To appear in *Dynamical Systems: An International Journal*
Vol. 00, No. 00, Month 20XX, 1–18

Usc/fibered entropy structure and applications.

David Burguet,
LPMA Universite Paris 6
4 Place Jussieu
75252 Paris Cedex 05
david.burguet@upmc.fr

(*Received 00 Month 20XX; final version received 00 Month 20XX*)

For a given metrizable space X we study continuity properties of the entropy as function not only of the measure but also of the dynamical system on X. We introduce the notion of robust tail entropy, which implies upper semicontinuity of the topological entropy but also stability of measures of maximal entropy (when the topological entropy is continuous). This gives, inter alia, simple proofs of results of Misiurewicz and Raith for multimodal interval maps. We also consider fibered entropy structures which allow us to investigate the symbolic extensions of (smooth) skew-product systems.

Keywords: entropy, maximal measures, symbolic extensions, asymptotic h-expansiveness

1. Introduction

Downarowicz [9] has introduced a master entropy invariant, called entropy structure, which recovers all previously known entropy quantities. An entropy structure is a non-decreasing sequence of nonnegative functions on the set of Borel invariant probability measures of a given dynamical system, which is converging pointwisely to the measure theoretical entropy. The convergence of this sequence reflects how entropy appears at smaller and smaller scales. In particular the default of uniform convergence of an entropy structure is known to be the tail entropy introduced by Misiurewicz to bound from above the default of upper semicontinuity of the measure theoretical entropy [18]. Also the topological symbolic extension entropy, which is the infimum of the topological entropy of all symbolic extensions of the system (topological extensions which are subshifts with a finite alphabet), is characterized by the entropy structure [1].

In this short note we introduce the concept of upper semicontinuous entropy structure of a compact metrizable space X. It is a sequence of functions defined at any Borel probability measure, which is invariant for some continuous dynamics on X, and upper semicontinuous on this set. Moreover its restriction to the set of invariant measures of any given system on X defines an entropy structure of this system. This upper semicontinuous entropy structure allows us to analyse entropy of general families of dynamical systems. We derive from it a notion of robust asymptotic h-expansiveness, which implies upper semicontinuity of the topological entropy and of the set of measures of maximal entropy when the topological entropy is also (lower semi-) continuous. We also define a fibered Newhouse local entropy to study the symbolic extensions of a skew-product.

Using this general framework we recover some known properties of entropy in the context of multimodal interval maps and of C^∞ smooth maps on a manifold. We also deduce some new results in the theory of symbolic extensions : any skew-product of C^r interval maps with $r > 1$ have symbolic extensions. Also on any manifold any compact family of C^∞ maps may be encoded using a unique subshift by preserving the entropy of invariant measures.

2. Upper semicontinuous entropy structure

In the whole paper X will denote a compact metrizable space. We consider the topological space $C(X)$ of continuous dynamical systems, $T : X \to X$, with the usual C^0 topology. The set $\mathcal{M}(X)$ of Borel probability measures on X is endowed with the weak $*$-topology (then $\mathcal{M}(X)$ is compact). For any $T \in C(X)$ we let $\mathcal{M}(X,T)$ be the closed subset of $\mathcal{M}(X)$ of T-invariant measures. We also consider the closed subset $\mathcal{M}*C(X)$ of $\mathcal{M}(X)\times C(X)$ given by $\mathcal{M}*C(X) := \{(\mu,T),\ \mu \in \mathcal{M}(X,T)\}$.

For $(\mu,T) \in \mathcal{M} * C(X)$, the Kolmogorov-Sinai entropy $h(\mu,T)$ of (μ,T) is the supremum, over all finite Borel partitions P, of $h_P(\mu,T) := \inf_n \frac{1}{n} H_\mu(P^n)$, where P^n is the joined partition $P^n = \bigvee_{k=0}^{n-1} T^{-k}P$ and $H_\mu(P^n) := \sum_{A\in P^n} -\mu(A)\log\mu(A)$. When $(P_k)_k$ is a refining sequence of partitions (i.e. $P_{k+1} > P_k$ for all k) whose diameter goes to zero then the functions h_{P_k} is converging pointwisely to h on $\mathcal{M} * C(X)$. Recall also that for $l > k$ we have $(h_{P_l} - h_{P_k})(\mu,T) = h_{P_l|P_k}(\mu,T) := \inf_n \frac{1}{n} H_\mu(P_l^n|P_k^n)$, with $H_\mu(P_l^n|P_k^n) = \sum_{A\in P_k^n} \mu(A) H_{\mu_A}(P_l^n)$ and $\mu_A = \frac{\mu(A\cap .)}{\mu(A)}$ for any $(\mu,T) \in \mathcal{M} * C(X)$ (see [24]).

The sequence $(h_{P_k})_k$ is in general not an entropy structure [9] : it does not reflect correctly the topological properties of entropy, e.g. it does not allow to recover the topological tail entropy and symbolic extension entropy.

If X is zero-dimensional and $(P_k)_k$ are clopen partitions, then $(h_{P_k})_k$ defines an entropy structure for any $T \in C(X)$ when restricted to $\mathcal{M}(X,T)$. Moreover $h_{P_l} - h_{P_k}$ is upper semicontinuous on $\mathcal{M}(X,T)$ for any $l > k$. In fact $h_{P_l} - h_{P_k}$ is upper semicontinuous on the whole set $\mathcal{M}*C(X)$. Indeed, for any n and k, for any $A_0, ..., A_{n-1} \in P_k$ and for any (μ,T), we have $\lim_{(\nu,S)\to(\mu,T)} \nu(A_0\cap ...\cap S^{-(n-1)}A_{n-1}) = \mu(A_0\cap ...\cap T^{-(n-1)}A_{n-1})$: for S close enough to T the following equality holds $A_0\cap ...\cap S^{-(n-1)}A_{n-1} = A_0\cap ...\cap T^{-(n-1)}A_{n-1}$ and then as this last set is clopen the function $\mu \mapsto \mu(A_0 \cap ... \cap T^{-(n-1)}A_{n-1})$ is continuous on $\mathcal{M}(X)$. Consequently the function $h_{P_l} - h_{P_k} = \inf_n \frac{1}{n} H_.(P_l^n|P_k^n)$ is upper semicontinuous on $\mathcal{M} * C(X)$ as an infimum of continuous functions.

Definition 2.1: A nondecreasing sequence of nonnegative functions $(h_k)_k$ on $\mathcal{M} * C(X)$ is called an upper semicontinuous entropy structure of X, when the restriction of $(h_k)_k$ to $\mathcal{M}(X,T)$ defines an entropy structure for any $T \in C(X)$ and when moreover $h_l - h_k$ are upper semicontinuous for any $l > k$.

Recall a topological extension is principal when it preserves the entropy of measures. It is known that any topological system admits a principal zero-dimensional extension [1, 15],[11]. For a general topological system (Z,R) an entropy structure $(g_l)_l$ is a nondecreasing sequence of nonnegative functions converging pointwisely to the measure theoretical entropy, such that for any principal zero dimensional extension $\pi : (X,T) \to (Z,R)$ the lifted functions $(g_l(\pi.))_l$ satisfies $\lim_{k,l\to+\infty} \sup_{\mu\in\mathcal{M}(X,T)} |g_l(\pi.) - h_{P_k}|(\mu) = 0$, where $(P_k)_k$ is some refining sequence of clopen partitions of X whose diameter goes to zero as above.

Using the entropy with respect to a family of continous functions introduced in [9] we prove now that upper semicontinuous entropy structures always exist.

Proposition 2.2: *Any compact metrizable space X admits an upper semicontinuous entropy structure.*

We first recall the definition of the entropy by Downarowicz with respect to a family of continous functions. For a continuous map $f : X \to [0,1]$ we let A_f be the partition of $X \times [0,1]$ given by the three disjoint subsets $\{(x,t),\ f(x) > t\}$, $\{(x,t),\ f(x) < t\}$ and $\{(x,t),\ f(x) = t\}$. Then for any finite family $\mathcal{F}$ of such maps, we let $A_{\mathcal{F}}$ be the joined partition $\bigvee_{f\in\mathcal{F}} A_f$. The entropy $h_{\mathcal{F}}(\mu,T)$ of $(\mu,T) \in \mathcal{M} * C(X)$ with respect to $\mathcal{F}$ is then defined as the measure theoretical entropy of the product of μ with the Lebesgue measure λ of the unit interval for the product of T with the identity map $Id_{[0,1]}$ of the interval with respect to the partition $A_{\mathcal{F}}$:

$$h_{\mathcal{F}}(\mu,T) := h_{A_{\mathcal{F}}}(\mu \times \lambda, T \times Id_{[0,1]}).$$

By choosing suitable partitions of unity one may fix a nondecreasing sequence of finite families $(\mathcal{F}_k)_k$ such that diameter of the partitions $(A_{\mathcal{F}_k})_k$ of $X \times [0,1]$ goes to zero when k goes to infinity. In particular the entropy h is the nondecreasing pointwise limit of $(h_k)_k = (h_{\mathcal{F}_k})_k$.

In [9] it is proved that the restriction of $(h_k)_k$ to $\mathcal{M}(X,T)$ is an entropy structure for any $T \in C(X)$. As the sequence $(\mathcal{F}_k)_k$ is nondecreasing we may write the difference $h_l - h_k$ as

$$(h_l - h_k)(\mu,T) := \inf_n \frac{1}{n} H_{\mu\times\lambda}(A^n_{\mathcal{F}_l} | A^n_{\mathcal{F}_k}).$$

Thus the upper semicontinuity of $h_l - h_k$ follows from the following lemma :

Lemma 2.3: *For any n, k and any $B_0, ..., B_{n-1} \in A_{\mathcal{F}_k}$, the following map is continuous:*

$$\begin{aligned} &\mathcal{M} * C(X) \to [0,1], \\ &(\mu,T) \mapsto (\mu \times \lambda)\left(B_0 \cap T^{-1}B_1 \cap ... \cap T^{-(n-1)}B_{n-1}\right). \end{aligned}$$

Proof. Fix $(\mu,T) \in \mathcal{M} * C(X)$. The iterated partition $(A_{\mathcal{F}_k})^n$ is just the partition $A_{\mathcal{F}_k^n}$ where $\mathcal{F}_k^n = \{f \circ T^k,\ 0 \le k \le n-1 \text{ and } f \in \mathcal{F}_k\}$. Thus the boundary of $(A_{\mathcal{F}_k})^n$ is contained in the graphs of the functions $f, f\circ T, ..., f\circ T^{n-1}$ with $f \in \mathcal{F}_k$, which have zero $(\mu\times\lambda)$-measure. In particular there is a closed neighborhood O of $B_0 \cap ... \cap T^{-(n-1)}B_{n-1}$ with almost the same $(\mu\times\lambda)$-measure. Now for any S which is C^0-close enough to T we have $B_0 \cap ... \cap S^{-(n-1)}B_{n-1} \subset O$. Finally we get

$$\begin{aligned} \limsup_{(\nu,S)\to(\mu,T)} (\nu\times\lambda)(B_0 \cap ... \cap S^{-(n-1)}B_{n-1}) &\le \limsup_{\nu\to\mu} (\nu\times\lambda)(O), \\ &\le (\mu\times\lambda)(O) \simeq (\mu\times\lambda)(B_0 \cap ... \cap T^{-(n-1)}B_{n-1}). \end{aligned}$$

This proves the upper semicontinuity. But the complementary set of $B_0 \cap T^{-1}B_1 \cap ... \cap T^{-(n-1)}B_{n-1}$ may be written as a finite disjoint union of intersections of this form, so that in fact we have proved the continuity. □

For any compact subset D of $C(X)$, we may consider the skew-product system S_D on

$X \times D$ defined by $S_D(x,T) = (Tx,T)$. Then, when considering the functions of $\mathcal{F}_k$ as functions on $X \times D$, the sequence $(h_{\mathcal{F}_k})_k$ defines an entropy structure on $\mathcal{M}(S_D, X \times D)$ of S_D, but also an entropy structure of T, when restricted to $\mathcal{M}(X,T)$. Moreover these functions have upper semicontinuous differences. In the next section we investigate the entropy structure of general skew-product systems.

3. Fibered Entropy structure for skew-products

We consider now a skew-product continuous map S on $X \times Y$ (with Y another compact metrizable space) over T, i.e. there exists a continuous family $(S_x)_{x\in X}$ of $C(Y)$ such that $S(x,y) = (T(x), S_x(y))$ for any $(x,y) \in X \times Y$. Let $(\mathcal{F}_k^X)_k$ and $(\mathcal{F}_k^Y)_k$ be nondecreasing sequences of functions, as defined above, for the compact sets X and Y respectively. We also denote by $(h_k^X)_k = (h_{\mathcal{F}_k^X})_k$ and $(h_k^Y)_k = (h_{\mathcal{F}_k^Y})_k$ the associated entropy structures. We then consider for any k the family $\mathcal{G}_k := \{f \times g : X \times Y \to [0,1]^2,\ (f,g) \in \mathcal{F}_k^X \times \mathcal{F}_k^Y\}$ and the associated partition $A_{\mathcal{G}_k}$ of $(X \times [0,1]) \times (Y \times [0,1])$ given by $A_{\mathcal{G}_k} = \left(A_{\mathcal{F}_k^X} \times (Y \times [0,1])\right) \vee \left((X \times [0,1]) \times A_{\mathcal{F}_k^Y}\right)$. Finally we let for any $\mu \in \mathcal{M}(X \times Y, S)$:

$$h_k(\mu, S) := h_{A_{\mathcal{G}_k}}(\mu \times \lambda^2, S \times Id^2_{[0,1]}).$$

Again any element of $A^n_{\mathcal{G}_k}$ has boundaries with zero measure for $\mu \times \lambda^2$. It follows then from Lemma 7.1.2 of [9] that $(h_k)_k$ defines an entropy structure of $(X \times Y, S)$. Let π be the factor map from $X \times Y$ to X and let $\mu := \int \mu_x d\pi\mu(x)$ be the desintegration of μ with respect to π (see [10] p.35 for the notion of disintegration of a measure). For any $(t,s) \in [0,1]^2$ and $k \in \mathrm{N}$ we let $A^t_{\mathcal{F}_k^X}$ be the partition of $A_{\mathcal{F}_k^X} \cap (X \times \{t\})$ of $X \times \{t\} \simeq X$ and $A_{\mathcal{G}_k^{(t,s)}}$ be the partition of $A_{\mathcal{G}_k} \cap (X \times Y \times \{(t,s)\})$ of $X \times Y \times \{(t,s)\} \simeq X \times Y$.

Then we have for all S-invariant measure μ, by Abramov-Rokhlin formula (see Theorem 2.6.3 of [10]),

$$\begin{aligned} h_k(\mu,S) &= h_k^X(\pi\mu, T) + h_{A_{\mathcal{G}_k}|A_{\mathcal{F}_k^X}\times Y\times[0,1]}\left(\mu \times \lambda^2, S \times Id^2_{[0,1]}\right), \\ &= h_k^X(\pi\mu, T) + \int_{[0,1]^2} h_{A^{(t,s)}_{\mathcal{G}_k}|A^t_{\mathcal{F}_k^X}\times Y}(\mu, S)\, d\lambda^2(s,t). \end{aligned}$$

We have for any $(t,s) \in [0,1]^2$ with $\mathcal{B}^X$ the Borel σ-algebra of X and $h_{.|\pi^{-1}\mathcal{B}^X}$ the conditional entropy with respect to $\pi^{-1}\mathcal{B}^X$ (see [10]) :

$$h_{A^{(t,s)}_{\mathcal{G}_k}|A^t_{\mathcal{F}_k^X}\times Y}(\mu, S) \geq h_{A^{(t,s)}_{\mathcal{G}_k}|\pi^{-1}\mathcal{B}^X}(\mu, S).$$

and by applying again Abramov-Rokhlin formula

$$h_{A^{(t,s)}_{\mathcal{G}_k}|\pi^{-1}\mathcal{B}^X}(\mu, S) = \lim_n \frac{1}{n} \int H_{\mu_x}\left((A^{(t,s)}_{\mathcal{G}_k})^n\right) d\pi\mu(x).$$

Finally recall that if we let $h^{fib}(\mu,S) = \sup_P h_{P|\pi^{-1}\mathcal{B}^X}(\mu,S)$ where the supremum is taken over all finite Borel partitions P of $X \times Y$ then we have

$$h(\mu,S) = h(\pi\mu,T) + h^{fib}(\mu,S). \tag{1}$$

By analogy with the theory of entropy structure of Downarowicz the sequence $(h_k^{fib})_k := \left(h_{A_{\mathcal{G}_k}|\pi^{-1}\mathcal{B}^X}\right)_k = \left(\int_{[0,1]^2} h_{A_{\mathcal{G}_k}^{(t,s)}|\pi^{-1}\mathcal{B}^X} d\lambda^2(t,s)\right)_k$ is said to be a fibered entropy structure. It converges pointwisely to h^{fib} and reflects somehow the emergence of entropy in the fibers at arbitrarily small scales.

We will now relate this fibered entropy structure with an analogy of Newhouse's entropy (see [22],[9]) for fibered entropy, which is convenient to estimate for smooth dynamical systems. As $T\mu_x = \mu_{Tx}$ for $\pi\mu$-almost all x we get in fact by subadditive Kingsman's theorem that $\lim_n \frac{1}{n} H_{\mu_x}((A_{\mathcal{G}_k}^{(t,s)})^n)$ exists for $\pi\mu$-almost all x and $\lim_n \frac{1}{n}\int H_{\mu_x}((A_{\mathcal{G}_k}^{(t,s)})^n) d\pi\mu(x) = \int \lim_n \frac{1}{n} H_{\mu_x}((A_{\mathcal{G}_k}^{(t,s)})^n) d\pi\mu(x)$.

We define now the "fibered Newhouse local entropy" of an ergodic S-invariant measure μ. For $\delta > 0$ and $n \in \mathrm{N}$, a set $E \subset X \times Y$ is said (n,δ)-separated when for any $x \neq y$ in E there is $0 \leq k < n$ with $d(S^k x, S^k y) \geq \epsilon$. Then for $x \in X$, $y \in Y$, $\epsilon > 0$ and $F_x \subset \pi^{-1}(x)$ a Borel set, we let:

$$\begin{aligned}
H^x(n,\delta|y,F_x,\epsilon) &:= \log\max\Big\{\sharp E \ : \ E \subset F_x \cap \bigcap_{k=0}^{n-1} S^{-k}\left(B(S^k(x,y),\epsilon)\right) \\
&\qquad\qquad \text{and } E \text{ is a } (n,\delta)\text{-separated set}\Big\},\\
H^x(n,\delta|F_x,\epsilon) &:= \sup_{y\in F_x} H^x(n,\delta|y,F_x,\epsilon),\\
h^x(\delta|F_x,\epsilon) &:= \limsup_{n\to+\infty} \frac{1}{n} H^x(n,\delta|F_x,\epsilon),\\
h^x(Y|F_x,\epsilon) &:= \lim_{\delta\to 0} h^x(\delta|F_x,\epsilon),
\end{aligned}$$

For any Borel set $F \subset X \times Y$ we denote by F_x the intersection $F_x = F \cap \pi^{-1}(x)$. Finally we let

$$h_{New}^{fib}(\mu,S,\epsilon) := \lim_{\alpha\to 1} \inf_{F,\ \mu(F)>\alpha} \int_X h^x(Y|F_x,\epsilon) d\pi\mu(x).$$

For non ergodic measures ν, we consider the harmonic extension, i.e. with $\nu = \int \mu dM_\nu(\mu)$ being the ergodic decomposition of ν, we let :

$$h_{New}^{fib}(\nu,S,\epsilon) = \int h_{New}^{fib}(\mu,S,\epsilon) dM_\nu(\mu).$$

Lemma 3.1: *For any k there is l such that for all $\mu \in \mathcal{M}(X\times Y,S)$, with ϵ_k being the diameter of $A_{\mathcal{G}_k} \cap X \times Y \times [0,1]^2$,*

$$\left(h^{fib} - h_k^{fib}\right)(\mu,S) \leq h_{New}^{fib}(\mu,S,\epsilon_k)$$

This lemma is the fibered version of Theorem 1.1 in [22] and the proof follows the same (standard) lines.

Proof. It is enough to consider ergodic measures μ as the left and right members are harmonic. In the definition of h^{fib} one may consider partitions P with boundary of null μ-measure. Let $\gamma > 0$. Let P be such a partition with $h^{fib}(\mu, S) \simeq h_{P|\pi^{-1}\mathcal{B}^X}(\mu, S)$. We fix a set F with $\mu(F) > \alpha > 1 - \frac{\gamma}{\log \sharp P}$. Let $\delta' > 0$ be such that $\mu(\partial^{\delta'} P) < \frac{\gamma}{\log \sharp P}$ where $\partial^{\delta'} P$ denotes the closed δ'-neighborhood of ∂P. Let also $\delta = \frac{1}{2}\min_{A \neq B \in P} d(A, B \setminus \partial^{\delta'} P) > 0$. By the ergodic theorem $\left(\frac{\sharp\{0 \leq k \leq n,\ S^k z \in \partial^{\delta'} P\}}{n}\right)_n$ goes to $\mu(\partial^{\delta'} P)$ for μ-almost every z. Now when $S^k z \notin \partial^{\delta'} P$ there exists a unique element A_k of P such that $S^k y \in A_k$ for all y in the dynamical ball $B_n(S, z, \delta) := \bigcap_{0 \leq l < n} S^{-l} B(S^l z, \delta)$ with $n > k$. Consequently there is a subset F' of F with μ-measure larger than α, such that for any $z \in F'$ the dynamical ball $B_n(S, z, \delta)$ intersects only $e^{\gamma n}$ elements of P^n for large n. Now one only needs to show that for all x, t, s:

$$\limsup_n \frac{1}{n} H_{\mu_x}\left(P^n | (A_{\mathcal{G}_k}^{(t,s)})^n\right) \leq h^x(\delta | F'_x, \epsilon_k) + \gamma + \mu_x(X \setminus F') \log \sharp P. \tag{2}$$

Indeed we obtain then by integrating over $X \times [0,1]^2$:

$$h^{fib}(\mu, S) - h_{A_{\mathcal{G}_k}|\pi^{-1}\mathcal{B}^X}(\mu, S) \lesssim \int_X h^x(Y | F'_x, \epsilon_k) d\pi\mu(x) + 2\gamma.$$

By taking the infimum over F and then the limit when α goes to one we conclude that $h^{fib}(\mu) - h(\mu, A_{\mathcal{G}_k} | \pi^{-1}\mathcal{B}^X) \leq h^{fib}_{New}(\mu, \epsilon_k)$.

We go back to the proof of (2), which follows from the following inequalities, by letting Q be the partition $Q = \{F', X \setminus F'\}$:

$$\begin{aligned} H_{\mu_x}\left(P^n | (A_{\mathcal{G}_k}^{(t,s)})^n\right) &\leq H_{\mu_x}(Q) + H_{\mu_x}\left(P^n | (A_{\mathcal{G}_k}^{(t,s)})^n \vee Q\right), \\ &\leq \log 2 + \mu_x(X \setminus F') H_{\mu_x^{X \setminus F'}}\left(P^n | (A_{\mathcal{G}_k}^{(t,s)})^n\right) + \mu_x(F') H_{\mu_x^{F'}}\left(P^n | (A_{\mathcal{G}_k}^{(t,s)})^n\right), \\ &\leq \log 2 + \mu_x(X \setminus F') n \log \sharp P + \sup_{A \in (A_{\mathcal{G}_k}^{(t,s)})^n} \log \sharp\left\{B \in P^n,\ B \cap F'_x \neq \emptyset \text{ and } B \subset A\right\}. \end{aligned}$$

Then if E is a (n, δ) separated set with maximal cardinality in F'_x then the balls $B_n(S, z, \delta)$ for $z \in E$ are covering F'_x. Therefore we have from the choice of F':

$$H_{\mu_x}\left(P^n | (A_{\mathcal{G}_k}^{(t,s)})^n\right) \leq \log 2 + \mu_x(X \setminus F') n \log \sharp P + \gamma n + H^x(n, \delta | F'_x, \epsilon_k).$$

Finally by taking the limsup in n we get

$$\limsup_n \frac{1}{n} H_{\mu_x}\left(P^n | (A_{\mathcal{G}_k}^{(t,s)})^n\right) \leq h^x(Y | F'_x, \epsilon_k) + \gamma + \mu_x(X \setminus F') \log \sharp P.$$

□

An affine upper semicontinuous map $E : \mathcal{M}(X, T) \to \mathrm{R}$ is called a superenvelope of (X, T) if and only if $\lim_k \widetilde{E - h_k} = E - h$, where $\tilde{f}$ denotes the upper semicontinuous envelope of f, that is $\tilde{f}(\mu) = \limsup_{\nu \to \mu,\ \nu \in \mathcal{M}(X,T)} f(\nu)$. We define fibered superenvelope as affine upper semicontinuous maps $E : \mathcal{M}(X \times Y, S) \to \mathrm{R}$ with $\lim_k \widetilde{E - h_k^{fib}} = E - h^{fib}$.

Corollary 3.2: *When E^X is a superenvelope of (X, T) and E^{fib} is a fibered superenvelope then the sum $E^X + E^{fib}$ defines a superenvelope of the skew-product $(X \times Y, S)$.*

Proof. This follows from the following series of inequalities :

$$\begin{aligned}
E^X + E^{fib} - h_k &\leq E^X + E^{fib} - h_k^X - h_k^{fib},\\
E^X + \widetilde{E^{fib}} - h_k &\leq \widetilde{E^X - h_k^X} + \widetilde{E^{fib} - h_k^{fib}},\\
\lim_k E^X + \widetilde{E^{fib}} - h_k &\leq E^X - h^X + E^{fib} - h^{fib},\\
&\leq E^X + E^{fib} - h.
\end{aligned}$$

□

4. Robust tail entropy

We first recall the tail entropy of a topological system $T \in C(X)$. Fix $\epsilon > 0$. Given $x \in X$, $n \in \mathrm{N}$, denote the n-step dynamical ball $B_n(T, x, \epsilon)$ consisting of all such points $y \in X$ that

$$d(T^i y, T^i x) < \epsilon, \quad i = 0, 1, ..., n-1.$$

Following Bowen we define the ϵ-tail entropy $h^*(T, \epsilon)$ as follows :

$$h^*(T, \delta, \epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} r_n(T, B_n(T, x, \epsilon), \delta),$$

where $r_n(T, G, \delta)$ denotes the maximal cardinality of a (n, δ)-separated set inside a subset G of X,

$$h^*(T, \epsilon) = \lim_{\delta\to 0} h^*(T, \delta, \epsilon).$$

The tail entropy $h^*(T)$ of T is then given by

$$h^*(T) = \lim_{\epsilon\to 0} h^*(T, \epsilon).$$

Let $\mathcal{C} \subset C(X)$ be a topological space of continuous dynamical systems on X such that the topology on $\mathcal{C}$ is stronger than the C^0 topology. We introduce a new quantity, $h^*_{\mathcal{C}}(T)$ for $T \in \mathcal{C}$, which estimates the tail entropy of S for $S \in \mathcal{C}$ arbitrarily close to T in the topology of $\mathcal{C}$:

$$h^*_{\mathcal{C}}(T) = \lim_{\epsilon\to 0} \limsup_{S\to T} h^*(S, \epsilon) = \lim_{\epsilon\to 0} \inf_U \sup_{S\in U} h^*(S, \epsilon),$$

where the infimum holds over all $\mathcal{C}$-neighborhoods U of T. As the limit in ϵ is also an infimum we may invert the limit and the infimum in the above definition so that:

$$h^*_{\mathcal{C}}(T) = \inf_U \lim_{\epsilon\to 0} \sup_{S\in U} h^*(S, \epsilon).$$

Finally we define the tail entropy $h^*_{\mathcal{C}}$ of $\mathcal{C}$ as:

$$h^*_{\mathcal{C}} = \sup_{T\in\mathcal{C}} h^*_{\mathcal{C}}(T).$$

As the map $T \mapsto \limsup_{S \to T} h^*(S, \epsilon)$ is upper semicontinuous on $\mathcal{C}$ for any ϵ the function $T \mapsto h^*_{\mathcal{C}}(T)$ is also upper semicontinuous as an infimum of upper semicontinuous functions. Moreover by Proposition 2.4 of [1] we can invert the supremum in T with the limit in ϵ when $\mathcal{C}$ is compact, that is :

$$\sup_{T \in \mathcal{C}} \lim_{\epsilon \to 0} \limsup_{S \to T} h^*(S, \epsilon) = \lim_{\epsilon \to 0} \sup_{T \in \mathcal{C}} \limsup_{S \to T} h^*(S, \epsilon).$$

By definition the left member is just $h^*_{\mathcal{C}}$ whereas in the right member the term $\sup_{T \in \mathcal{C}} \limsup_{S \to T} h^*(S, \epsilon)$ is just equal to $\sup_{S \in \mathcal{C}} h^*(S, \epsilon)$. Thus we have for compact spaces $\mathcal{C}$:

$$h^*_{\mathcal{C}} = \lim_{\epsilon \to 0} \sup_{S \in \mathcal{C}} h^*(S, \epsilon).$$

For general topological spaces $\mathcal{C}$ we only get $h^*_{\mathcal{C}} \leq \lim_{\epsilon \to 0} \sup_{S \in \mathcal{C}} h^*(S, \epsilon)$. One can not invert the limit in ϵ and the supremum in the right member of the above inequality, indeed we may have $h^*_{\mathcal{C}} > \sup_{S \in \mathcal{C}} \lim_{\epsilon \to 0} h^*(S, \epsilon) = \sup_{S \in \mathcal{C}} h^*(S)$, see Remark 6.2.

We let $\mathcal{M} * \mathcal{C} \subset \mathcal{M} * C(X)$ be the topological subspace of $\mathcal{M}(X) \times \mathcal{C}$ given by pairs (μ, T) with $T \in \mathcal{C}$ and $\mu \in \mathcal{M}(X, T)$. We consider an upper semicontinuous entropy structure $(h_k)_k$ on X. Finally for a function $f : \mathcal{M} * \mathcal{C} \to \mathrm{R}$ we let $\tilde{f}$ be its upper semicontinuous envelope, i.e. $\tilde{f}(\mu, T) := \limsup_{(\nu, S) \to (\mu, T)} f(\nu, S)$. We have the following variational principles:

Proposition 4.1:

$$h^*_{\mathcal{C}}(T) = \lim_k \sup_{\mu \in \mathcal{M}(X,T)} \widetilde{h - h_k}(\mu, T) = \sup_{\mu \in \mathcal{M}(X,T)} \lim_k \widetilde{h - h_k}(\mu, T)$$

and

$$h^*_{\mathcal{C}} = \sup_{(\mu,T) \in \mathcal{M} * \mathcal{C}} \lim_k \widetilde{h - h_k}(\mu, T).$$

Remark 4.2: Observe that when $\mathcal{C}$ is compact then it follows from the tail variational principle [9][3] and the second equality that $h^*_{\mathcal{C}} = h^*(S_{\mathcal{C}})$, where $S_{\mathcal{C}}$ is the skew-product map on $X \times \mathcal{C}$ defined at the end of Section 2.

Proof. The second equality follows straightforwardly from the first one, which we prove now. Firstly we may again invert the supremum and the limit as the limit in k is nonincreasing and the functions $\widetilde{h - h_k}$ are upper semicontinuous on the compact set $\mathcal{M}(X, T)$.

We need the following lemma.

Lemma 4.3: *Let $(h_k)_k$ be the upper semicontinuous entropy structure on X given in the proof of Proposition 2.2. Then there exist two nonincreasing sequences $(\epsilon_k)_k$ and $(\epsilon'_k)_k$ of real positive numbers going to zero such that for any $T \in C(X)$,*

$$\left(1 - \frac{1}{2^k}\right) h^*(T, \epsilon'_k) - \frac{1}{2^k} \leq \sup_{\mu \in \mathcal{M}(X,T)} (h - h_k)(\mu, T) \leq h^*(T, \epsilon_k).$$

Proof. With the notations of Section 2 we have $h_k(\mu, T) = h_{A_{\mathcal{F}_k}}(\mu \times \lambda, T \times Id_{[0,1]})$. The second inequality follows from the standard following fact whose proof is left to the reader (Hint: argue as in the proof of Lemma 3.1).

Fact. *Let $T \in C(X)$ and $\epsilon > 0$ then for any finite partition P of X with diameter less than ϵ and for any $\mu \in \mathcal{M}(X,T)$, we have:*

$$(h - h_P)(\mu) \leq h^*(T, \epsilon).$$

Indeed we have then $\sup_{\mu \in \mathcal{M}(X,T)}(h - h_k)(\mu) \leq h^*(T \times Id_{[0,1]}, \epsilon_k) = h^*(T, \epsilon_k)$ with ϵ_k being the diameter of the partition $A_{\mathcal{F}_k}$ of $X \times [0,1]$.

Let us prove now the first inequality. For any $\mu \in \mathcal{M}(X)$ and $k \in \mathrm{N}$ we have $(\lambda \times \mu)(\partial A_{\mathcal{F}_k}) = 0$. Thus there is $\epsilon''_k > 0$ such that the closed ϵ''_k-neighborhood of $A_{\mathcal{F}_k}$ has $(\lambda \times \mu)$-measure less than $\frac{1}{2^k \log \sharp A_{\mathcal{F}_k}}$ for any μ in $\mathcal{M}(X)$. Finally we let $\epsilon'_k = \min_{A \neq B \in A_{\mathcal{F}_k}} d(A, B \setminus \partial^{\epsilon''_k} A_{\mathcal{F}_k})$. We fix now $T \in C(X)$ and we will prove $\left(1 - \frac{1}{2^{k/2}}\right) h^*(T, \epsilon'_k) - \frac{2}{2^{k/2}} \leq \sup_{\mu \in \mathcal{M}(X,T)}(h - h_k)(\mu)$. We follow the lines of the proof of the tail variational principle in [3]. First let $\delta > 0$ be such that $h^*(T, \epsilon'_k) \simeq h^*(T, \delta, \epsilon'_k)$ and consider then for all n a point $x_n \in X$ such that $r_n(T, B_n(T, x_n, \epsilon'_k), \delta)$ is maximal. Finally we let $(\xi, \nu) \in \mathcal{M}(X,T)^2$ be a weak limit of $\left(\frac{1}{n}\sum_{k=0}^{n-1}(T^k \delta_{x_n}, T^k \mu_n)\right)_n$ with $\mu_n = \frac{1}{\sharp E_n}\sum_{x \in E_n} \delta_x$ where E_n a (n, δ)-separated set in $B_n(T, x_n, \epsilon'_k)$ with $\sharp E_n = r_n(T, B_n(T, x_n, \epsilon'_k), \delta)$. Let l be such that the diameter of $A_{\mathcal{F}_l}$ (and thus $A^y_{\mathcal{F}_l}$ for any $y \in [0,1]$) is less than δ. Recall we have

$$(h_l - h_k)(\nu) = \int_{[0,1]} (h_{A^y_{\mathcal{F}_l}} - h_{A^y_{\mathcal{F}_k}})(\nu, T) d\lambda(y).$$

As the boundary of $A^y_{\mathcal{F}_k}$ for all k has zero ν-boundary for almost every y, we have as in [3] for such y

$$(h_{A^y_{\mathcal{F}_l}} - h_{A^y_{\mathcal{F}_k}})(\nu, T) \geq \limsup_n \frac{1}{n} H_{\mu_n}\left(\left(A^y_{\mathcal{F}_l}\right)^n \mid \left(A^y_{\mathcal{F}_k}\right)^n\right),$$

To conclude it is enough to check that the right member is larger than $h^*(T, \epsilon'_k) - 2/2^{k/2}$ for y in a set of measure larger than $1 - \frac{1}{2^{k/2}}$. It follows from the choice of ϵ''_k that for any y in a set E with $\lambda(E) > 1 - \frac{1}{2^{k/2}}$ we have $\xi(\partial^{\epsilon''_k} A^y_{\mathcal{F}_k}) < \frac{1}{2^{k/2} \log \sharp A_{\mathcal{F}_k}}$. We may also assume $\left(\frac{1}{n}\sum_{k=0}^{n-1} T^k \delta_{x_n}\right)\left(\partial^{\epsilon''_k} A^y_{\mathcal{F}_k}\right) < \frac{1}{2^{k/2} \log \sharp A_{\mathcal{F}_k}}$ for large enough n with y in a set $E' \subset E$ independent of n with $\lambda(E') > 1 - \frac{1}{2^{k/2}}$. In particular by arguing as in the proof of Lemma 3.1 the dynamical ball $B_n(T, x_n, \epsilon'_k)$ intersects only $e^{n/2^{k/2}}$ elements of $\left(A^y_{\mathcal{F}_k}\right)^n$ for any $y \in E'$. Let $\mathcal{B}^{y,n}_k$ be the subcollection of $\left(A^y_{\mathcal{F}_k}\right)^n$ satisfying this property. We have $\mu_n(\bigcup_{A \in \mathcal{B}^{y,n}_k} A) = 1$ because μ_n is supported on $B_n(T, x_n, \epsilon'_k)$ and therefore with $\mu^A_n = \frac{\mu_n(A \cap .)}{\mu_n(A)}$

$$\limsup_n \frac{1}{n} H_{\mu_n}\left(\left(A^y_{\mathcal{F}_l}\right)^n \mid \left(A^y_{\mathcal{F}_k}\right)^n\right) = \limsup_n \frac{1}{n} \sum_{A \in \left(A^y_{\mathcal{F}_k}\right)^n} \mu_n(A) H_{\mu^A_n}\left(\left(A^y_{\mathcal{F}_l}\right)^n\right),$$

$$\geq \limsup_n \frac{1}{n} \sum_{A \in \mathcal{B}^{y,n}_k,\ \mu_n(A) > e^{-2n/2^{k/2}}} \mu_n(A) H_{\mu^A_n}\left(\left(A^y_{\mathcal{F}_l}\right)^n\right).$$

Then for any $A \in \mathcal{B}_k^{y,n}$ with $\mu_n(A) > e^{-2n/2^{k/2}}$ we have

$$H_{\mu_n^A}\left(\left(A^y_{\mathcal{F}_l}\right)^n\right) \geq -\frac{2n}{2^{k/2}} + \log \sharp E_n = -\frac{2n}{2^{k/2}} + \log r_n(T, B_n(T, x_n, \epsilon'_k), \delta).$$

We conclude that

$$\begin{aligned}\limsup_n \frac{1}{n} H_{\mu_n}\left(\left(A^y_{\mathcal{F}_l}\right)^n \mid \left(A^y_{\mathcal{F}_k}\right)^n\right) &\geq \limsup_n \left(-\frac{2}{2^{k/2}} + \frac{1}{n}\log r_n(T, B_n(T, x_n, \epsilon'_k), \delta)\right) \sum_{A,\ \mu_n(A) > e^{-2n/2^{k/2}}} \mu_n(A),\\ &\geq \limsup_n \left(-\frac{2}{2^{k/2}} + \frac{1}{n}\log r_n(T, B_n(T, x_n, \epsilon'_k), \delta)\right)(1 - e^{-n/2^{k/2}}),\\ &\geq -\frac{2}{2^{k/2}} + h^*(T, \epsilon'_k).\end{aligned}$$

□

We finish now the proof of Proposition 4.1. Fix a neighborhood U of T in $\mathcal{C}$. According to the above Lemma, we have

$$\begin{aligned}\sup_{\mu \in \mathcal{M}(X,T)} \widetilde{h - h_k}(\mu, T) &\leq \sup_{(\nu,S),\ S \in U} (h - h_k)(\nu, S),\\ &\leq \sup_{S \in U} h^*(S, \epsilon_k).\end{aligned}$$

Thus by taking the infimum over U and then the limit in k we conclude that

$$\lim_k \sup_{\mu \in \mathcal{M}(X,T)} \widetilde{h - h_k}(\mu, T) \leq h^*_{\mathcal{C}}(T).$$

For the converse inequality, we consider for any $S \in \mathcal{C}$ a measure $\nu_S \in \mathcal{M}(X, S)$ such that $(h - h_k)(\nu_S, S) \geq (1 - \frac{1}{2^k})h^*(S, \epsilon'_k) - \frac{1}{2^k}$. Now if $\mu_T \in \mathcal{M}(X, T)$ is a weak limit of $(\nu_S)_S$ when S goes to T in $\mathcal{C}$, then

$$\begin{aligned}\sup_{\mu \in \mathcal{M}(X,T)} \widetilde{h - h_k}(\mu, T) &\geq \widetilde{h - h_k}(\mu_T, T),\\ &\geq \limsup_{S \to T}(h - h_k)(\nu_S, S),\\ &\geq \inf_U \sup_{S \in U}\left(1 - \frac{1}{2^k}\right) h^*(S, \epsilon'_k) - \frac{1}{2^k}.\end{aligned}$$

We conclude the proof of the converse inequality by taking the limit in k. □

Definition 4.4: A dynamical system T on X is said to be

- h-expansive if $h^*(T, \epsilon) = 0$ for some $\epsilon > 0$;
- asymptotically h-expansive when $h^*(T) = 0$;
- $\mathcal{C}$-stably asymptpotically h-expansive when $T \in \mathcal{C}$ and $h^*_{\mathcal{C}}(T) = 0$.

The family $\mathcal{C}$ of dynamical systems on X is said to be asymptpotically h-expansive when $h^*_{\mathcal{C}} = 0$.

5. Upper semicontinuity of entropy and continuity of measure of maximal entropy

Asymptotic h-expansiveness was introduced by Misiurewicz in [18]. One important consequence is the upper semicontinuity of the measure theoretical entropy for a given system.

Theorem 5.1: *(Misiurewicz) Let (X, T) be an asymptotically h-expansive system. Then the measure theoretical entropy $h(., T) : \mathcal{M}(X, T) \to \mathrm{R}$ is an upper semicontinuous function, i.e.*

$$\limsup_{\nu \to \mu} h(\nu, T) \leq h(\mu, T).$$

Let $\mathcal{C}$ be as in the above section. For a $\mathcal{C}$-stably asymptotic h-expansive we obtain the following generalization.

Theorem 5.2: *Let T be $\mathcal{C}$-stably asymptpotically h-expansive. Then the measure theoretical entropy $h : \mathcal{M} * \mathcal{C} \to \mathrm{R}$ is upper semicontinuous at (μ, T), i.e.*

$$\limsup_{(\nu,S)\to(\mu,T),\ (\nu,S)\in\mathcal{M}*\mathcal{C}} h(\nu, S) \leq h(\mu, T).$$

Proof. By the variational principle (Proposition 4.1) and upper semicontinuity of h_k on $\mathcal{M} * C(X) \supset \mathcal{M} * \mathcal{C}$ we have using the previous notation ˜ for the upper semicontinuous envelope :

$$\begin{aligned}\tilde{h}(\mu, T) &\leq \tilde{h_k}(\mu, T) + \widetilde{h - h_k}(\mu, T),\\ &\leq h_k(\mu, T) + \sup_{\mu\in\mathcal{M}(X,T)} \widetilde{h - h_k}(\mu, T).\end{aligned}$$

By taking the limit in k we conclude the proof:

$$\tilde{h}(\mu, T) \leq h(\mu, T) + h^*_{\mathcal{C}}(T) = h(\mu, T).$$

□

Observe that Theorem 5.1 follows from the previous theorem by taking $\mathcal{C} = \{T\}$. One may also deduce upper semicontinuity of the topological entropy.

Corollary 5.3: *Let T be $\mathcal{C}$-stably asymptpotically h-expansive. Then*

$$\limsup_{S \xrightarrow{\mathcal{C}} T} h_{top}(S) \leq h_{top}(T).$$

Let T be an asymptotically h-expansive system on X. Then the set $\mathcal{M}_{max}(T) = \{\mu \in \mathcal{M}(X, T),\ h(\mu, T) = h_{top}(T)\}$ is a non-empty compact convex subset of $\mathcal{M}(X, T) \subset \mathcal{M}(X)$. We will see that if $\mathcal{C}$ is asymptotically h-expansive and the topological entropy is (lower semi-) continuous on $\mathcal{C}$ then $T \mapsto \mathcal{M}_{max}(T)$ is upper semicontinuous on $\mathcal{C}$ for the Hausdorff topology on $\mathcal{M}(X)$.

Corollary 5.4: *Assume that T is $\mathcal{C}$-stably asymptpotically h-expansive and that the topological entropy on $\mathcal{C}$ is lower semi-continuous at $T \in \mathcal{C}$. Then any weak limit of $\mu_S \in \mathcal{M}_{max}(S)$ when S goes to T belongs to $\mathcal{M}_{max}(T)$.*

Proof. Let $\mu = \lim_S \mu_S$ be such a weak limit. We have by Theorem 5.2 and lower semi-continuity of the topological entropy :

$$\begin{aligned} h(\mu, T) &\geq \limsup_{S \to T} h(\mu_S, S); \\ &\geq \limsup_{S \to T} h_{top}(S); \\ &\geq h_{top}(T). \end{aligned}$$

□

Corollary 5.5: *Assume moreover that T has a unique measure of maximal entropy μ_T then any $\mu_S \in \mathcal{M}_{max}(S)$ is converging to μ when S goes to T.*

Note that all statements of this section may be applied to topological, measure theoretical pressure and equilibrium states associated to a continuous potential.

6. Applications

We illustrate now our abstract theory with two examples : multimodal interval maps and C^r-smooth maps with $r > 1$. There are other contexts, that we will not develop here, where the above abstract results may be applied : e.g. piecewise affine maps [7], C^1 maps far from homoclinic tangencies [13],...

6.1. *Multimodal maps of the interval*

Let $\mathcal{M}_k^r([0,1])$, with $r = 0$ or 1, be the set of C^r interval maps f, which admits a partition of $[0,1]$ into k intervals such that f is weakly monotone on each element of this partition.

We say $x \in [0,1]$ is a turning point of an interval map f when there exist $0 \leq a < b \leq x \leq c < d \leq 1$ such that f is constant on $[b,c]$ and strictly monotone both on $[a,b]$ and $[c,d]$ but in the opposite sense.

Theorem 6.1: *(Misiurewicz-Szlenk)[20] Let $f \in M_k^0([0,1])$ such that the image of any turning point is not turning. Then f is $M_k^0([0,1])$-stably asymptotic h-expansive.*

Proof. Let $\delta > 0$ and let f be as above. We fix $p \in \mathrm{N}$ with $\frac{\log 2}{p} < \delta$. For any piecewise monotone map g we let $L(g) > 0$ be the minimal distance between two turning points of g, which are not in a common interval of constancy. As the image by f of any turning point is not turning, the map $g \mapsto L(g^k)$ is continuous at f for any k. This concludes the proof of the theorem as for any $\epsilon < L(g^p)$,

$$h^*(g, \epsilon) \leq \frac{1}{p} h^*(g^p, \epsilon) \leq \frac{\log 2}{p} < \delta.$$

Indeed the inequality $h^*(g^p, \epsilon) \leq \log 2$ follows from the fact that any dynamical ball $B_n(g^p, x, \epsilon)$ intersect only 2^n n-monotone branches of g^p and that the cardinality of any (n, δ)-separated set lying in a n-monotone branch is less than n/δ. □

Remark 6.2: It is not true that $M_k^0([0,1])$ is stably asymptotic h-expansive altough any $T \in M_k^0([0,1])$ is asymptotically h-expansive. It is enough to observe that the topological entropy is not upper semicontinuous on $M_k^0([0,1])$. For $k = 2$, one can for example consider the continuous piecewise affine maps $T = \min(x, 1-x)$ and T_n, with $T_n = T$

outside $[1/2-1/n, 1/2+1/n]$ and T_n be the rescaled usual 2-tent map on $[1/2-1/n, 1/2+1/n]$. One easily checks that $h_{top}(T_n) = \log 2$ and $T_n \xrightarrow{n\to+\infty} T$. However any compact subset $\mathcal{C}$ of $M_k^0([0,1])$, such that $g \mapsto L(g^k)$ is continuous on $\mathcal{C}$ for any k, is asymptotically $\mathcal{C}$-expansive.

As the topological entropy is lower semicontinuous at any $f \in M_k^0([0,1])$ we recover according to Corollary 5.4 the following result of P. Raith [23]:

Corollary 6.3: *Assume $f \in M_k^0([0,1])$ such that the image of any turning point is not turning. Then any weak limit of measures of maximal entropy μ_g of $g \in M_k^0([0,1])$, when g goes to f, is a measure of maximal entropy of f.*

For $f \in M_k^1([0,1])$, we let $w(f', \epsilon)$ be the modulus of continuity of f', i.e.

$$w(f', \epsilon) := \sup_{|x-y|<\epsilon} |f'(x) - f'(y)|.$$

It was proved in [4] (proof of Theorem D) that for any $\epsilon > 0$ we have

$$h^*(f, \epsilon) \leq \frac{\log k}{|\log w(f', \epsilon)|}.$$

Consequently we have

Theorem 6.4: *Any $f \in M_k^1([0,1])$ is $M_k^1([0,1])$-stably asymptotic h-expansive. Morevoer any compact subset of $M_k^1([0,1])$ is asymptotically h-expansive.*

By smoothing the example in the previous remark one easily checks that $M_k^1([0,1])$ is not asymptotically h-expansive. The above theorem together with Corollary 5.3 give a new proof of the following statement due to M. Misiurewicz:

Corollary 6.5: *(Misiurewicz)[21] The topological entropy is (upper) semicontinuous on $M_k^1([0,1])$.*

6.2. C^∞ *smooth systems*

Building on previous works of Yomdin and Gromov [25], Buzzi has proved that C^∞ maps on a compact smooth manifold are asymptoticlly h-expansive. In fact it follows from his proof that:

Theorem 6.6: *(Yomdin, Buzzi) Let M be a compact smooth Riemanian manifold of dimension d and let $r \in \mathbb{N}$. Then any bounded subset $\mathcal{C}$ of $C^r(M)$ endowed with the usual C^r topology satisfies for any $T \in \mathcal{C}$:*

$$h^*_{\mathcal{C}}(T) \leq \frac{dR(T)}{r},$$

with $R(T) := \lim_n \frac{1}{n} \log^+ \|DT^n\|_\infty$. In particular any compact subset of $C^\infty(M)$ is asymptotically h-expansive.

Remark 6.7: The tail entropy may be defined for nonautonomous dynamical systems $(T_n : X \to X)_n$ by following straightforwardly the definition of Section 4 with $T^n = T_n \circ ... T_1$. Then the above theorem also holds true in this context [8] (provided each T_n lies in the C^r bounded set $\mathcal{C}$).

Then one can deduce from the above theorem and Corollary 5.4 the following general result of continuity of maximal measures (which is new to our knowledge):

Corollary 6.8: *Assume S is converging to T in the C^∞ topology on M and that $\liminf_{S\to T} h_{top}(S) \geq h_{top}(T)$ then any weak limit of measure of maximal entropy μ_S of S is a measure of maximal entropy of T.*

Remark 6.9: When it is unique the measure of maximal entropy is continuous with T. For example the unique measure of maximal entropy μ_T is continuous with T in the set of C^∞ transitive interval maps or surface diffeomorphisms. Indeed the topological entropy is lower semicontinuous in these contexts by respective results of Misiurewicz [16] and Katok [14]. Moreover uniqueness was proved in [8] and [6].

It was proved by Boyle and the Fiebig's [2] that any asymptoticaly h-expansive system has a principal symbolic extension. Applying this result to our skew-product map $S_{\mathcal{C}}$ for a compact subset $\mathcal{C}$ of $C^\infty(M)$, which is asymptotically h-expansive by Remark 4.2, we may encode any dynamics of $\mathcal{C}$ in a single subshift:

Corollary 6.10: *Let $\mathcal{C}$ be a compact subset of $C^\infty(M)$, then there is a principal symbolic extension of $S_{\mathcal{C}}$. In other words there is a subshift Y and disjoint subshifts $(Y_T)_{T\in\mathcal{C}}$ with $Y = \bigsqcup_{T\in\mathcal{C}} Y_T$ and a continuous surjective map $\pi : Y \to M$ such that the restriction of π to Y_T is a principal symbolic extension of T.*

Using fibered entropy structures developped in Section 3 we generalize now the above corollary to smooth skew-products.

6.3. *C^r smooth systems*

Boyle and Downarowicz have completely characterized the entropy in a symbolic extension. Let $E : \mathcal{M}(X,T) \to \mathrm{R}$ be an affine continuous map, then recall that E is called a superenvelope if and only if $\lim_k \widetilde{E - h_k} = E - h$. They showed that for any symbolic extension $\pi : (Y,S) \to (X,T)$ the map $h^\pi : \mathcal{M}(X,T) \to \mathrm{R}$ defined as $h^\pi(\mu) = \sup_{\nu,\ \pi\nu=\mu} h(\nu)$ is a superenvelope and that for any superenvelope E there is a symbolic extension π with $h^\pi = E$.

It was proved for C^r interval maps [12] and for C^r surface diffeomorphism T [5] with $r > 1$ that $\mu \mapsto h(\mu,T) + \frac{\chi^+(\mu,T)}{r-1} = h(\mu,T) + \frac{1}{r-1}\int \chi^+(x,T)d\mu(x)$, with $\chi^+(x,T) := \limsup_n \frac{1}{n}\log^+ \|D_x T^n\|$ the largest positive Lyapunov exponent at a point $x \in M$, is a superenvelope of (X,T).

We consider here a skew-product map $S \in C(X\times M)$ over $T \in C(X)$ with M a smooth manifold, such that $x \mapsto S_x$ from X to $C^r(M)$, with $r > 1$, is continuous. We let χ^+_{fib} be the maximal positive fibered Lyapunov exponent defined for any $\mu \in \mathcal{M}(X\times M, S)$ as $\chi^+_{fib}(\mu,S) = \int \chi^+_{fib}(x,y)d\mu(x)$ with $\chi^+_{fib}(x,y) = \limsup_n \frac{1}{n}\log^+ \|D_y S_{T^{n-1}x} \circ ... \circ S_x\|$ at any $(x,y) \in X\times M$. By subadditivity we have $\chi^+_{fib}(\mu,S) = \inf_n \int \frac{1}{n}\log^+ \|D_y S_{T^{n-1}x} \circ ... \circ S_x\| d\mu(x,y)$ and when ν is ergodic, then $\left(\frac{1}{n}\log^+ \|D_y S_{T^{n-1}x} \circ ... \circ S_x\|\right)_n$ is converging to $\chi^+_{fib}(\nu,S)$ for ν-almost all (x,y). Observe that $\mu \mapsto \chi^+_{fib}(\mu,S)$ is an harmonic upper semicontinuous function on $\mathcal{M}(X\times M, S)$.

Theorem 6.11: *Assume $M = \mathrm{T}^1$ is a circle. Then the function $\mu \mapsto h^{fib}(\mu,S) + \frac{1}{r-1}\chi^+_{fib}(\mu,S)$ is a fibered superenvelope of $(X\times M, S)$.*

The proof in [5] applies straightforwardly in this fibered context by using the quantities h^{fib}_{New} and χ^+_{fib}. Indeed as in Remark 6.7 the proof is based on a reparametrization lemma of dynamical balls which applies to the nonautonomous dynamical systems $(S_{T^l x})_l$ for any $x \in X$. We give now more details.

Sketch of proof. We first recall the Reparametrization Lemma of [5] in the context of circle maps. Let $\mathcal{T} = (T_n)_n : \mathrm{T}^1 \to \mathrm{T}^1$ be a bounded sequence of $\mathcal{C}^1$ circle maps (we endow the circle T^1 with some Riemannian metric $\|.\|$). For any $\chi > 0$, $\gamma > 0$ and $C > 1$, we consider the set $\mathcal{H}^n_{\mathcal{T}}(\chi, \gamma, C)$ of points of T^1 whose exponential growth of the derivative of the n-first iterations of $\mathcal{T}$ is almost equal to χ, i.e. with $T^j = T_{j-1} \circ ... \circ T_0$:

$$\mathcal{H}^n_{\mathcal{T}}(\chi, \gamma, C) := \left\{ y \in \mathrm{T}^1 \ : \ \forall 1 \leq j \leq n, \ C^{-1} e^{(\chi - \gamma) j} \leq \|D_y T^j\| \leq C e^{(\chi+\gamma) j} \right\}.$$

We also denote $H : [1, +\infty[\to \mathrm{R}$ the function defined by $H(t) = -\frac{1}{t}\log(\frac{1}{t}) - (1 - \frac{1}{t})\log(1 - \frac{1}{t})$. Moreover $[x]$ is the usual integer part of $x \in \mathrm{R}$ if $x > 0$ and zero if not.

Reparametrization Lemma: *Let $\mathcal{T} = (T_n)_n : \mathrm{T}^1 \to \mathrm{T}^1$ be a sequence of $\mathcal{C}^r$ maps with $r > 1$ lying in a C^r bounded set $\mathcal{C}$. There exist $\epsilon = \epsilon(\mathcal{C}) > 0$ depending only on $\mathcal{C}$ and a universal constant $A = A(r) > 0$ depending only on r with the following properties.*

For all $\chi > 0$, $\gamma > 0$ and $C > 1$, for all $y \in \mathrm{T}^1$, for all positive integers n, there exists a family $\mathcal{F}_n$ of C^∞ maps from $[0, 1]$ to T^1, such that we have with $\lambda^+_n(y, \mathcal{T}) := \frac{1}{n}\sum_{j=0}^{n-1} \log^+ \|D_{T^j y} T_j\|$ and some constant $B > 0$ (independent of n):

- $\forall \psi \in \mathcal{F}_n, \ \forall 0 \leq l \leq n, \ \|D(T^l \circ \psi)\|_\infty \leq 1,$
- $\mathcal{H}^n_{\mathcal{T}}(\chi, \gamma, C) \cap (B_{n+1}(\mathcal{T}, y, \epsilon)) \subset \bigcup_{\psi \in \mathcal{F}_n} \psi([0, 1]),$
- $\log \sharp \mathcal{F}_n \leq \frac{1}{r-1}\left(1 + H([\lambda^+_n(y, \mathcal{T}) - \chi] + 3)\right)\left(\lambda^+_n(y, \mathcal{T}) - \chi\right) n + An + B.$

We sketch now the proof of Theorem 6.11. We fix a measure $\mu \in \mathcal{M}(X \times \mathrm{T}^1, S)$ and we will show there is an integer k_μ such that for ν close to μ we have

$$\left(h^{fib} - h^{fib}_{k_\mu}\right)(\nu, S) \lesssim \frac{1}{r-1}\left(\chi^+_{fib}(\mu, S) - \chi^+_{fib}(\nu, S)\right).$$

Equivalently the map $\mu \mapsto h^{fib}(\mu, S) + \frac{\chi^+_{fib}(\mu, S)}{r-1}$ is a fibered superenvelope. By Lemma 8.2.14 of [10] it is enough to consider ergodic measures ν. We fix an integer p_μ with $\chi^+_{fib}(\mu, S) \simeq \frac{1}{p_\mu}\int \log^+ \|D_y S_{T^{p_\mu - 1}x} \circ ... \circ S_x\| d\mu(x, y)$. Let $\tilde{\nu}$ be an ergodic component of ν under S^{p_μ}. Then by the Ergodic Theorem the sequence $\left(\lambda^+_n(y, \mathcal{T}_x)\right)_n$, with $\mathcal{T}_x = (S_{T^{np_\mu - 1}x} \circ ... \circ S_{T^{(n-1)p_\mu}x})_n$, is converging $\tilde{\nu}$-almost surely to $\int \log^+ \|D_y S_{T^{p_\mu - 1}x} \circ ... \circ S_x\| d\tilde{\nu}(x, y)$. For any $\alpha < 1$ there exists therefore a subset $\tilde{F}$ of $X \times \mathrm{T}^1$ with $\tilde{\nu}(\tilde{F}) > \alpha$ such that the sequences $\left(\lambda^+_n(y, \mathcal{T}_x)\right)_n$, and $\left(\frac{1}{n}\log^+ \|D_y S_{T^{(n-1)p_\mu}x} \circ ... \circ S_x\|\right)_n$ are converging uniformly in $(x, y) \in \tilde{F}$ respectively to $\int \log^+ \|D_y S_{T^{p_\mu - 1}x} \circ ... \circ S_x\| d\tilde{\nu}(x, y)$ and $\chi^+_{fib}(x, y)$. Then by arguing as in [5] we get by applying the Reparametrization lemma to $\mathcal{T}_x$ for any x that for $\epsilon_\mu = \epsilon\left((S^{p_\mu}_x)_{x \in X}\right)$:

$$\begin{aligned} h^x(\mathrm{T}^1 | \tilde{F}_x, \epsilon_\mu) &\lesssim \lim_n \sup_{y \in \tilde{F}_x} \lambda^+_n(y, \mathcal{T}_x) - \chi^+_{fib}(\tilde{\nu}, S^{p_\mu}), \\ &\lesssim \frac{1}{r-1}\left(\int \log^+ \|D_y S_{T^{p_\mu - 1}x} \circ ... \circ S_x\| d\tilde{\nu}(x, y) - \chi^+_{fib}(\tilde{\nu}, S^{p_\mu})\right). \end{aligned}$$

and then by integrating in x the left member with respect to $\pi\tilde{\nu}$ and by letting α go to

1, we get :

$$h_{New}^{fib}(\tilde{\nu}, S^{p_\mu}, \epsilon_\mu) \lesssim \frac{1}{r-1}\left(\int \log^+ \|D_y S_{T^{p_\mu -1}x}\circ ... \circ S_x\| d\tilde{\nu}(x,y) - \chi_{fib}^+(\tilde{\nu}, S^{p_\mu})\right).$$

By harmonicity of $h^{fib}(., S^{p_\mu}, \epsilon_\mu)$ we have by summing over all the ergodic components $\tilde{\nu}$ of ν under S^{p_μ} :

$$\begin{aligned} h_{New}^{fib}(\nu, S^{p_\mu}, \epsilon_\mu) &\lesssim \frac{1}{r-1}\left(\int \log^+ \|D_y S_{T^{p_\mu -1}x}\circ ... \circ S_x\| d\nu(x,y) - \chi_{fib}^+(\nu, S^{p_\mu})\right), \\ &\lesssim \frac{1}{r-1}\left(\int \log^+ \|D_y S_{T^{p_\mu -1}x}\circ ... \circ S_x\| d\mu(x,y) - \chi_{fib}^+(\nu, S^{p_\mu})\right), \\ &\lesssim \frac{p_\mu}{r-1}\left(\chi_{fib}^+(\mu, S) - \chi_{fib}^+(\nu, S)\right). \end{aligned}$$

This concludes the proof of the theorem as by Lemma 3.1 there is an integer k_μ such that :

$$\begin{aligned} \left(h^{fib} - h_{k_\mu}^{fib}\right)(\nu, S) &= \frac{1}{p_\mu}\left(h^{fib} - h_{k_\mu}^{fib}\right)(\nu, S^{p_\mu}), \\ &\leq \frac{1}{p_\mu} h_{New}^{fib}(\nu, S^{p_\mu}, \epsilon_\mu). \end{aligned}$$

□

Let $\mathrm{T}^n = \underbrace{\mathrm{S}^1 \times ... \times \mathrm{S}^1}_{n\,\times}$ be the n-torus. A map $T \in C^r(\mathrm{T}^n)$ with $r > 1$ is said to be a **skew-product of circle maps** when T is of the form $T(x_1, ..., x_n) = (f_1(x_1), f_2(x_1, x_2), ..., f_n(x_1, ..., x_n))$ and for any $0 \leq k \leq n-1$ and any $x_1, ..., x_k \in \mathrm{T}^k$ the map $x_{k+1} \mapsto f_{k+1}(x_1, ..., x_k, x_{k+1})$ is a C^r circle map.

It is conjectured that any C^r map with $r > 1$ on a compact smooth manifold admits a symbolic extension and that $h + \frac{\sum_i \chi_i^+}{r-1}$ is a superenvelope, where $\sum_i \chi_i^+$ denotes the sum of the positive Lyapunov exponents. It follows from the above theorem that this conjecture holds for a C^r skew-product of interval maps with $r > 1$.

Corollary 6.12: *Let T be a C^r skew-product of interval maps with $r > 1$, then $h + \frac{\sum_i \chi_i^+}{r-1}$ is a superenvelope of T.*

Proof. We argue by induction on n. As already noticed the case $n = 1$ was proved in [12]. By Theorem 6.11 with $X = \mathrm{T}^{n-1}$ and $M = \mathrm{T}^1$ the map $\mu \mapsto h^{fib}(\mu, S) + \frac{\chi_{fib}^+(\mu, S)}{r-1}$ is a fibered superenvelope. By induction hypothesis the function $h^X(\pi.) + \frac{1}{r-1}\sum_{i \leq n-1} \chi_i^+$ is a superenvelope of $(x_1, ..., x_{n-1}) \mapsto (f_1(x_1), f_2(x_1, x_2), ..., f_{n-1}(x_1, ..., x_{n-1}))$. By Proposition 3.2 we conclude that the following map is a superenvelope of T :

$$h^{fib} + \frac{\chi_{fib}^+}{r-1} + h^X(\pi.) + \sum_{i \leq n-1} \chi_i^+ = h + \frac{1}{r-1}\sum_{i \leq n} \chi_i^+.$$

□

Theorem 6.13: *Assume $r = \infty$. Then $(h_k^{fib})_k$ is converging uniformly to zero. In particular $h^{fib} + E(\pi.)$ is a super envelope of $X \times M$ for any superenvelope E of (X, T).*

Proof. Clearly we have $h_{New}^{fib}(., \epsilon_k) \leq \sup_{x \in X} h^*((S_{T^l x})_l, \epsilon_k)$. According to Remark 6.7 (note that $\{S_x, \ x \in X\}$ is a compact subset of $C^\infty(M)$) the term $\sup_{x \in X} h^*\left((S_{T^l x})_l, \epsilon_k\right)$ is going to zero when k goes to infinity. We conclude the proof by applying Corollary 3.2. □

References

[1] M. Boyle and T. Downarowicz, *The entropy theory of symbolic extensions*, Invent. Math. 156 (2004), no. 1, 119-161.

[2] M. Boyle, D. Fiebig, U. Fiebig, *Residual entropy, conditional entropy and subshift covers*, Forum Math. 14, no. 5, p. 713757, 2002.

[3] D. Burguet, *A direct proof of the tail variational principle and its extension to maps*, Erg. Theory and Dyn.Systems (29), p 357-369, 2009.

[4] D. Burguet, G.Liao, J. Yang, *Asymptotic h-expansiveness of C^∞ maps*, Proc. Lond. Math. Soc. (3) 111 (2015), no. 2, 381-419.

[5] D. Burguet, *Symbolic extensions in intermediate smoothness on surfaces*, Ann. Sci. c. Norm. Supr. (4) 45 (2012), no. 2, 337-362.

[6] J. Buzzi, S. Crovisier, O. Sarig, *Finiteness of maximal measures for C^r surface diffeomorphisms*, in preparation.

[7] J. Buzzi, *Maximal entropy measures for piecewise affine surface homeomorphisms*, Ergodic Theory Dynam. Systems 29 (2009), no. 6, 1723-1763.

[8] J. Buzzi, *Intrinsic ergodicity for smooth interval maps*, Israel J. Math., 100 (1997), 125-161.

[9] T. Downarowicz, *Entropy structure*, J. Anal. Math. 96 (2005), 57-116.

[10] T. Downarowicz, *Entropy in dynamical systems*, New Mathematical Monographs, 18. Cambridge University Press, Cambridge, 2011.

[11] T. Downarowicz and D. Huczek, *Zero-dimensional principal extensions*, Acta Appl. Math. 126, p. 117129, 2013.

[12] T. Downarowicz and A. Maass, Smooth interval maps have symbolic extensions, *Invent. Math.*, **176**: 617-636, 2009.

[13] G. Liao, M. Viana, and J. Yang, *Entropy conjecture for diffeomorphisms away from tangencies*, J. Eur. Math. Soc., **6**, 2043-2060, 2013.

[14] A. Katok, *Lyapounov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. I.H.E.S. 51 (1980), 137-173.

[15] E. Lindenstrauss, *Mean Dimension, Small Entropy Factors, and an Embedding Theorem*, Publ. Math. IHES 89, p. 227-262, 1999.

[16] M. Misiurewicz, *Horseshoes for continuous mappings of an interval*, in "Dynamical Systems", Liguori Editore, Napoli 1980, 127-135.

[17] M. Misiurewicz, *On non-continuity of topological entropy*, Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys. 19 (1971), 319-320.

[18] M. Misiurewicz, *Topological conditional entropy* Studia Mathematica 55 (1976), 175-200.

[19] M. Misiurewicz, *Jumps of entropy in one dimension*, Fund. Math. 132 (1989), no. 3, 215-226.

[20] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. 67 (1980), no. 1, 45-63.

[21] M. Misiurewicz, *Continuity of entropy revisited*, Dynamical systems and applications, World Sci. Ser. Appl. Anal. 4 (1995), 495-503.
[22] S.Newhouse, *Continuity properties of entropy*, Annals of Math. 129 (1989) 215-235.
[23] P. Raith, *Stability of the maximal measure for piecewise monotonic interval maps*, Ergodic Theory Dynam. Systems 17 (1997), 14191436.
[24] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.
[25] Y.Yomdin, *Volume growth and entropy*, Israel J.Math. 57 (1987), 285-300.