Lecture 1 and 2: Modern entropy theory of topological systems

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13th February 2017

(X, T) a topological system, i.e.

- (X, d) a compact metrique space,
- $T : X \to X$ continuous.

Dynamical ball : $x \in X$, $n \in \mathbb{N} \cup \{\infty\}$, $\epsilon > 0$,

$$B_T(x, n, \epsilon) = \bigcap_{0 \leq k < n} T^{-k} B(T^k x, \epsilon)$$

with $B(x, \epsilon) = \{y,\ d(x, y) < \delta\}$.

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Subshift (Y, S) :

Y closed shift subset of $\{1, ..., K\}^{\mathbb{Z}}$ for $K \in \mathbb{N}^*$ invariant by the shift S.

With the metric d defined as $d(x, y) = \sum_{i \in \mathbb{Z}} \frac{\delta_{x_i = y_i}}{3^i}$

$$B_S(x, n, 1) := \{y \in Y, \ y_i = x_i \text{ for } 0 \leq i < n\}.$$

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Topological entropy :

$$h_{top}(T,\epsilon) := \limsup_n \frac{1}{n} \min\{\sharp C, \bigcup_{x\in C} B(x,n,\epsilon) = X\},$$

$$h_{top}(T) = \lim_{\epsilon\to 0} h_{top}(T,\epsilon).$$

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$$h_{top}(S) = \log K.$$

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$(X, \mathcal{B}, T, \mu)$ a measure preserving system
P finite Borel partition

$$H_\mu(P) = -\sum_{A \in P} \mu(A) \log \mu(A)$$

The entropy $h(\mu, P)$ of μ w.r.t. P :

$$\begin{aligned} h(\mu, P) &= \inf_n \frac{1}{n} H_\mu\Big(\bigvee_{k=0,\dots,n-1} T^{-k}P\Big), \\ &= \lim_n \frac{1}{n} H_\mu\Big(\bigvee_{k=0,\dots,n-1} T^{-k}P\Big). \end{aligned}$$

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(X, T) topological system

$$\mathcal{M}(X, T) := \{\mu \text{ mesure de proba T-invariante}\}$$

is a compact convex set with the set $\mathcal{M}_e(X, T)$ of ergodic measures as extremal points.

Ergodic decomposition : $\mathcal{M}(X, T)$ is a Choquet simplex, i.e.

$\forall \mu \exists M_\mu$ supported on $\mathcal{M}_e(X, T)$ s.t.

$$f(\mu) = \int f(\nu) dM_\mu(\nu) \text{ for all affine real continuous } f.$$

Any Choquet simplex may be realized as the set of invariant measures of a dynamical system.

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$$h(\mu) = \int h(\nu) dM_\mu(\nu).$$

Variational principle :

$$h_{top}(f) = \sup_{\mu \in \mathcal{M}(X,T)} h(\mu).$$

Entropy functions have been completely characterized for topological systems.

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(X, T) invertible topological system

Two sided dynamical ball :
$x \in X$, $n \in \mathbb{N} \cup \{\infty\}$, $\epsilon > 0$,

$$B_T^*(x, n, \epsilon) = \bigcap_{0 \le |k| < n} T^{-k} B(T^k x, \epsilon)$$

(X, T) **strongly expansive :**

$$\forall x \in X, \; B_T^*(x, \epsilon, \infty) = \{x\}.$$

Example : Subshifts.

The dimension of X is finite, even 0 when T is minimal.

C^1 robust expansive systems are the Axiom A systems.

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Topological Tail entropy :

$$h^*(T, \epsilon) := \sup_{x \in X} h_{top}(B_T^{(*)}(x, \epsilon, \infty))$$

$$h^*(T) := \lim_{\epsilon \to 0} h^*(T, \epsilon).$$

Measure theoretical Tail entropy : X zero-dimensional
$(P_k)_k$ sequence of clopen partitions with diameters going to 0

$$u(\mu) = \lim_k \limsup_{\nu \to \mu} [h(\nu) - h(\nu, P_k)]$$

Tail variational principle :

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$\forall n \in \mathbb{N}$,

$$Per_n(X,T) := \{x,\ \ T^n x = x \text{ and } T^m x \neq x \text{ for } 0 < m < n\}.$$

Periodic growth :

$$p(T) := \limsup_n \frac{1}{n} \log \sharp Per_n(X,T).$$

Local periodic growth :

$$p^*(T) := \lim_k \limsup_n \frac{1}{n} \sup_{A^n \in P_k^n} \log \sharp Per_n(X,T) \cap A^n.$$

Measure theoretical local periodic growth :

$$p^*(\mu) := \lim_k \limsup_{\nu \to \mu,\ \nu(Per)=1} \int p_k^*(x) d\nu(x)$$

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Asymptotical h-expansiveness :

$$h^*(T) = 0.$$

C^1 robustly h-expansive are the diffeos C^1 far from homoclinic tangencies.

Examples : C^∞ maps !

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$$\limsup_{\nu \to \mu} [h(\nu) - h(\mu)] \leq u(\mu).$$

U.s.c. of the entropy function for (X, T) a. h-expansive :

$h : \mathcal{M}(X, T) \to \mathbb{R}$ is u.s.c. and thus

$$\mathcal{M}_{max}(T) = \{\mu \in \mathcal{M}(X, T), \ h(\mu) = h_{top}(T)\}$$

is a non empty compact set.

$$p(T) \leq h_{top}(T) + p^*(T).$$

Equidistribution of periodic points for (X, T) a. p-expansive :

Assume also $+\infty > p(T) \geq h_{top}(T)$ then

- $p(T) = h_{top}(T)$,
- any weak limit of $\left(\frac{1}{\sharp Per_n(X,T)} \sum_{x \in Per_n(X,T)} \delta_x\right)_n$ is a measure of maximal entropy.

Preuve

(X, T) topological system

Zero-dimensional extension :
(Y, S) zero-dim. system with $\pi : Y \to X$

- π surjective,
- $\pi \circ S = T \circ \pi$.

Principal zero-dimensional extension :
Zero dimensional extension $\pi(Y, S) \to (X, T)$ s.t.
$\forall \mu \in \mathcal{M}(Y, S), \; h_T(\pi\mu) = h_S(\mu)$.

Every topological system admits a zero-dimensional principal extension.

Symbolic Extension :
(Y, S) subshift with $\pi : Y \to X$

- π surjective,
- $\pi \circ S = T \circ \pi$.

$$\implies h_{top}(T) < +\infty$$

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Symbolic extension $\pi(Y, S) \to (X, T)$ s.t.
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$$\implies h : \mathcal{M}(X, T) \to \mathbb{R} \text{ is u.s.c.}$$

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Ergodic generators : $(X, \mathcal{B}, T, \mu)$ be a measure preserving system a partition P of X is an ergodic generator when

$$\left(\bigvee_{|k|<n} T^k P\right)_n \quad \text{generates (up to null sets) } \mathcal{B}$$

Uniform generators : (X, T) topological invertible system a Borel partition P of X is a uniform generator when

$$diam\left(\bigvee_{|k|<n} T^k P\right) \xrightarrow{n} 0.$$

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Existence of finite ergodic generators :

An ergodic m.p. system $(X, \mathcal{B}, T, \mu)$ has a generator

$$\Longleftrightarrow$$

$$h(\mu) < +\infty$$

Existence of finite uniform generator : For a 0-dim top. (X, T)

- clopen uniform generators $\Longleftrightarrow$ strongly expansive,
- 0-boundary uniform generators $\Longleftrightarrow$ a. h and a. p expansive, Preuve
- (Borel) uniform generator $\Longleftrightarrow$ symbolic extension and $p(T) < \infty$.

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Entropy function w.r.t. a symbolic extension :
$\pi : (Y, S) \to (X, T)$,

$$\begin{aligned} h_\pi : \mathcal{M}(X,T) &\to \mathbb{R} \\ \mu &\mapsto \sup_{\nu,\ \pi\nu=\mu} h_S(\nu). \end{aligned}$$

Supperenveloppe : (X, T) 0-dim. system,
$(P_k)_k$ sequence of clopen partitions with diameters going to 0. An

u.s.c. affine function $E : \mathcal{M}(X,T) \to \mathbb{R}$ is a superenvelope when

$$\begin{aligned} &\forall \delta > 0,\ \forall \mu \in \mathcal{M}(X,T), \\ &\exists k_\mu \text{ and } \mathcal{V}_\mu \text{ neighborhood of } \mu \text{ s.t.} \\ &\forall \nu \in \mathcal{V}_\mu,\ h(\nu) - h_{k_\mu}(\nu) \le E(\mu) - E(\nu) + \delta \end{aligned}$$

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Symbolic extension theorem :

$$E + hhgm = h_\pi \text{ for some symbolic extension } \pi$$

$$\Longleftrightarrow$$

$$E \text{ is a superenvelope.}$$

Preuve sens facile

Embedding theorem :

$$E = h_\pi \text{ for some symbolic extension } \pi$$

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$$E \text{ is a superenvelope with } E \geq p^*.$$

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S.e. operator : $\mathcal{S}(X,T)$ the set of u.s.c. real nonnegative function on $\mathcal{M}(X,T)$,
$\widetilde{f}$ the u.s.c. envelope of f and $h_k = h(., P_k)$.

$$\begin{aligned} \mathcal{T} : \mathcal{S}(X,T) &\rightarrow \mathbb{R}^+, \\ f &\mapsto \lim_k f + \widetilde{h - h_k}. \end{aligned}$$

The superenvelopes are the fixed point of the nondecreasing operator $\mathcal{T}$ of the complete lattice given by $\mathcal{S}(X,T) \cup \{\infty\}$. The function $h_{s.e.}$ is the smallest fixed point of $\mathcal{T}$ and it is the transfinite limit of $(\mathcal{T}^\alpha(0))_\alpha$. Moreover $u = \mathcal{T}(0)$.

Order of accumulation : smallest countable ordinal with $\mathcal{T}^\alpha(0) = h_{s.e.}$.

Any countable ordinal is realized by a topological system.

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Example with large $h_{s.e.}$: Collections of periodic measures $(\mathcal{P}_k)_k$ and collection of ergodic measures $(\mathcal{M}_k)_k$ s.t. :

- $h(\mu) > a_k > 0$ for all $\mu \in \mathcal{M}_k$,
- for all k any $\mu \in \mathcal{M}_k$ is a weak-* limit of periodic measures in $\mathcal{P}_k$,
- for all k any $\mu \in \mathcal{P}_k$ is a weak-* limit of measures in $\mathcal{M}_{k+1}$,

$$\text{then } \forall \mu \in \mathcal{M}_0,\ h_{s.e.}(\mu) \geq \sum_k a_k.$$

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