Lecture 3 and 4: Entropy of C^r smooth systems via semi-algebraic tools

David Burguet

17th February 2017

Set, maps, degrees. Set and functional version with polynomial bound. General case, curve case with given derivative, action on the tangent bundle.

M compact smooth manifold,
$\mathcal{H} := \{ f \in \mathit{Diff}^1(M)$ admits an homoclinic tangency $\}$.

Any $f \in \mathit{Diff}^1(M) \setminus \overline{\mathcal{H}}$ is h-expansive

and thus has a principal symbolic extension.

Any generic $f \in \mathcal{H}$ has no symbolic extension.

M compact smooth manifold,
$\mathcal{H} := \{f \in Diff^1(M)$ admits an homoclinic tangency $\}$.

Any $f \in Diff^1(M) \setminus \overline{\mathcal{H}}$ is h-expansive

and thus has a principal symbolic extension.

Any generic $f \in \mathcal{H}$ has no symbolic extension.

M compact smooth manifold,
$\mathcal{H} := \{ f \in \mathit{Diff}^1(M) \text{ admits an homoclinic tangency } \}$.

Any $f \in \mathit{Diff}^1(M) \setminus \overline{\mathcal{H}}$ is h-expansive

and thus has a principal symbolic extension.

Any generic $f \in \mathcal{H}$ has no symbolic extension.

M of dimension d, $f \in C^r(M)$, $r \geq 1$

Growth rate of the derivative :

$$R(f) := \lim_n \frac{1}{n} \sup_x \log^+ \|D_x f^n\|.$$

Tail entropy of C^r maps :

Theorem

$$h^*(f) = \frac{dR(f)}{r}.$$

In particular for $r = \infty$,

$$h^*(f) = 0.$$

Any C^∞ dynamical system has a measure of maximal entropy.

M of dimension d, $f \in C^r(M)$, $r \geq 1$

Growth rate of the derivative :

$$R(f) := \lim_n \frac{1}{n} \sup_x \log^+ \|D_x f^n\|.$$

Tail entropy of C^r maps :

Theorem

$$h^*(f) = \frac{dR(f)}{r}.$$

In particular for $r = \infty$,

$$h^*(f) = 0.$$

Any C^∞ dynamical system has a measure of maximal entropy.

M of dimension d, $f \in C^r(M)$, $r \geq 1$

Growth rate of the derivative :

$$R(f) := \lim_n \frac{1}{n} \sup_x \log^+ \|D_x f^n\|.$$

Tail entropy of C^r maps :

Theorem

$$h^*(f) = \frac{dR(f)}{r}.$$

In particular for $r = \infty$,

$$h^*(f) = 0.$$

Any C^∞ dynamical system has a measure of maximal entropy.

Theorem

Assume f is real analytic then there exists a constant $C > 0$ such that for any $\epsilon > 0$

$$h^*(f, \epsilon) \leq C \frac{\log(|\log \epsilon|)}{|\log \epsilon|}.$$

Explicit rate of convergence in ultradifferentiable classes which are sharp in many case.

But for the real analytic classe we conjecture that the optimal rate is in $1/|\log \epsilon|$. Surface real-analytic example with such a rate (non degenerate homoclinic tangency).

Theorem

Assume f is real analytic then there exists a constant $C > 0$ such that for any $\epsilon > 0$

$$h^*(f,\epsilon) \leq C \frac{\log(|\log \epsilon|)}{|\log \epsilon|}.$$

Explicit rate of convergence in ultradifferentiable classes which are sharp in many case.

But for the real analytic classe we conjecture that the optimal rate is in $1/|\log \epsilon|$. Surface real-analytic example with such a rate (non degenerate homoclinic tangency).

Theorem

Assume f is real analytic then there exists a constant $C > 0$ such that for any $\epsilon > 0$

$$h^*(f, \epsilon) \leq C \frac{\log(|\log \epsilon|)}{|\log \epsilon|}.$$

Explicit rate of convergence in ultradifferentiable classes which are sharp in many case.

But for the real analytic classe we conjecture that the optimal rate is in $1/|\log \epsilon|$. Surface real-analytic example with such a rate (non degenerate homoclinic tangency).

$f \in Diff^1(M)$, $\mu \in \mathcal{M}(M, f)$,

Lyapunov exponents :

$$\forall 0 < k \leq d,\ \chi^+ := \sum_{0<i\leq k} \chi_i^+(\mu) = \int \lim_n \frac{1}{n} \log^+ \|\Lambda^k D_x f^n\| \, d\mu(x)$$

Upper semi-continuity

$$\begin{aligned} \mathcal{M}(M,F) &\to \mathbb{R}, \\ \mu &\mapsto \sum_{0<i\leq d} \chi_i^+(\mu) \end{aligned}$$

defines a u.s.c. function.

$f \in Diff^1(M)$, $\mu \in \mathcal{M}(M,f)$,
Lyapunov exponents :

$$\forall 0 < k \leq d,\ \chi^+ := \sum_{0<i\leq k} \chi_i^+(\mu) = \int \lim_n \frac{1}{n} \log^+ \|\Lambda^k D_x f^n\| \ d\mu(x)$$

Upper semi-continuity

$$\begin{aligned} \mathcal{M}(M,F) &\rightarrow \mathbb{R}, \\ \mu &\mapsto \sum_{0<i\leq d} \chi_i^+(\mu) \end{aligned}$$

defines a u.s.c. function.

C^r interval/circle map or surface diffeomorphism f

Theorem

$h + \frac{\chi^+}{r-1}$ est une superenvelope.

The entropy function is a difference of u.s.c. functions.

C^r interval/circle map or surface diffeomorphism f

Theorem

$h + \frac{\chi^+}{r-1}$ *est une superenvelope.*

The entropy function is a difference of u.s.c. functions.

For $\delta > 0$,

$$Per_n^{\delta} := \{p \in Per_n, \chi_1(p) > \delta > -\delta > \chi_2(p)\}$$

$$p^*_{\delta}(f) \quad := \quad \limsup_n \frac{1}{n} \log \sup_x \sharp B(x, n, \epsilon) \cap Per_n^{\delta}.$$

a. δ-expansiveness for $r = \infty$:

Theorem

$$p^*_{\delta}(f) = 0$$

For $\delta > 0$,

$$Per_n^\delta := \{p \in Per_n, \chi_1(p) > \delta > -\delta > \chi_2(p)\}$$

$$p_\delta^*(f) \quad := \quad \limsup_n \frac{1}{n} \log \sup_x \sharp B(x,n,\epsilon) \cap Per_n^\delta.$$

a. δ-expansiveness for $r = \infty$:

Theorem

$$p_\delta^*(f) = 0$$

.

For $\delta > 0$,

$$Per_n^\delta := \{p \in Per_n, \chi_1(p) > \delta > -\delta > \chi_2(p)\}$$

$$p_\delta^*(f) \quad := \quad \limsup_n \frac{1}{n} \log \sup_x \sharp B(x, n, \epsilon) \cap Per_n^\delta.$$

a. δ-expansiveness for $r = \infty$:

Theorem

$$p_\delta^*(f) = 0$$

.

Semi-algebraic set :

$$A = \bigcup_{finite} \{P_1 > 0, ..., P_r > 0, P_{r+1} = 0\}.$$

Degree of A : $deg(A) = \sum_i deg(P_i)$.

For $\phi :]0,1[^d \to \mathbb{R}^d$ we let

$$\|\phi\|_r := \max_{\alpha,\ |\alpha| \leq r} \sup_{x \in]0,1[^d} \|\partial^\alpha f(x)\|$$

Theorem

Let $A \subset [0,1]^d$ be a semi-algebraic set. There exists a collection of finite maps $(\phi_i :]0,1[^d \to A)_{i \in I}$ s.t.

- $A = \bigcup_{i \in I} \phi_i(]0,1[^d)$,
- $\|\phi_i\|_r \leq 1$,
- $\sharp I \leq C(deg(P), d)$.

Semi-algebraic set :

$$A = \bigcup_{finite} \{P_1 > 0, ..., P_r > 0, P_{r+1} = 0\}.$$

Degree of A : $deg(A) = \sum_i deg(P_i)$.

For $\phi :]0,1[^d \to \mathbb{R}^d$ we let

$$\|\phi\|_r := \max_{\alpha,\ |\alpha| \leq r} \sup_{x \in]0,1[^d} \|\partial^\alpha f(x)\|$$

Theorem

Let $A \subset [0,1]^d$ be a semi-algebraic set. There exists a collection of finite maps $(\phi_i :]0,1[^d \to A)_{i \in I}$ s.t.

- $A = \bigcup_{i \in I} \phi_i(]0,1[^d)$,
- $\|\phi_i\|_r \leq 1$,
- $\sharp I \leq C(deg(P), d)$.

Semi-algebraic set :

$$A = \bigcup_{finite} \{P_1 > 0, ..., P_r > 0, P_{r+1} = 0\}.$$

Degree of A : $deg(A) = \sum_i deg(P_i)$.

For $\phi :]0,1[^d \to \mathbb{R}^d$ we let

$$\|\phi\|_r := \max_{\alpha,\ |\alpha| \leq r} \sup_{x \in]0,1[^d} \|\partial^\alpha f(x)\|$$

Theorem

Let $A \subset [0,1]^d$ be a semi-algebraic set. There exists a collection of finite maps $(\phi_i :]0,1[^d \to A)_{i \in I}$ s.t.

- $A = \bigcup_{i \in I} \phi_i(]0,1[^d)$,
- $\|\phi_i\|_r \leq 1$,
- $\sharp I \leq C(deg(P), d)$.

Theorem

Let $\sigma :]0,1[^d \to \mathbb{R}^d$ be a C^r map. Then there exists a collection of finite maps $(\phi_i :]0,1[^d \to \mathbb{R}^d)_{i \in I}$ s.t.

- $\sigma^{-1}(B(0,1)) \subset \bigcup_{i \in I} \phi_i(]0,1[^d)$,
- $\|\sigma \circ \phi_i\|_r \leq 1$,
- $\sharp I \leq C(deg(P), d) \|\sigma\|_r^{d/r}$.

Newhouse local entropy and local volume growth

$\sigma : [0,1]^k \to M$ of class $\mathcal{C}^r$ with $1 \leq k \leq d$, $\epsilon > 0$ and $F \subset M$,

$$v(\sigma,\epsilon,F) := \limsup_n \frac{1}{n} \ln^+ \sup_{x \in F} \int_{\sigma^{-1}B(x,n,\epsilon)} \|\Lambda^k D_y(T^n \circ \sigma)\|_k \, d\lambda(y)$$

$\nu \in \mathcal{M}_e(M,T)$,

$$v(\nu,\epsilon) := \lim_{\alpha \to 1} \inf_{\nu(F_\alpha) > \alpha} \sup_{\substack{\sigma:[0,1]^{l_\nu} \to M \\ \max_{k=1,\dots,r} \|D^k\sigma\|_\infty \leq 1}} v(\sigma,\epsilon,F_\alpha)$$

Theorem (New)

$\forall \epsilon > 0$, $\forall \nu \in \mathcal{M}_e(M,T)$,

$$h^{New}(\nu,\epsilon) \leq v(\nu,\epsilon)$$

Newhouse local entropy and local volume growth

$\sigma : [0,1]^k \to M$ of class $\mathcal{C}^r$ with $1 \leq k \leq d$, $\epsilon > 0$ and $F \subset M$,

$$v(\sigma, \epsilon, F) := \limsup_n \frac{1}{n} \ln^+ \sup_{x \in F} \int_{\sigma^{-1}B(x,n,\epsilon)} \|\Lambda^k D_y(T^n \circ \sigma)\|_k d\lambda(y)$$

$\nu \in \mathcal{M}_e(M, T)$,

$$v(\nu, \epsilon) := \lim_{\alpha \to 1} \inf_{\nu(F_\alpha) > \alpha} \sup_{\substack{\sigma : [0,1]^{l_\nu} \to M \\ \max_{k=1,\dots,r} \|D^k \sigma\|_\infty \leq 1}} v(\sigma, \epsilon, F_\alpha)$$

Theorem (New)

$\forall \epsilon > 0$, $\forall \nu \in \mathcal{M}_e(M, T)$,

$$h^{New}(\nu, \epsilon) \leq v(\nu, \epsilon)$$

Newhouse local entropy and local volume growth

$\sigma : [0,1]^k \to M$ of class $\mathcal{C}^r$ with $1 \leq k \leq d$, $\epsilon > 0$ and $F \subset M$,

$$v(\sigma, \epsilon, F) := \limsup_n \frac{1}{n} \ln^+ \sup_{x \in F} \int_{\sigma^{-1}B(x,n,\epsilon)} \|\Lambda^k D_y(T^n \circ \sigma)\|_k d\lambda(y)$$

$\nu \in \mathcal{M}_e(M, T)$,

$$v(\nu, \epsilon) := \lim_{\alpha \to 1} \inf_{\nu(F_\alpha) > \alpha} \sup_{\substack{\sigma:[0,1]^{l_\nu} \to M \\ \max_{k=1,\dots,r} \|D^k\sigma\|_\infty \leq 1}} v(\sigma, \epsilon, F_\alpha)$$

Theorem (New)

$\forall \epsilon > 0$, $\forall \nu \in \mathcal{M}_e(M, T)$,

$$h^{New}(\nu, \epsilon) \leq v(\nu, \epsilon)$$

Newhouse local entropy and local volume growth

$\sigma : [0,1]^k \to M$ with $1 \leq k \leq d$, $\epsilon > 0$, $F \subset M$ and $\chi > 0$.

$$v(\sigma, \epsilon, F, \chi) := \limsup_n \frac{1}{n} \ln^+ \sup_{x \in F} \int_{\sigma^{-1}B(x,n,\epsilon) \cap \{\frac{1}{n}\log \|D(T^n \circ \sigma)\| \simeq \chi\}} \|\Lambda^k(D_y T^n \circ \sigma)\|_k d\lambda$$

$\nu \in \mathcal{M}_e(M,T)$,

$$v(\nu, \epsilon) := \lim_{\alpha \to 1} \inf_{\nu(F_\alpha) > \alpha} \sup_{\substack{\sigma:[0,1]^{l_\nu} \to M \\ \max_{k=1,\dots,r} \|D^k\sigma\|_\infty \leq 1}} v(\sigma, \epsilon, F_\alpha, \chi_1(\nu))$$

Theorem (New)

$\forall \epsilon > 0$, $\forall \nu \in \mathcal{M}_e(M,T)$,

$$h^{New}(\nu, \epsilon) \leq v(\nu, \epsilon)$$

Proposition

Let $(\epsilon_k)_k$ be a sequence of positive numbers decreasing to 0. Then the sequence of functions $h - h^{New}(., \epsilon_k)$ defines an entropy structure.

To prove the Main Proposition for $l_\nu = 1$ it is enough to prove the following one

Proposition

$\forall \mu \in \mathcal{M}(M, T)\ \exists \delta_\mu > 0\ \exists \epsilon_\mu \in \mathbb{N}$
$\forall \nu \in \mathcal{M}_e(M, T)$ *with* $dist(\nu, \mu) < \delta_\mu$ *and* $l_\nu = 1$

$$v(\nu, \epsilon_\mu) \lesssim \frac{1}{r-1}\left(\overline{\chi_1^+}(\mu) - \chi_1^+(\nu)\right)$$

Proposition

Let $(\epsilon_k)_k$ be a sequence of positive numbers decreasing to 0. Then the sequence of functions $h - h^{New}(., \epsilon_k)$ defines an entropy structure.

To prove the Main Proposition for $l_\nu = 1$ it is enough to prove the following one

Proposition

$\forall \mu \in \mathcal{M}(M, T)$ $\exists \delta_\mu > 0$ $\exists \epsilon_\mu \in \mathbb{N}$
$\forall \nu \in \mathcal{M}_e(M, T)$ *with* $dist(\nu, \mu) < \delta_\mu$ *and* $l_\nu = 1$

$$v(\nu, \epsilon_\mu) \lesssim \frac{1}{r-1}\left(\overline{\chi_1^+}(\mu) - \chi_1^+(\nu)\right)$$

Reparametrization Lemma

Lemma

Let $T : M \to M$ *be a* $\mathcal{C}^r$ *map with* $r > 1$, *then* $\exists \epsilon > 0$ *s.t.*
$\forall \sigma : [0,1] \to M$ *with* $\max_{1 \leq k \leq r} \|D^k \sigma\| \leq 1$
$\forall \chi > 0\ \forall n \in \mathbb{N}\ \forall x \in M$
$\exists \mathcal{F}_n = (\phi_n : [0,1] \to [0,1])$ *a family of affine maps satisfying*

- $\|(T^n \circ \sigma \circ \phi_n)'\|_\infty \leq 1$;
- $\bigcup_{\phi_n \in \mathcal{F}_n} \phi_n([0,1]) \supset \sigma^{-1}\left(B(x,n,\epsilon) \cap \{\frac{1}{n} \log \|(T^n \circ \sigma)'\| \simeq \chi\}\right)$;
- $\frac{\log \sharp \mathcal{F}_n}{n} \lesssim \frac{1}{r-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} \log^+ \|D_{T^k x} T\| - \chi\right)$.

Reparametrization Lemma

Lemma

Let $T : M \to M$ *be a* $\mathcal{C}^r$ *map with* $r > 1$, *then* $\exists \epsilon > 0$ *s.t.*
$\forall \sigma : [0,1] \to M$ *with* $\max_{1 \leq k \leq r} \|D^k \sigma\| \leq 1$
$\forall \chi > 0 \ \forall n \in \mathbb{N} \ \forall x \in M$
$\exists \mathcal{F}_n = (\phi_n : [0,1] \to [0,1])$ *a family of affine maps satisfying*

- $\|(T^n \circ \sigma \circ \phi_n)'\|_\infty \leq 1$;
- $\bigcup_{\phi_n \in \mathcal{F}_n} \phi_n([0,1]) \supset \sigma^{-1}\left(B(x,n,\epsilon) \cap \{\frac{1}{n} \log \|(T^n \circ \sigma)'\| \simeq \chi\}\right)$;
- $\frac{\log \sharp \mathcal{F}_n}{n} \lesssim \frac{1}{r-1}\left(\frac{1}{n} \sum_{k=0}^{n-1} \log^+ \|D_{T^k x} T\| - \chi\right)$.

Reparametrization Lemma

Lemma

Let $T : M \to M$ *be a* $\mathcal{C}^r$ *map with* $r > 1$, *then* $\exists \epsilon > 0$ *s.t.*
$\forall \sigma : [0,1] \to M$ *with* $\max_{1 \leq k \leq r} \|D^k \sigma\| \leq 1$
$\forall \chi > 0 \; \forall n \in \mathbb{N} \; \forall x \in M$
$\exists \mathcal{F}_n = (\phi_n : [0,1] \to [0,1])$ *a family of affine maps satisfying*

- $\|(T^n \circ \sigma \circ \phi_n)'\|_\infty \leq 1$;
- $\bigcup_{\phi_n \in \mathcal{F}_n} \phi_n([0,1]) \supset \sigma^{-1}\left(B(x,n,\epsilon) \cap \{\frac{1}{n} \log \|(T^n \circ \sigma)'\| \simeq \chi\}\right)$;
- $\frac{\log \sharp \mathcal{F}_n}{n} \lesssim \frac{1}{r-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} \log^+ \|D_{T^k x} T\| - \chi\right)$.

Reparametrization Lemma

Lemma

Let $T : M \to M$ *be a* $\mathcal{C}^r$ *map with* $r > 1$*, then* $\exists \epsilon > 0$ *s.t.*
$\forall \sigma : [0,1] \to M$ *with* $\max_{1 \leq k \leq r} \|D^k \sigma\| \leq 1$
$\forall \chi > 0 \; \forall n \in \mathbb{N} \; \forall x \in M$
$\exists \mathcal{F}_n = (\phi_n : [0,1] \to [0,1])$ *a family of affine maps satisfying*

- $\|(T^n \circ \sigma \circ \phi_n)'\|_\infty \leq 1$;
- $\bigcup_{\phi_n \in \mathcal{F}_n} \phi_n([0,1]) \supset$
  $\sigma^{-1}\left(B(x,n,\epsilon) \cap \{\frac{1}{n} \log \|(T^n \circ \sigma)'\| \simeq \chi\}\right)$;
- $\frac{\log \sharp \mathcal{F}_n}{n} \lesssim \frac{1}{r-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} \log^+ \|D_{T^k x} T\| - \chi\right)$.

Proof of the Main Proposition assuming the Reparametrization Lemma

Let $\mu \in \mathcal{M}(M, T)$, we want to prove there exist $\delta_\mu > 0$ and $\epsilon_\mu > 0$ s.t. for ergodic measure ν δ_μ-close to μ

$$v(\nu, \epsilon_\mu) \lesssim \frac{1}{r-1}\left(\overline{\chi_1^+}(\mu) - \chi_1^+(\nu)\right)$$

- Choice of δ_μ :

$$\overline{\chi_1^+}(\mu) = \inf_n \frac{1}{n} \int \log^+ \|D_x T^n\| d\mu(x) \simeq \int \log^+ \|D_x T\| d\mu(x)$$

Choose $\delta_\mu > 0$ s.t. for ν δ_μ-close to μ so that

$$\int \log^+ \|D_x T\| d\nu(x) \simeq \int \log^+ \|D_x T\| d\mu(x)$$

Proof of the Main Proposition assuming the Reparametrization Lemma

Let $\mu \in \mathcal{M}(M, T)$, we want to prove there exist $\delta_\mu > 0$ and $\epsilon_\mu > 0$ s.t. for ergodic measure ν δ_μ-close to μ

$$v(\nu, \epsilon_\mu) \lesssim \frac{1}{r-1}\left(\overline{\chi_1^+}(\mu) - \chi_1^+(\nu)\right)$$

- Choice of δ_μ :

$$\overline{\chi_1^+}(\mu) = \inf_n \frac{1}{n}\int \log^+ \|D_x T^n\| d\mu(x) \simeq \int \log^+ \|D_x T\| d\mu(x)$$

Choose $\delta_\mu > 0$ s.t. for ν δ_μ-close to μ so that

$$\int \log^+ \|D_x T\| d\nu(x) \simeq \int \log^+ \|D_x T\| d\mu(x)$$

Proof of the Main Proposition assuming the Reparametrization Lemma

Let $\mu \in \mathcal{M}(M, T)$, we want to prove there exist $\delta_\mu > 0$ and $\epsilon_\mu > 0$ s.t. for ergodic measure ν δ_μ-close to μ

$$v(\nu, \epsilon_\mu) \lesssim \frac{1}{r-1}\left(\overline{\chi_1^+}(\mu) - \chi_1^+(\nu)\right)$$

- Choice of δ_μ :

$$\overline{\chi_1^+}(\mu) = \inf_n \frac{1}{n} \int \log^+ \|D_x T^n\| d\mu(x) \simeq \int \log^+ \|D_x T\| d\mu(x)$$

Choose $\delta_\mu > 0$ s.t. for ν δ_μ-close to μ so that

$$\int \log^+ \|D_x T\| d\nu(x) \simeq \int \log^+ \|D_x T\| d\mu(x)$$

Proof of the Main Proposition assuming the Reparametrization Lemma

Let $\mu \in \mathcal{M}(M, T)$, we want to prove there exist $\delta_\mu > 0$ and $\epsilon_\mu > 0$ s.t. for ergodic measure ν δ_μ-close to μ

$$v(\nu, \epsilon_\mu) \lesssim \frac{1}{r-1}\left(\overline{\chi_1^+}(\mu) - \chi_1^+(\nu)\right)$$

- Choice of δ_μ :

$$\overline{\chi_1^+}(\mu) = \inf_n \frac{1}{n} \int \log^+ \|D_x T^n\| d\mu(x) \simeq \int \log^+ \|D_x T\| d\mu(x)$$

Choose $\delta_\mu > 0$ s.t. for ν δ_μ-close to μ so that

$$\int \log^+ \|D_x T\| d\nu(x) \simeq \int \log^+ \|D_x T\| d\mu(x)$$

- Choice of ϵ_μ : Apply the Reparametrization Lemma to T, you get ϵ_μ such that for all subset F of M, for all $\sigma : [0,1] \to M$ with $\max_{1\leq k\leq r} \|D^k\sigma\| \leq 1$,

$$v(\sigma, \epsilon_\mu, F, \chi_1^+(\nu)) \lesssim \overline{\lim}_n \frac{1}{n} \sup_{x\in F} \frac{1}{r-1} \left(\frac{1}{n} \sum_{k=0}^{n-1} \log^+ \|D_{T^k x} T\| - \chi_1^+(\nu)\right.$$

- Choice of F_α : For all $\alpha < 1$ we choose by the Ergodic Theorem a Borel set F_α with $\nu(F_\alpha) > \alpha$ s.t. $\left(\frac{1}{n}\sum_{k=0,\dots,n-1} \log^+ \|D_{T^k x} T\|\right)_n$ converges uniformly in $x \in F$ to $\int \log^+ \|D_x T\| d\nu(x)$.

- Conclusion : We get finally for ergodic measures ν δ_μ-close to μ,

- Choice of ϵ_μ : Apply the Reparametrization Lemma to T, you get ϵ_μ such that for all subset F of M, for all $\sigma : [0,1] \to M$ with $\max_{1\leq k\leq r} \|D^k\sigma\| \leq 1$,

$$v(\sigma, \epsilon_\mu, F, \chi_1^+(\nu)) \lesssim \overline{\lim}_n \frac{1}{n} \sup_{x\in F} \frac{1}{r-1} \left(\frac{1}{n} \sum_{k=0}^{n-1} \log^+ \|D_{T^k x} T\| - \chi_1^+(\nu$$

- Choice of F_α : For all $\alpha < 1$ we choose by the Ergodic Theorem a Borel set F_α with $\nu(F_\alpha) > \alpha$ s.t. $\left(\frac{1}{n}\sum_{k=0,\dots,n-1} \log^+ \|D_{T^k x} T\|\right)_n$ converges uniformly in $x \in F$ to $\int \log^+ \|D_x T\| d\nu(x)$.

- Conclusion : We get finally for ergodic measures ν δ_μ-close to μ,

- Choice of ϵ_μ : Apply the Reparametrization Lemma to T, you get ϵ_μ such that for all subset F of M, for all $\sigma : [0,1] \to M$ with $\max_{1\leq k\leq r} \|D^k\sigma\| \leq 1$,

$$v(\sigma, \epsilon_\mu, F, \chi_1^+(\nu)) \lesssim \overline{\lim}_n \frac{1}{n} \sup_{x\in F} \frac{1}{r-1} \left(\frac{1}{n} \sum_{k=0}^{n-1} \log^+ \|D_{T^k x} T\| - \chi_1^+(\nu\right.$$

- Choice of F_α : For all $\alpha < 1$ we choose by the Ergodic Theorem a Borel set F_α with $\nu(F_\alpha) > \alpha$ s.t. $\left(\frac{1}{n}\sum_{k=0,\dots,n-1} \log^+ \|D_{T^k x} T\|\right)_n$ converges uniformly in $x \in F$ to $\int \log^+ \|D_x T\| d\nu(x)$.

- Conclusion : We get finally for ergodic measures ν δ_μ-close to μ,

Sketch of Proof of the Reparametrization Lemma

For a $\mathcal{C}^r$ curve $\sigma : [0,1] \to \mathbb{R}^d$, we want estimate the local length, i.e the length of $\sigma|_{\sigma^{-1}(B(0,1))}$.

1. Yomdin's approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of smooth semi-algebraic maps s.t.*

- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi \in \mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp\mathcal{F} \leq C(r,d)$.

To estimate the local length of a $\mathcal{C}^r$ general curve, we proceed as follows. Cut the interval $[0,1]$ into $[\|D^r\sigma\|^{\frac{1}{r}}] + 1$ subintervals of size less than $\frac{1}{\|D^r\sigma\|^{\frac{1}{r}}}$. Reparametrize these intervals by affine

Sketch of Proof of the Reparametrization Lemma

For a $\mathcal{C}^r$ curve $\sigma : [0,1] \to \mathbb{R}^d$, we want estimate the local length, i.e the length of $\sigma|_{\sigma^{-1}(B(0,1))}$.

1. Yomdin's approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of smooth semi-algebraic maps s.t.*

- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi \in \mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp \mathcal{F} \leq C(r,d)$.

To estimate the local length of a $\mathcal{C}^r$ general curve, we proceed as follows. Cut the interval $[0,1]$ into $[\|D^r\sigma\|^{\frac{1}{r}}] + 1$ subintervals of size less than $\frac{1}{\|D^r\sigma|^{\frac{1}{r}}}$. Reparametrize these intervals by affine

Sketch of Proof of the Reparametrization Lemma

For a $\mathcal{C}^r$ curve $\sigma : [0,1] \to \mathbb{R}^d$, we want estimate the local length, i.e the length of $\sigma|_{\sigma^{-1}(B(0,1))}$.

1. Yomdin's approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of smooth semi-algebraic maps s.t.*

- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi \in \mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp \mathcal{F} \leq C(r,d)$.

To estimate the local length of a $\mathcal{C}^r$ general curve, we proceed as follows. Cut the interval $[0,1]$ into $[\|D^r\sigma\|^{\frac{1}{r}}] + 1$ subintervals of size less than $\frac{1}{\|D^r\sigma|^{\frac{1}{r}}}$. Reparametrize these intervals by affine

Sketch of Proof of the Reparametrization Lemma

For a $\mathcal{C}^r$ curve $\sigma : [0,1] \to \mathbb{R}^d$, we want estimate the local length, i.e the length of $\sigma|_{\sigma^{-1}(B(0,1))}$.

1. Yomdin's approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of smooth semi-algebraic maps s.t.*

- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi \in \mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp \mathcal{F} \leq C(r,d)$.

To estimate the local length of a $\mathcal{C}^r$ general curve, we proceed as follows. Cut the interval $[0,1]$ into $[\|D^r\sigma\|^{\frac{1}{r}}] + 1$ subintervals of size less than $\frac{1}{\|D^r\sigma|^{\frac{1}{r}}}$. Reparametrize these intervals by affine

Sketch of Proof of the Reparametrization Lemma

For a $\mathcal{C}^r$ curve $\sigma : [0,1] \to \mathbb{R}^d$, we want estimate the local length, i.e the length of $\sigma|_{\sigma^{-1}(B(0,1))}$.

1. Yomdin's approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of smooth semi-algebraic maps s.t.*

- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi \in \mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp\mathcal{F} \leq C(r,d)$.

To estimate the local length of a $\mathcal{C}^r$ general curve, we proceed as follows. Cut the interval $[0,1]$ into $[\|D^r\sigma\|^{\frac{1}{r}}] + 1$ subintervals of size less than $\frac{1}{\|D^r\sigma\|^{\frac{1}{r}}}$. Reparametrize these intervals by affine

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

2. New approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of affine maps s.t.*

-
- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$;
- $\bigcup_{\phi \in \mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$;
- $\sharp\mathcal{F} \leq C(r,d)$.

Lemma

Assume $\forall t, s \in [0,1],\ \|\sigma'(t) - \sigma'(s)\| \leq \frac{\|\sigma'\|_\infty}{3}$ *then* $\exists \mathcal{F}$ *a family of (affine) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1$;

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

2. New approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of affine maps s.t.*

-
- $\|(\sigma\circ\phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$;
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$;
- $\sharp\mathcal{F} \leq C(r,d)$.

Lemma

Assume $\forall t, s \in [0,1], \ \|\sigma'(t) - \sigma'(s)\| \leq \frac{\|\sigma'\|_\infty}{3}$ *then* $\exists\mathcal{F}$ *a family of (affine) maps s.t.*

- $\|D(\sigma\circ\phi)\|_\infty \leq 1$;

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

2. New approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of affine maps s.t.*

-
- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp\mathcal{F} \leq C(r,d)$.

Lemma

Assume $\forall t,s \in [0,1],\ \|\sigma'(t) - \sigma'(s)\| \leq \frac{\|\sigma'\|_\infty}{3}$ *then* $\exists \mathcal{F}$ *a family of (affine) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1$ *;*

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

2. New approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of affine maps s.t.*

-
- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp\mathcal{F} \leq C(r,d)$.

Lemma

Assume $\forall t,s \in [0,1],\ \|\sigma'(t) - \sigma'(s)\| \leq \frac{\|\sigma'\|_\infty}{3}$ *then* $\exists \mathcal{F}$ *a family of (affine) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1$ *;*

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

2. New approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists\mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of affine maps s.t.*

-
- $\|(\sigma\circ\phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$;
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$;
- $\sharp\mathcal{F} \leq C(r,d)$.

Lemma

Assume $\forall t,s\in[0,1]$, $\|\sigma'(t)-\sigma'(s)\| \leq \frac{\|\sigma'\|_\infty}{3}$ *then* $\exists\mathcal{F}$ *a family of (affine) maps s.t.*

- $\|D(\sigma\circ\phi)\|_\infty \leq 1$;

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

2. New approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of affine maps s.t.*

-
- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp\mathcal{F} \leq C(r,d)$.

Lemma

Assume $\forall t, s \in [0,1],\ \|\sigma'(t) - \sigma'(s)\| \leq \frac{\|\sigma'\|_\infty}{3}$ *then* $\exists \mathcal{F}$ *a family of (affine) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1$ *;*

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

2. New approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of affine maps s.t.*

-
- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp\mathcal{F} \leq C(r,d)$.

Lemma

Assume $\forall t, s \in [0,1],\ \|\sigma'(t) - \sigma'(s)\| \leq \frac{\|\sigma'\|_\infty}{3}$ *then* $\exists \mathcal{F}$ *a family of (affine) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1$ *;*

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

2. New approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of affine maps s.t.*

-
- $\|(\sigma\circ\phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp\mathcal{F} \leq C(r,d)$.

Lemma

Assume $\forall t, s \in [0,1],\ \|\sigma'(t) - \sigma'(s)\| \leq \frac{\|\sigma'\|_\infty}{3}$ *then* $\exists \mathcal{F}$ *a family of (affine) maps s.t.*

- $\|D(\sigma\circ\phi)\|_\infty \leq 1$ *;*

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

2. New approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of affine maps s.t.*

-
- $\|(\sigma \circ \phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp\mathcal{F} \leq C(r,d)$.

Lemma

Assume $\forall t, s \in [0,1],\ \|\sigma'(t) - \sigma'(s)\| \leq \frac{\|\sigma'\|_\infty}{3}$ *then* $\exists \mathcal{F}$ *a family of (affine) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1$ *;*

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

2. New approach

Lemma (Gromov-Yomdin)

Assume $\|(\sigma)^{(r)}\|_\infty \leq 1$ then $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ a family of affine maps s.t.

-
- $\|(\sigma\circ\phi)^{(k)}\|_\infty \leq 1$ *for* $k = 1, ..., r$ *;*
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1))$ *;*
- $\sharp\mathcal{F} \leq C(r,d)$*.*

Lemma

Assume $\forall t, s \in [0,1],\ \|\sigma'(t) - \sigma'(s)\| \leq \frac{\|\sigma'\|_\infty}{3}$ *then* $\exists\mathcal{F}$ *a family of (affine) maps s.t.*

- $\|D(\sigma\circ\phi)\|_\infty \leq 1$ *;*

Lemma (Gromov-Yomdin)

Assume $\|D^r\sigma\|_\infty \leq 1 \leftarrow a$ *then* $\exists\mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of (s.a.) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1 \leftarrow \|D(\sigma' \circ \phi)\|_\infty \leq \frac{a}{6}$;
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1)) \leftarrow (\sigma')^{-1}(B(0,2a))$;
- $\sharp\mathcal{F} \leq C(r,d)$.

We conclude that the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$ is bounded from above by $[\left(\frac{\|D^r\sigma\|_\infty}{a}\right)^{\frac{1}{r-1}}]+1$ up to a multiplicative constant.

Lemma (Gromov-Yomdin)

Assume $\|D^r\sigma\|_\infty \leq 1 \leftarrow a$ *then* $\exists\mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of (s.a.) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1 \leftarrow \|D(\sigma' \circ \phi)\|_\infty \leq \frac{a}{6}$;
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1)) \leftarrow (\sigma')^{-1}(B(0,2a))$;
- $\sharp\mathcal{F} \leq C(r,d)$.

We conclude that the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$ is bounded from above by

$[\left(\frac{\|D^r\sigma\|_\infty}{a}\right)^{\frac{1}{r-1}}]+1$ up to a multiplicative constant.

Lemma (Gromov-Yomdin)

Assume $\|D^r\sigma\|_\infty \leq 1 \leftarrow a$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of (s.a.) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1 \leftarrow \|D(\sigma' \circ \phi)\|_\infty \leq \frac{a}{6}$;
- $\bigcup_{\phi \in \mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1)) \leftarrow (\sigma')^{-1}(B(0,2a))$;
- $\sharp\mathcal{F} \leq C(r,d)$.

We conclude that the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$ is bounded from above by $[\left(\frac{\|D^r\sigma\|_\infty}{a}\right)^{\frac{1}{r-1}}]+1$ up to a multiplicative constant.

Lemma (Gromov-Yomdin)

Assume $\|D^r\sigma\|_\infty \leq 1 \leftarrow a$ *then* $\exists\mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of (s.a.) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1 \leftarrow \|D(\sigma' \circ \phi)\|_\infty \leq \frac{a}{6}$;
- $\bigcup_{\phi\in\mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1)) \leftarrow (\sigma')^{-1}(B(0,2a))$;
- $\sharp\mathcal{F} \leq C(r,d)$.

We conclude that the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$ is bounded from above by $\left[\left(\frac{\|D^r\sigma\|_\infty}{a}\right)^{\frac{1}{r-1}}\right]+1$ up to a multiplicative constant.

Lemma (Gromov-Yomdin)

Assume $\|D^r\sigma\|_\infty \leq 1 \leftarrow a$ *then* $\exists\mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of (s.a.) maps s.t.*

- $\|D(\sigma \circ \phi)\|_\infty \leq 1 \leftarrow \|D(\sigma' \circ \phi)\|_\infty \leq \frac{a}{6}$;
- $\bigcup_{\phi \in \mathcal{F}} \phi([0,1]) \supset \sigma^{-1}(B(0,1)) \leftarrow (\sigma')^{-1}(B(0,2a))$;
- $\sharp\mathcal{F} \leq C(r,d)$.

We conclude that the volume of $\sigma|_{\sigma^{-1}(B(0,1)) \cap \{\frac{a}{2} \leq \|\sigma'\| \leq 2a\}}$ is bounded from above by

$[\left(\frac{\|D^r\sigma\|_\infty}{a}\right)^{\frac{1}{r-1}}] + 1$ up to a multiplicative constant.

The induction

We are working in local small charts along the orbit of x :

$$T_n = \exp^{-1}_{T^n x}(\epsilon^{-1}.) \circ T \circ \exp_{T^{n-1}x}(\epsilon.)$$

One can choose ϵ small enough $\|D^s T_n\|_\infty \leq 1$ for all $2 \leq s \leq r$.

We put $T^n = T_n \circ T_{n-1} \circ ... \circ T_1$. Then compute the r-derivative of $T^n \circ \sigma$.

Painful ! It involves in particular the s-derivative of $T^{n-1} \circ \sigma$ of order less than r, but they are related with the r derivative by using Landau-Kolmogorov type inequalities.

The induction

We are working in local small charts along the orbit of x :

$$T_n = \exp^{-1}_{T^n x}(\epsilon^{-1}.) \circ T \circ \exp_{T^{n-1}x}(\epsilon.)$$

One can choose ϵ small enough $\|D^s T_n\|_\infty \leq 1$ for all $2 \leq s \leq r$.

We put $T^n = T_n \circ T_{n-1} \circ \ldots \circ T_1$. Then compute the r-derivative of $T^n \circ \sigma$.

Painful ! It involves in particular the s-derivative of $T^{n-1} \circ \sigma$ of order less than r, but they are related with the r derivative by using Landau-Kolmogorov type inequalities.

The induction

We are working in local small charts along the orbit of x :

$$T_n = \exp^{-1}_{T^n x}(\epsilon^{-1}.) \circ T \circ \exp_{T^{n-1}x}(\epsilon.)$$

One can choose ϵ small enough $\|D^s T_n\|_\infty \leq 1$ for all $2 \leq s \leq r$.

We put $T^n = T_n \circ T_{n-1} \circ \ldots \circ T_1$. Then compute the r-derivative of $T^n \circ \sigma$.

Painful ! It involves in particular the s-derivative of $T^{n-1} \circ \sigma$ of order less than r, but they are related with the r derivative by using Landau-Kolmogorov type inequalities.