now explain our model :

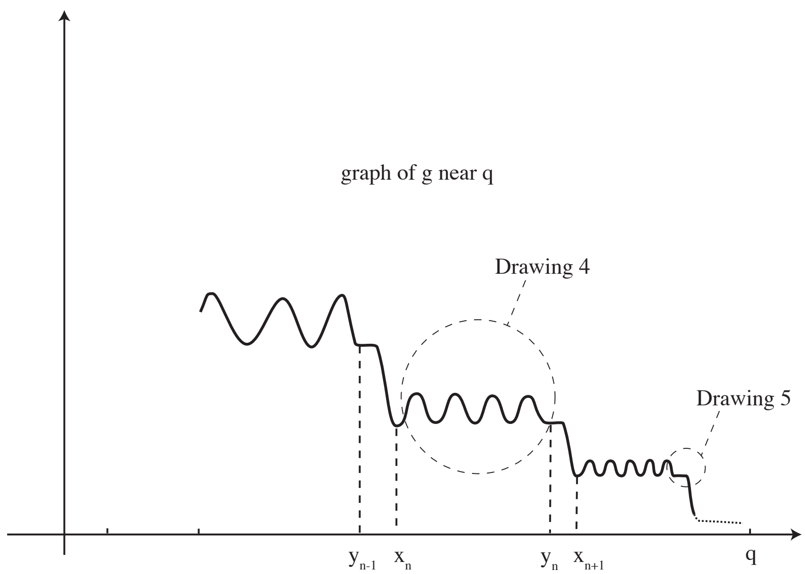
Proposition 1 *Let $\epsilon > 0$, $\lambda > 1$, $0 \leq p < p' < q' < q \leq 1$ and let $f : [0,1] \to [0,1]$ be a $\mathcal{C}^r$ interval map, such that p is a periodic point of f of period P and $f(q) = p$, $f^{(k)}(q) = f^{(k)}(q') = 0$ for $k = 1, ..., r$. We also assume there exists an integer S such that :*

- $f^S(p) = p$, *i.e. S is a multiple of P ;*
- f^S *is increasing and affine on $[p, p']$ with slope λ ;*
- $q \in f^S([p, p'])$.

Then there exists a $\mathcal{C}^r$ interval map g such that $f = g$ outside the interval $]q', q[$, $\|f'\|_\infty = \|g'\|_\infty$ and $\|f - g\|_r \leq \epsilon$. Moreover there exist a strictly increasing sequence of integers $(T_n)_{n\in\mathbb{N}}$, a strictly increasing sequence of even integers $(N_n)_{n\in\mathbb{N}}$, a sequence of intervals $([x_n, y_n])_{n\in\mathbb{N}}$ and a sequence of linear horseshoes $(J^n)_{n\in\mathbb{N}}$ such that :

- $f^{(l)}(x_n) = f^{(l)}(y_n) = 0$ *for all $n \in \mathbb{N}$ and $l = 1, ..., r$;*

- $J^n \subset [x_n, y_n] \subset [q', q]$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} y_n = q$;
- J^n *is a linear* $(N_n, ST_n + 1)$ *horseshoe for* g *with slope* $\frac{\lambda^{ST_n}}{n^2(3n^2N_n)^{r-1}}$;
- $\lim_{n\to+\infty} h_{top}(J^n) = \lim_{n\to+\infty} \frac{\log N_n}{T_n} = \frac{\log \lambda}{r}$;
- *Invariant probability measures supported by* H_{J^n} *converge to the periodic measure associated to* p *when* n *goes to infinity.*



Drawing 2 : Accumulation of small horsehoes

PROOF :

Let n_0 be large enough such that $q - \frac{1}{n_0} > q'$. For all $n \geq n_0$ we put g equal to a N_n zig zag of height $\frac{1}{n^2(2n^2N_n)^r}$ on $[x_n, y_n] := [q - \frac{1}{n}, q - \frac{1}{n} + \frac{1}{3n^2}]$ as described on the above picture (Drawing 2).

More precisely for all $i = 0, ..., N_n - 1$ the map g is affine on the interval $J_i^n := [x_n + (i + \frac{1}{4})\frac{1}{3n^2N_n}, x_n + (i + \frac{3}{4})\frac{1}{3n^2N_n}]$ with slope $\frac{(-1)^i}{n^2(3n^2N_n)^{r-1}}$ and $g(J_1^n) = ... = g(J_{N_n}^n) = [a_n, a_n + \frac{1}{2n^2(3n^2N_n)^r}]$ (we will specify a_n later). Then according to Lemma 4 (2), one can extend g on the whole interval $[x_n, y_n]$ such that :

- $g^{(k)}(x_n + i\frac{1}{3n^2N_n}) = 0$ for $k = 1, ..., r$ and for $i = 0, ..., N_n$;
- $\|(g)_{/[x_n,y_n]}\|_r \leq \frac{4^{r-1}}{M_2 n^2}$;
- $g(y_n) = g(x_n) \in [a_n - \frac{1}{M_2 n^2(3n^2N_n)^r}, a_n]$.

We choose a_n such that f maps $[a_n, a_n + \frac{1}{2n^2(3n^2N_n)^r}]$ on the expanding part $[p, p']$ of f^S during a time T_n and then comes back on $[x_n, y_n]$, that is $f^{ST_n}([a_n, a_n + \frac{1}{2n^2(3n^2N_n)^r}]) \supset [x_n, y_n]$. We choose $T_n > T_{n-1}$ minimal for this property. This can be done because $q \in f^S([p, p'])$. In this way we obtain for all integers $n > n_0$ a linear horseshoe $J^n = (J_1^n, ..., J_{N_n}^n)$ for g^{ST_n+1}. The condition on T_n is :

$$\frac{1}{n^2} \leq \lambda^{T_n} \frac{1}{2n^2(3n^2N_n)^r} \leq \lambda\frac{1}{n^2} \tag{8}$$

and the condition on a_n is :

$$f^{ST_n}(a_n) = x_n$$

that is :

$$\lambda^{T_n}(a_n - p) = x_n - p$$

We deduce from the inequality (8) that $\lim_{n\to+\infty}\frac{\log N_n}{T_n} = \frac{\log\lambda}{r}$. One can also replace N_n by $N_n - 1$ to ensure N_n is even.

By using Lemma 4(1) one can now extend g in a $\mathcal{C}^r$ way on the whole interval $[x_{n_0}, q]$ such that for all $n \geq n_0$:

$$\begin{aligned}
\|g_{/[y_n,x_{n+1}]}\|_r &\leq \frac{1}{M_1}\left|\frac{g(y_n) - g(x_{n+1})}{(y_n - x_{n+1})^r}\right| \\
&\leq \frac{6^r}{M_1}n^{2r}\left(a_n - a_{n+1} + \frac{1}{M_2 n^2(3n^2N_n)^r}\right) \\
&\leq \frac{6^r}{M_1}n^{2r}\left(\lambda^{-T_n} + \frac{1}{M_2 n^2(3n^2N_n)^r}\right) \leq \frac{6^r(1+1/M_2)}{M_1N_n^r}
\end{aligned}$$

We extend g in a $\mathcal{C}^r$ way on $[q', x_{n_0}]$ by putting for all $x \in [q', x_{n_0}]$:

$$g(x) = f(q') + \frac{g(x_{n_0}) - f(q')}{f(q) - f(q')}\left(f\left(\frac{q-q'}{x_{n_0} - q'}(x - q') + q'\right) - f(q')\right)$$

One checks easily that $g(q') = f(q')$ and $g^{(k)}(q') = g^{(k)}(x_{n_0}) = 0$ for $k = 1, ..., r$.

We conclude the proof by choosing n_0 large enough such that $\frac{6^r(1+1/M_2)}{M_1N_{n_0}^r} \leq \epsilon$ and $\|(f-g)_{/[q',x_{n_0}]}\|_r \leq \epsilon$.

□

3.3 Proof of Theorem 2

We build a collection of $\mathcal{C}^r$ maps $(g_k)_{k\in\mathbb{N}\cup\{\infty\}}$ defined on $[0,1]$ fixing 0 and with first derivative bounded by 5. For all $l \in \mathbb{N}$ and for all $(i_1, ..., i_{2l+2}) \in \mathbb{N}^{2l+2}$ there exist points $p_{i_1,...,i_{2l+2}}$, intervals $[x_{i_1,...,i_{2l+1}}, y_{i_1,...,i_{2l+1}}]$, collections of disjoint closed intervals $J_{i_1,...,i_{2l+1}}$, integers $P_{i_{2l+2}}$, $T_{i_{2l+1}}$ and even integers $N_{i_{2l+1}}$ such that for all $k \in \mathbb{N}\cup\{\infty\}$ and all integers $0 \leq l \leq k$ we have :

- $p_{i_1,...,i_{2l+2}} \in J'_{i_1,...,i_{2l+1}}$ is a periodic point of $g_k^{S_{i_1,...,i_{2l+1}}}$ of period $P_{i_{2l+2}}$;
- $f^{(m)}(x_{i_1,...,i_{2l+1}}) = f^{(m)}(y_{i_1,...,i_{2l+1}}) = 0$ for $m = 1, ..., r$;
- $J_{i_1,...,i_{2l+1}} \subset [x_{i_1,...,i_{2l+1}}, y_{i_1,...,i_{2l+1}}] \subset [x_{i_1,...,i_{2l-1}}, y_{i_1,...,i_{2l-1}}]$ [3] ;
- $J_{i_1,...,i_{2l+1}}$ is a quasi linear $(N_{i_{2l+1}}, S_{i_1,...,i_{2l}}T_{i_{2l+1}} + 1)$ horseshoe for g_k and $J_{i_1,...,i_{2k+1}}$ is a linear horseshoe ;
- $\lim_{i_{2l+1}\to+\infty}\|g_{k/[x_{i_1,...,i_{2l+1}},y_{i_1,...,i_{2l+1}}]}\|_r = 0$.

[3] the last inclusion holds only for $l \neq 0$

where the integers $S_{i_1,...,i_m}$ for $m \leq 2k+1$ are defined inductively in the following way : $S_{i_1} = T_{i_1}$, $S_{i_1,...,i_{2l+1}} = S_{i_1,...,i_{2l}} \times T_{i_{2l+1}} + 1$ and $S_{i_1,...,i_{2l+2}} = S_{i_1,...,i_{2l+1}} \times P_{i_{2l+2}}$. Remark that the integer $S_{i_1,...,i_{2l+2}}$ is the period of $p_{i_1,...,i_{2l+2}}$ for g_k.

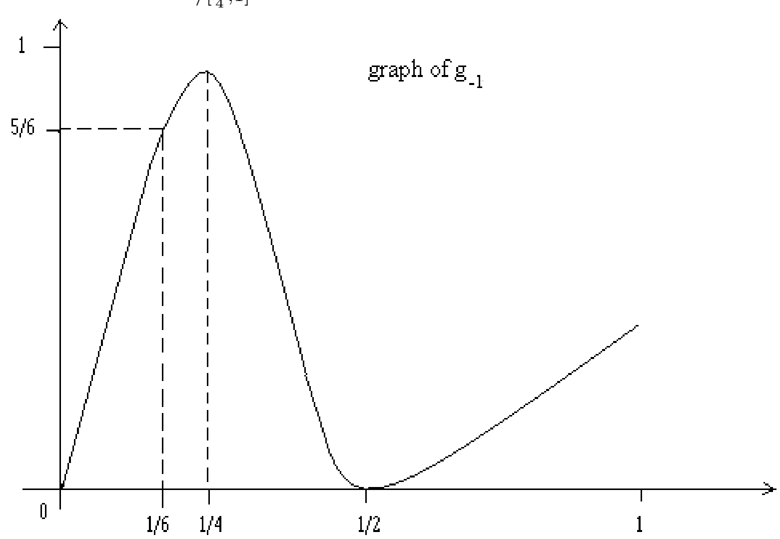
These periodic points and horseshoes can also be arranged to satisfy the following properties so that one can apply Lemma 2 (to simplify the notations we write $H_{i_1,...,i_{2l+1}}$ instead of $H_{J_{i_1,...,i_{2l+1}}}$ and $H'_{i_1,...,i_{2l+1}}$ instead of $H_{J'_{i_1,...,i_{2l+1}}}$) :

for all $k \in \mathbb{N} \cup \{\infty\}$ and for all integers $0 \leq l \leq k$,

1. the sequence of periodic measures $(\gamma_{p_{i_1,...,i_{2l+2}}})_{i_{2l+2}\in\mathbb{N}}$ converges to the measure of maximal entropy of $H'_{i_1,...,i_{2l+1}}$ when $i_{2l+2} \to +\infty$;
2. measures supported by $H'_{i_1,...,i_{2l+1}}$ converge to $\gamma_{p_{i_1,...,i_{2l}}}$ when $i_{2l+1} \to +\infty$ (measures supported by H'_{i_1} converge to δ_0 when $i_1 \to +\infty$) ;
3. for all $\epsilon > 0$ there exists an integer I_k such that :
$$\forall i_{2l+1} > I_k \ \exists x \in [0,1] \text{ s.t. } H'_{i_1,...,i_{2l+1}} \subset \bigcap_{n\in\mathbb{N}} B_{g_k}(x,n,\epsilon) \ ;$$
4. $\lim_{i_1\to+\infty} ... \lim_{i_{2l+1}\to+\infty} h(H'_{i_1,...,i_{2l+1}}, g_k) = \dfrac{\log 5}{r^{l+1}}$.

One deduces easily from the above assertions 1-4 that the map g_∞ satisfies the assumptions 1-4 of Lemma 2. Then by applying this lemma for g_∞, we get for all integers n: $u_n(\delta_0) \geq \left(\sum_{k=1}^n \frac{1}{r^k}\right) \log \|g'_\infty\|_\infty > 0$. The converse inequalities $u_n(\delta_0) \leq \left(\sum_{k=1}^n \frac{1}{r^k}\right) \log \|g'_\infty\|_\infty$ follow from 5 of Theorem 1. This concludes the proof of Theorem 2 with $f_r := g_\infty$.

We explain now the construction of the sequence $(g_n)_{n\in\mathbb{N}}$ and the map g_∞. We first consider a $\mathcal{C}^r$ interval map g_{-1}, such that $g_{-1}(0) = 0$, $g_{-1}(\frac{1}{2}) = 0$, $\|g'_{-1}\|_\infty = 5$, g_{-1} is affine with slope $\lambda = 5$ on $[0, \frac{1}{6}]$ ($\frac{1}{2} \in g_{-1}([0, \frac{1}{6}]) = [0, \frac{5}{6}]$) and $g_{-1}^{(k)}(\frac{1}{2}) = g_{-1}^{(k)}(\frac{1}{4}) = 0$ for $k = 1, ..., r$. We can assume moreover that $\|(g_{-1})'_{/[\frac{1}{4},1]}\|_\infty < 4$. See Drawing 3.



Drawing 3

One can apply Proposition 1 to the map g_{-1} with $\epsilon = 1$, $S = 1$, $\lambda = 5$, $p = 0$, $p' = \frac{1}{6}$, $q' = \frac{1}{4}$, $q = \frac{1}{2}$ and get a map g_0 (with $\|g'_0\|_\infty \leq 5$) which admits a sequence of horseshoes $(H_{i_1})_{i_1\in\mathbb{N}}$ and sequences of periodic points $(p_{i_1,i_2})_{i_1,i_2\in\mathbb{N}}$ satisfying all the above required conditions (1), (2), (3), (4) for $k=0$.

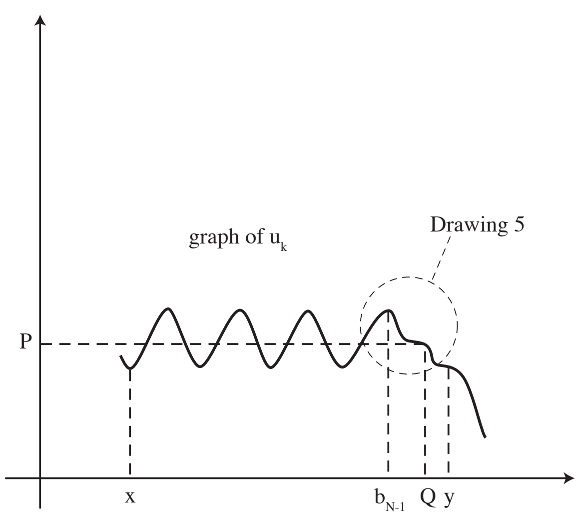
Assume that g_k is already built and define g_{k+1}.

The horseshoes $J_{i_1,...,i_{2k+1}} = (J_1, ..., J_{N_{i_{2k+1}}})$ are linear $(N_{i_{2k+1}}, S_{i_1,...,i_{2k+1}})$ horseshoes for g_k. To get g_{k+1} from g_k we only change g_k on $[x_{i_1,...,i_{2k+1}}, y_{i_1,...,i_{2k+1}}]$ with i_{2k+1} large enough such that the modulus of the r derivative of g_k on $[x_{i_1,...,i_{2k+1}}, y_{i_1,...,i_{2k+1}}]$ is less than $\frac{M_1^2}{2^{k+r}}$. Let us consider one such horseshoe $J_{i_1,...,i_{2k+1}}$ and we denote it $J=(J_1,...,J_N)$ (We also use the simplified notations $H := H_J$, $H' := H_{J'}$, $[x,y] := [x_{i_1,...,i_{2l+1}}, y_{i_1,...,i_{2l+1}}]$ and $S := S_{i_1,...,i_{2k+1}}$).

First step : Recall H' is a linear (N,S) horseshoe for g_k. By applying Lemma 3 there exists a sequence of periodic points $(p_n)_{n\in\mathbb{N}}$ in J' with periods $(SP_n)_{n\in\mathbb{N}}$ for g_k and a sequence of points $(p'_n)_{n\in\mathbb{N}}$ in J' such that :

- the periodic measures γ_{p_n} converge to the measure of maximal entropy of $H_{J'}$;
- the sequence $(g_k(p_n))_{n\in\mathbb{N}}$ is monotone ;
- $g_k^{SP_n}$ is increasing and affine on $[p_n, p'_n]$;
- $J_N \subset g_k^{SP_n}([p_n, p'_n])$.

Let P denote the limit of $(g_k(p_n))_{n\in\mathbb{N}}$. On the last branch $J_N \notin J'$ of the horseshoe J we create a tangency of order r with the horizontal line $\{(x', P),\ x' \in [0,1]\}$ at the point $Q = g_k^{-1}(P) \bigcap J_N$ by applying Lemma 4(1) to g_k on $[b_{N-1}, Q]$ and on $[Q, y]$. We recall $J_{N-1} < b_{N-1} < J_N$ and $f^{(l)}(b_{N-1}) = 0$ for $l = 1, ..., r$. We get a new map u_k. The norm $\|\|_r$ changed only on $[b_{N-1}, y]$ in the following way : $\|(u_k)_{[b_{N-1},y]}\|_r \leq M_1^{-1}\|(g_k)_{[b_{N-1},y]}\|_r$. Indeed $|g_k(b_{N-1}) - g_k(Q)| \leq \|g_k\|_r |b_{N-1} - Q|^r$ and $|g_k(y) - g_k(Q)| \leq \|g_k\|_r |y-Q|^r$. As N is even, the map g_k is non-increasing on J_N (see Definition 2). Remark that u_k is again non-increasing on J_N and the family of intervals $(J_1, ..., J_N)$ is a quasi linear (N,S) horseshoe for u_k.



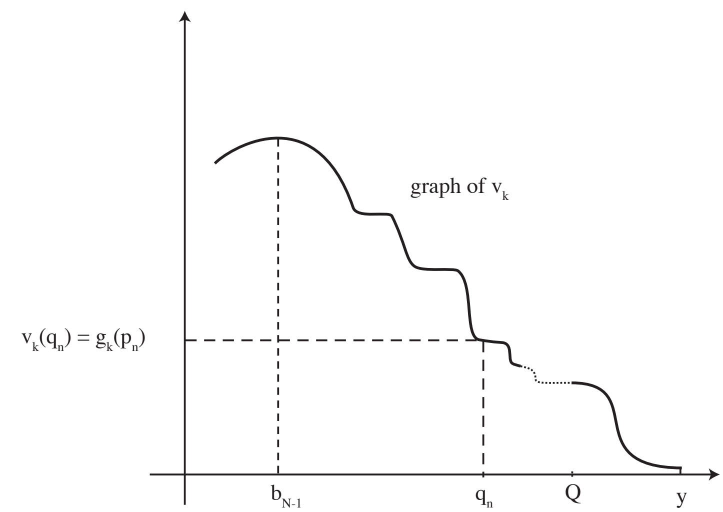
Drawing 4 : first step

Second step : Let us assume the sequence $(g_k(p_n))_{n\in\mathbb{N}}$ is converging non-increasingly. Let q_n denote the point of J_N such that $u_k(q_n) = u_k(p_n) = g_k(p_n)$. Since u_k is non-increasing on

J_N, the sequence q_n is increasing. By extracting a subsequence one can assume that $|q_n - Q| < 2(|q_n - q_{n+1}|)$. Finally we put $q_0 = b_{N-1}$. We create tangencies of order r with the horizontal line $\{(x', g_k(p_n)),\ x' \in [0,1]\}$ at the point q_n by applying again Lemma 4(1) to $[a,b] = [q_n, q_{n+1}]$ and $[d,c] = [u_k(q_{n+1}), u_k(q_n)]$ for all integers n. This can be done by preserving almost the norm $\|\|_r$ of u_k. In fact

$$|u_k(q_{n+1}) - u_k(q_n)| \leq \|(u_k)_{[x,y]}\|_r |q_n - Q|^r < 2^r \|(u_k)_{[x,y]}\|_r |q_n - q_{n+1}|^r$$

We get a new map which is again $\mathcal{C}^r$ with horizontal tangencies of order r at each point q_n. We denote by v_k this new map ; we have $\|(v_k)_{/[x,y]}\|_r < 2^r M_1^{-1} \|(u_k)_{/[x,y]}\|_r < 2^r M_1^{-2} \|(g_k)_{/[x,y]}\|_r < 1/2^k$.



Drawing 5 : second step

If the sequence $(g_k(p_n))_{n\in\mathbb{N}}$ converges increasingly, we can create in the same way horizontal tangencies of order r on $[Q,y]$ accumulating on Q. In the following we assume always $(g_k(p_n))_{n\in\mathbb{N}}$ converges non-increasingly. The rest of the construction is completely similar in the increasing case.

Third step : According to Lemma 3 there exists p'_n such that $v_k^{SP_n}$ is affine on $[p_n, p'_n]$ with slope λ equal to $s(H)^{P_n}$ and $J_N \subset v_k^{SP_n}([p_n, p'_n])$. By applying Proposition 1 with $\epsilon = \frac{1}{2^k}$, "$S = SP_n$", $\lambda = s(H)^{P_n}$, $p = p_n$, $p' = p'_n$, $q' = q_n$ and $q = q_{n+1}$, one can create small horseshoes accumulating on p_n for all integers n to get finally g_{k+1}. We have created in this way a sequence of new horseshoes for each $J = J_{i_1,\dots,i_{2k+1}}$. Coming back to the initial notations this sequence of new horseshoes and their associated intervals are denoted $J_{i_1,\dots,i_{2k+3}}$ and $[x_{i_1,\dots,i_{2k+3}}, y_{i_1,\dots,i_{2k+3}}]$. We also denote by $T_{i_{2k+3}}$ and $N_{i_{2k+3}}$ the integers such that $J_{i_1,\dots,i_{2k+3}}$ is a $N_{i_{2k+3}}$ horseshoe for $g_{k+1}^{T_{i_{2k+3}} S_{i_1,\dots,i_{2k+2}}+1}$. Finally p_n and P_n are respectively denoted by $p_{i_1,\dots,i_{2k+2}}$ and $P_{i_{2k+2}}$. It follows easily from the construction that the new horseshoes $J_{i_1,\dots,i_{2k+3}}$ are $(N_{i_{2k+3}}, S_{i_1,\dots,i_{2k+3}})$ linear horseshoes for g_{k+1} and that the previous horseshoes $J_{i_1,\dots,i_{2l+1}}$ for $l < k$ are modified only on their last branch and therefore are again $(N_{i_{2l+1}}, S_{i_1,\dots,i_{2l+1}})$ quasi linear horseshoes for g_{k+1}. By Proposition 1 the slope of the horseshoe $J_{i_1,\dots,i_{2k+3}}$ is related with the slope of the horseshoe $J_{i_1,\dots,i_{2k+1}}$ in the following way :

$$s(J_{i_1,\dots,i_{2k+3}}) = \frac{s(J_{i_1,\dots,i_{2k+1}})^{P_{i_{2k+2}} T_{i_{2k+3}}}}{i_{2k+3}^2 (3 i_{2k+3}^2 N_{i_{2k+3}})^{r-1}} \tag{9}$$

Notice that the modifications to get g_{k+1} from g_k are made only on the intervals $[x,y] = [x_{i_1,\dots,i_{2k+1}}, y_{i_1,\dots,i_{2k+1}}]$ where the moduli of the derivatives of order $\leq r$ of g_k are less $\frac{M_1^2}{2^{k+r}}$. Therefore

$$\begin{aligned}
\|g_{k+1} - g_k\|_r &\leq \sup_{[x,y]} \|(g_{k+1} - g_k)_{\ /[x,y]}\|_r \\
&\leq \sup_{[x,y]} \left(\|g_{k+1\ /[x,y]}\|_r + \|g_{k\ /[x,y]}\|_r\right) \\
&\leq \sup_{[x,y]} \left(\|g_{k+1\ /[x,y]}\|_r + \frac{M_1^2}{2^{k+r}}\right)
\end{aligned}$$

After the second step, we have $\|v_{k\ /[x,y]}\|_r < \frac{1}{2^k}$. Then by having applied Proposition 1 with $\epsilon = \frac{1}{2^k}$, we get $\|g_{k+1\ /[x,y]}\|_r \leq \frac{1}{2^k} + \|v_{k\ /[x,y]}\|_r < \frac{1}{2^{k-1}}$. We have finally ($M_1 \leq 1$):

$$\|g_{k+1} - g_k\|_r \leq \frac{1}{2^{k-2}}$$

Therefore the maps g_k converge uniformly in $\mathcal{C}^r$ topology to a $\mathcal{C}^r$ map g_∞. The claims (1),(2) and (3) of p.11 follow easily from the construction. Let us check the item (4).

3.3.1 Computation of the entropy

One can remark that the Lyapounov exponent $\lambda_{i_1,\ldots,i_{2k+2}}$ of the periodic point $p_{i_1,\ldots,i_{2k+2}}$ can be written in terms of the slope of $J_{i_1,\ldots,i_{2k+1}}$ in the following way :

$$\lambda_{i_1,\ldots,i_{2k+2}} = \frac{\log s(J_{i_1,\ldots,i_{2k+1}})}{S_{i_1,\ldots,i_{2k+1}}}$$

Now let us compute the topological entropy $h_{i_1,\ldots,i_{2k+1}}$ of the quasi linear $(N_{i_{2k+1}}, S_{i_1,\ldots,i_{2k+1}})$ horseshoe $J_{i_1,\ldots,i_{2k+1}}$ and the entropy $h'_{i_1,\ldots,i_{2k+1}}$ of the linear $(N_{i_{2k+1}} - 1, S_{i_1,\ldots,i_{2k+1}})$ horseshoe $J'_{i_1,\ldots,i_{2k+1}}$:

$$h_{i_1,\ldots,i_{2k+1}} = \frac{\log N_{i_{2k+1}}}{S_{i_1,\ldots,i_{2k+1}}}$$

$$h'_{i_1,\ldots,i_{2k+1}} = \frac{\log(N_{i_{2k+1}} - 1)}{S_{i_1,\ldots,i_{2k+1}}}$$

Since $N_{i_{2k+1}}$ grows exponentially, we have by taking the limit in i_{2k+1} :

$$\lim_{i_{2k+1}\to+\infty} h_{i_1,\ldots,i_{2k+1}} = \lim_{i_{2k+1}\to+\infty} h'_{i_1,\ldots,i_{2k+1}} \tag{10}$$

We have the two following relations according to equation (8) and to equation (9) respectively :

- Each element of $J_{i_1,\ldots,i_{2k+1}}$ spends enough time during the expanding and affine part to get a horseshoe (Equation (8) p.10):

$$\frac{1}{3i_{2k+1}^2} \leq e^{\lambda_{i_1,\ldots,i_{2k}} \times S_{i_1,\ldots,i_{2k}} T_{i_{2k+1}}} \times \frac{1}{2i_{2k+1}^2(3i_{2k+1}^2 N_{i_{2k+1}})^r} \leq e^{\lambda_{i_1,\ldots,i_{2k}}} \frac{1}{3i_{2k+1}^2}$$

  Then we get after a simple computation :

$$\frac{S_{i_1,\ldots,i_{2k+1}} - 2}{S_{i_1,\ldots,i_{2k+1}}} \times \frac{\lambda_{i_1,\ldots,i_{2k}}}{r} - \frac{\log(6i_{2k+1})}{S_{i_1,\ldots,i_{2k+1}}} \leq h_{i_1,\ldots,i_{2k+1}} = \frac{\log N_{i_{2k+1}}}{S_{i_1,\ldots,i_{2k+1}}} \leq \frac{S_{i_1,\ldots,i_{2k+1}} - 1}{S_{i_1,\ldots,i_{2k+1}}} \times \frac{\lambda_{i_1,\ldots,i_{2k}}}{r} - \frac{\log(6i_{2k+1})}{S_{i_1,\ldots,i_{2k+1}}} \tag{11}$$

- The Lyapounov exponent of $p_{i_1,...,i_{2k}}$ decreases with k because we spend more and more time (precisely $P_{i_{2k}}$) in the affine part with slope $\frac{1}{i_{2k-1}^2\left(3i_{2k-1}^2N_{i_{2k-1}}\right)^{r-1}}$ of the horseshoe $J_{i_1,...,i_{2k-1}}$ (Equation (9) p.13) :

$$e^{\lambda_{i_1,...,i_{2k}}\times S_{i_1,...,i_{2k}}} = e^{\lambda_{i_1,...,i_{2k-2}}\times\left((S_{i_1,...,i_{2k-1}}-1)P_{i_{2k}}\right)}\times\left(\frac{1}{i_{2k-1}^2\left(3i_{2k-1}^2N_{i_{2k-1}}\right)^{r-1}}\right)^{P_{i_{2k}}}$$

and we get therefore :

$$\lambda_{i_1,...,i_{2k}} = \frac{(S_{i_1,...,i_{2k-1}}-1)P_{i_{2k}}}{S_{i_1,...,i_{2k}}}\lambda_{i_1,...,i_{2k-2}} - (r-1)h_{i_1,...,i_{2k-1}} \tag{12}$$

Since $T_{i_{2k+1}}$ (and thus $S_{i_1,...i_{2k+1}}$) increases linearly in i_{2k+1} and $P_{i_{2k}}$ goes to infinity when i_{2k} goes to infinity, we obtain by taking successively the limits in $i_{2k+1}, i_{2k}, ..., i_1$ in (11) and (12):

$$\lim_{i_1\to+\infty}...\lim_{i_{2k+1}\to+\infty}h_{i_1,...,i_{2k+1}} = \frac{1}{r}\lim_{i_1\to+\infty}...\lim_{i_{2k}\to+\infty}\lambda_{i_1,...,i_{2k}}$$

$$\lim_{i_1\to+\infty}...\lim_{i_{2k}\to+\infty}\lambda_{i_1,...,i_{2k}} = \lim_{i_1\to+\infty}...\lim_{i_{2k-2}\to+\infty}\lambda_{i_1,...,i_{2k-2}} - (r-1)\lim_{i_1\to+\infty}...\lim_{i_{2k-1}\to+\infty}h_{i_1,...,i_{2k-1}}$$

By putting $\alpha_k := \lim_{i_1\to+\infty}...\lim_{i_{2k+1}\to+\infty}h_{i_1,...,i_{2k+1}}$ and $\beta_k =: \lim_{i_1\to+\infty}...\lim_{i_{2k}\to+\infty}\lambda_{i_1,...,i_{2k}}$, we have according to the previous equations :

$$\left\{\begin{array}{lcc}\alpha_k & = & \frac{\beta_k}{r}\\ \beta_k & = & \beta_{k-1}-(r-1)\alpha_{k-1}\end{array}\right.$$

We get $\beta_k = \beta_{k-1} - (r-1)\alpha_{k-1} = \beta_{k-1} - \frac{r-1}{r}\beta_{k-1} = \frac{\beta_{k-1}}{r}$. Moreover β_0 is the Lyapounov exponent of the fixed point 0, which is equal to $\log 5$. Therefore we conclude that $\beta_k = \frac{\log 5}{r^k}$ and $\alpha_k = \frac{\log 5}{r^{k+1}}$, i.e.

$$\lim_{i_1\to+\infty}...\lim_{i_{2k+1}\to+\infty}h_{i_1,...,i_{2k+1}} = \frac{\log 5}{r^{k+1}}$$

This concludes the proof of Theorem 2.

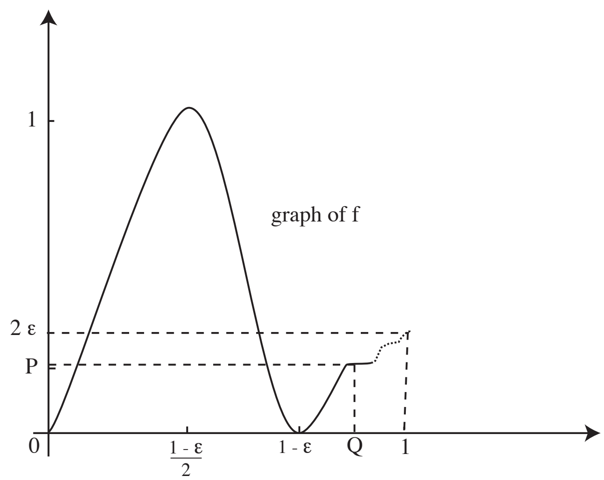
3.4 Proof of Theorem 3

We explain the modification of the previous example (Theorem 2) to get for every $\epsilon > 0$ an example of $\mathcal{C}^r$ interval map $f_{r,\epsilon}$ with $\|f'_{r,\epsilon}\|_\infty \geq 2$ and $h_{sex}(f_{r,\epsilon}) \geq \frac{r\log\|f'_{r,\epsilon}\|_\infty}{r-1} - \epsilon$.

Let f be a $\mathcal{C}^r$ map with the following properties :

- f is increasing on $[0, \frac{1-\epsilon}{2}] \cup [1-\epsilon, 1]$ and decreasing on $[\frac{1-\epsilon}{2}, 1-\epsilon]$;
- $f(0) = f(1-\epsilon) = 0$, $f(\frac{1-\epsilon}{2}) = 1$ and $f(1) = 2\epsilon$;
- f is affine on $[0, \frac{1}{2}-\epsilon]$ (resp. on $[\frac{1}{2}+\epsilon, 1-\frac{3\epsilon}{2}]$) with slope $\frac{2}{1-\epsilon}$ (resp. $-\frac{2}{1-\epsilon}$).

Clearly this map can be $\mathcal{C}^r$ extended such that $2 \leq \|f'\|_\infty \leq \frac{2}{1-\epsilon}$.



Drawing 6

The topological entropy of f is greater than $\log 2$, because $([\frac{\epsilon}{2}, \frac{1}{2}-\epsilon], [\frac{1}{2}+\epsilon, 1-\frac{3\epsilon}{2}])$ is a linear $(2,1)$ horseshoe for f. Let μ be a measure of maximal entropy of this horseshoe, there exists a sequence of periodic points $(p_n)_{n\in\mathbb{N}}$ such that the associated periodic measures are converging to μ and the sequence $(f(p_n))_{n\in\mathbb{N}}$ is converging to a point $P \in [\frac{\epsilon}{2}, 2\epsilon]$. Then one can apply the same process as in the last example to create quasi linear horseshoes accumulating on $q_n = (f_{/[1-\epsilon,1]})^{-1}(p_n)$ with $\lim_{n\to+\infty} q_n = Q \in f([\frac{\epsilon}{2}, \frac{1}{2}-\epsilon])$, such that the new map $f_{r,\epsilon}$ satisfies $u_\omega(\gamma_{p_n}) = \frac{\log \|f'_{r,\epsilon}\|_\infty}{r-1}$ without changing the supremum norm of the first derivative, that is $\|f'_{r,\epsilon}\|_\infty = \|f'\|_\infty$. Then we get :

$$\begin{aligned} h_{sex}(\mu) &\geq h(\mu) + \limsup_{n\to+\infty} u_\omega(\gamma_{p_n}) \\ &\geq \log 2 + \frac{\log \|f'_{r,\epsilon}\|_\infty}{r-1} \\ &\geq \frac{r\log \|f'_{r,\epsilon}\|_\infty}{r-1} + \log(1-\epsilon) \end{aligned}$$

3.5 About the convergence of the sequence $(u_n)_{n\in\mathbb{N}}$

By considering the example of Theorem 2, the examples of S.Newhouse and T.Downarowicz [10] and Conjecture 1, one can wonder if for a $\mathcal{C}^r$ $(r>1)$ map T defined on a smooth compact manifold M of dimension d we have for all $n \in \mathbb{N}$:

$$\sup_{\mu\in\mathcal{M}(M,T)} (u_{n+1}-u_n)(\mu) \leq d\frac{R(T)}{r^{n+1}} \tag{13}$$

If such inequalities hold for some dynamical system (X,T), it implies that the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly converging. Then one can easily prove that $u_{\omega+1} = u_\omega$, i.e. the order of accumulation of (X,T) is at most ω (recall the order of accumulation of (X,T) is the first ordinal α satisfying $u_\beta = u_\alpha$ for all $\beta > \alpha$). In the following we prove the inequalities (13) are false in general :

Theorem 4 *Let $r \geq 2$ be an integer. There exists a $\mathcal{C}^r$ interval map $f_r : [0,1] \to [0,1]$ such that*

$$\sup_{\mu\in\mathcal{M}([0,1],f_r)} (u_2-u_1)(\mu) > \frac{\log \|f'_r\|_\infty}{r^2}$$

PROOF : We modify the construction of the example of Theorem 2. Let us explain the main idea : in the previous construction of g_0 one can create horseshoes with entropy $< \frac{\log \|g_0'\|_\infty}{r}$ but with a slope bigger than in the estimates obtained for the example of Theorem 2. In the next step (construction of g_1), one can use this amount of expansion to get horseshoes with bigger entropy such that $(u_2 - u_1)(\delta_0) > \frac{\log \|g_1'\|_\infty}{r^2}$. We give now more details.

Let $\lambda > r \log 3$. We choose g_{-1} to be a map with three branches of monotonicity (in particular the topological entropy of g_0 is less than $\log 3$) such that g_{-1} is affine with slope $e^\lambda > 1$ on $[0, \frac{5}{6e^\lambda}]$ and $\|g_{-1}'\|_\infty = e^\lambda$. Fix some real a_0 such that $\log 3 < a_0 < \frac{\lambda}{r}$. Then following the construction of g_0, one can build the horseshoes $(J_{i_1})_{i_1 \in \mathbb{N}}$ such that $h_{top}(H_{J_{i_1}}) \leq a_0$ for all integers i_1 and $\lim_{i_1 \to +\infty} h_{top}(H_{J_{i_1}}) \leq a_0$. It follows easily that the tail entropy of g_0 is bigger or equal to a_0.

Let us prove now that the tail entropy of g_0 is equal to a_0. According to the variational principle for the tail entropy (Equation (2)) for the Katok entropy structure $\mathcal{H}^{Kat}_{\frac{1}{2}} = (h_k)_{k \in \mathbb{N}}$, we have

$$h^*(g_0) = \lim_{k \to +\infty} \sup_{\nu \in \mathcal{M}(X,T)} (h - h_k)(\nu) = \lim_{k \to +\infty} \sup_{\nu \in \mathcal{M}_e(X,T)} (h - h_k)(\nu)$$

The last equality follows from the harmonicity of $\mathcal{H}^{Kat}_{\frac{1}{2}}$. Therefore for all $\epsilon > 0$ there exists an integer k and an ergodic measure μ_ϵ such that $(h - h_k)(\mu_\epsilon) > h^*(g_0) - \epsilon$.

We show now that if x is a typical point for μ_ϵ with ϵ small enough then x must visit an interval of the form $[x_n, y_n]$, that is $\mu_\epsilon([x_n, y_n]) > 0$ for some integer n. Let $M := [0,1] - \bigcup_{n \in \mathbb{N}} [x_n, y_n]$. The map $(g_0)_{/M}$ can be extended on $[0,1]$ such that the extension has three branch of monotonicity. Then $\mu_\epsilon(M) = 1$ implies $h(\mu_\epsilon) \leq \log 3$ and therefore $(h - h_k)(\mu_\epsilon) \leq \log 3$: we get a contradiction for ϵ small enough.

Remark also that the topological entropy restricted to $[x_n, y_n]$ is equal to $h_{top}(H_{J_n})$. We conclude that $h(\mu) \leq \sup_{n \in \mathbb{N}} h_{top}(H_{J_n}) = a_0$ and therefore $h^*(g_0) = a_0$.

Now we follow the construction of g_1. According to equation (12) the Lyapounov exponents $(\lambda_{i_1,i_2})_{i_2 \in \mathbb{N}}$ of the periodic points $(p_{i_1,i_2})_{i_2 \in \mathbb{N}}$, whose associated measures converge to the measure of maximal entropy of $H_{J'_{i_1}}$, satisfies $\lambda_{i_1,i_2} = \frac{(T_{i_1}-1)P_{i_2}}{T_{i_1} P_{i_2}} \lambda - (r-1) h_{i_1}$ with $h_{i_1} \to a_0$ when i_1 goes to infinity. Therefore for large i_1 and i_2 we obtain the following estimate : $\lambda_{i_1,i_2} \simeq \lambda - (r-1)a_0 > \frac{\lambda}{r}$. Then, following the construction of g_1, one can build the horseshoes $H_{J_{i_1,i_2,i_3}}$ such that their topological entropy is almost equal to $\frac{\lambda_{i_1,i_2}}{r} > \frac{\lambda}{r^2}$. Arguing as above it is easily seen that the tail entropy of g_1 is equal to $\max(a_0, \frac{\lambda-(r-1)a_0}{r})$. Put $a_1 := \frac{\lambda-(r-1)a_0}{r} > \frac{\lambda}{r^2}$.

In the following we choose $a_0 = \frac{\lambda}{2r-1} < \frac{\lambda}{r}$. Then $a_1 = a_0$. According to Lemma 2 we have

$$u_2(\delta_0) \geq a_0 + a_1 = 2\frac{\lambda}{2r-1}$$

and

$$u_1(\delta_0) \geq a_0 = \frac{\lambda}{2r-1}$$

Moreover according to the variational principle for the tail entropy and the inequality $u_2 \leq 2u_1$, we get :

$$\sup_{\mu \in \mathcal{M}([0,1], g_1)} u_1(\mu) = h^*(g_1) = \frac{\lambda}{2r-1}$$

$$\sup_{\mu \in \mathcal{M}([0,1], g_1)} u_2(\mu) \leq 2h^*(g_1) = \frac{2\lambda}{2r-1}$$

Therefore we have finally $u_2(\delta_0) = \frac{2\lambda}{2r-1}$ and $u_1(\delta_0) = \frac{\lambda}{2r-1}$. Thus since $r \geq 2$,

$$(u_2 - u_1)(\delta_0) = \frac{\log \|g_1'\|_\infty}{2r-1} > \frac{\|g_1'\|_\infty}{r^2}$$

□

Our argument can be easily adapted to get, for every integer $n \geq 1$, a $\mathcal{C}^r$ $(r > 1)$ interval maps f_r with

$$\sup_{\mu \in \mathcal{M}([0,1], f_r)} (u_{n+1} - u_n)(\mu) \geq \frac{\log \|f_r'\|_\infty}{r^{n+1}}$$

Indeed one can modify the map g_n of the example of Theorem 2 in the same way as above such that we have with the notations introduced page 15 :

- α_k does not depend on $k = 0, ..., n$;
- $\alpha_n = \frac{\beta_n}{r}$;

Moreover we can again ensure that :

- $\beta_0 = \log \|g_n'\|_\infty$;
- $h^*(g_n) = \alpha_0$;
- $\beta_k = \beta_{k-1} - (r-1)\alpha_k$ for all $k = 0, ..., n$.

These properties imply easily that $\alpha_0 = \frac{\log \|g_n'\|_\infty}{r+n(r-1)}$. Moreover, by arguing as in the proof of Theorem 4 we get for all $k = 0, ..., n+1$:

$$\sup_{\mu \in \mathcal{M}([0,1], g_n)} u_k(\mu) = u_k(\delta_0) = \frac{k \log \|g_n'\|_\infty}{r + n(r-1)}$$

Then one can deduce an exemple of $\mathcal{C}^r$ interval map such that the sequence $(u_n)_{n\in\mathbb{N}}$ does not converge uniformly. Indeed one can modify the example of Theorem 3 in the following way. With the notations of the proof of Theorem 3 one can follow the above construction to ensure that $u_k(\gamma_{p_n}) = \frac{k \log \|g'\|_\infty}{r+(n(r-1))}$ for all $n \in \mathbb{N}$ and for all $k = 0, ..., n+1$. We get then for all $k \in \mathbb{N}$:

$$\sup_{\mu \in \mathcal{M}([0,1], g)} \left(\sup_{l \in \mathbb{N}} u_l - u_k \right)(\mu) = \frac{\log \|g'\|_\infty}{r-1}$$

In [5] the authors prove that any countable ordinal can be realized as the order of accumulation of a zero-dimensional dynamical system. The proof uses strongly the non-uniform convergence of subsequences $(u_{\alpha_n})_{n\in\mathbb{N}}$ of the transfinite sequence. Following the strategy of [5] we hope to prove the following conjecture :

Conjecture 2 *Let $r \in \mathbb{N}$ and let α be a countable ordinal. There exists a $\mathcal{C}^r$ interval map with order of accumulation equal to α.*

3.6 Higher dimensional examples

M.Boyle and T.Downarowicz proved the following formula for the symbolic extension entropy of a direct product :

Theorem 5 *(Theorem 3.2 of [2]) Let $T : X \to X$ and $S : Y \to Y$ be two dynamical systems with finite topological entropy, then :*

$$h_{sex}(T \times S) = h_{sex}(T) + h_{sex}(S) \tag{14}$$

Moreover for all ordinal α and for all $(\mu, \nu) \in \mathcal{M}(X,T) \times \mathcal{M}(Y,S)$,

$$u_\alpha(\mu \times \nu) \geq u_\alpha(\mu) + u_\alpha(\nu) \tag{15}$$

By considering product of the previous examples of interval maps, we get (non-invertible) examples of any dimension with large symbolic extension entropy :

Corollary 1 *Let $r, d \in \mathbb{N}^*$. There exists a $\mathcal{C}^r$ map $T_r : [0,1]^d \to [0,1]^d$ fixing $(0, ..., 0)$, such that for all integers $n \geq 1$:*

$$u_n(\delta_{(0,\ldots,0)}) = \left(\sum_{k=1}^{n} \frac{1}{r^k}\right) \log \|DT_r\|_\infty > 0$$

In particular :

- *if $r > 1$, then $u_\omega(\delta_{(0,\ldots,0)}) = \frac{\log \|DT_r\|_\infty}{r-1}$;*
- *if $r = 1$, then $u_\omega \equiv +\infty$ and therefore T_1 does not admit symbolic extensions.*

PROOF : Such an example has been already built for $d = 1$ (Theorem 2). Let g be such an interval map. We denote $g_d := \underbrace{g \times ... \times g}_{d\times}$. By the inequality (15) we have for all integers $n \geq 1$:

$$u_n^{g_d}(\delta_{(0,\ldots,0)}) \geq d u_n^g(\delta_0)$$

Also $\|Dg_d\|_\infty = \|Dg\|_\infty$. This concludes the proof.

□

Similarly by combining Theorem 5 and Theorem 3 we obtain :

Corollary 2 *Let $r \geq 2, d \geq 1$ be integers. For all $\epsilon > 0$ there exists a $\mathcal{C}^r$ map $T_{r,\epsilon} : [0,1]^d \to [0,1]^d$ with $\|DT_{r,\epsilon}\|_\infty \geq 2$ such that :*

$$h_{sex}(T_{r,\epsilon}) \geq \frac{dr \log \|DT_{r,\epsilon}\|_\infty}{r-1} - \epsilon$$

Appendix

We recall the definition of Katok's entropy. Let $0 < \lambda < 1$. Let $\epsilon > 0$ and μ be an ergodic measure.

$$h_\lambda^{Kat}(\mu, \epsilon) := \limsup_n \frac{\log \min \left\{\sharp C \mid \mu(\bigcup_{x \in C} B(x, \epsilon, n)) > \lambda\right\}}{n}$$

We extend this definition by harmonicity on the convex set of invariant measures. Katok proved in [12] that for any $0 < \lambda < 1$ and any invariant measure μ,

$$\lim_{\epsilon \to 0} h_\lambda^{Kat}(\mu, \epsilon) = h(\mu)$$

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