1. Entropy

Exercice 1. ***u.s.c. of K-S entropy w.r.t. zero-boundary partitions***.
Let (X,T) be a topological system. We consider a T-invariant Borel probability measure and a finite Borel partition P with $\mu(\partial P)=0$. Check that

$$\limsup_{\nu\to\mu} h(\nu,P)\leq h(\mu,P)$$

Exercice 2. ***Entropy w.r.t. a family of functions***.
Let (X,T) be a topological system.

- For $f: X\to[0,1]$ we let $\mathcal{A}_f$ be the two-set partition of $X\times[0,1]$ defined by $\mathcal{A}_f := \{(x,t),\ f(x)\geq t\}\cup\{(x,t),\ f(x)<t\}$. Show that for the Lebesgue measure λ on $[0,1]$ we have $\mu\times\lambda(\partial\mathcal{A}_f)=0$ when f is continuous.
- Show there is a nondecreasing sequence of finite family of continuous $(\mathcal{F}_k)$ so that $(\mathcal{A}_{\mathcal{F}_k})_k := \left(\bigvee_{f\in\mathcal{F}_k}\mathcal{A}_f\right)_k$ generates the Borel sigma algebra on $X\times[0,1]$ modulo $\mu\times\lambda$ null sets for any $\mu\in\mathcal{M}(X,T)$.
- Prove that the K-S entropy function is the pointwise limit of a nondecreasing sequence of nonneagtive function $(h_k)_k$ with $h_0\equiv 0$ and h_k-h_l u.s.c. for any $k\geq l$. Deduce that the set of zero-entropy invariant measures is a $\mathcal{G}_\delta$ subset of $\mathcal{M}(X,T)$.

Exercice 3. ***Topological entropy and periodic growth for subshifts***.
1. General subshift. For a subshift (Y,S) and for any positive integer n we let $\mathcal{W}_n$ denotes the set of n-words in Y). Prove the topological entropy of a subshift (Y,S) is given by

$$\mathsf{h}_{\mathsf{top}}(Y)=\limsup_n \frac{1}{n}\log\sharp\mathcal{W}_n.$$

Deduce from the above equality that

$$\mathsf{h}_{\mathsf{top}}(Y)\geq p(S).$$

2. Subshift of finite type. Let $A\in M_d(\{0,1\})$. We consider the subshift of finite type (Y_A,S) with adjacency matrix A. Show that

$$\mathsf{h}_{\mathsf{top}}(S)=\log\rho(A)$$

and

$$\mathsf{h}_{\mathsf{top}}(S)=p(S).$$

Does any C^∞ maps admit a principal extension which is a subshift of finite type?

Exercice 4. ***Invariant measures and K-S entropy of SFT***.
Let (Y,S) be a subshift of finite type. Prove the following properties :

- the periodic measures are dense in the set of probability invariant measures (endowed with the weak-$*$ topology),

- the continuity points of the entropy function of an SFT are the invariants measures of zero entropy, which forms a residual set.

Do these properties hold for general subshifts?

Exercice 5. ***l.s.c. of top. entropy for smooth systems***.
Let $(f_t)_t : X \to X$ be a flow with positive topological entropy. We also consider a continuous vector field Y on the circle $\mathbb{R}/\mathbb{Z}$ vanishing only at 1 and the associated flow $(\phi_s^Y)_s$ on $\mathbb{R}/\mathbb{Z}$.

By considering the parametrized family $s \in \mathbb{R}^+ \mapsto g_s$ of dynamical systems on $X \times \mathbb{R}/\mathbb{Z}$ defined by

$$\forall (x,t) \in X \times \mathbb{R}/\mathbb{Z},\ g_s(x,t) = (f_{\sin(2\pi t)}(x), \phi_s^Y(t)),$$

Show $\mathsf{h}_{\mathsf{top}}(g_0) > 0$ whereas $\mathsf{h}_{\mathsf{top}}(g_s) = 0$ for all $s > 0$. Give an example of a 4-dimensional manifold M such that the topological entropy is not lower semi-continuous on $Diff^\infty(M)$.

2. Tail entropy

Exercice 6. ***Power formula***.
Let (X,T) be a topological dynamical system. Prove that for any $n \in \mathbb{Z}$

$$h^*(T^n) = |n| h^*(T).$$

Exercice 7. ***The tail entropy bounds the defect of u.s.c.***.
Let (X,T) be a topological system. We consider a T-invariant Borel probability measure μ.

- Let P be a finite partition. Check that for any $\delta > 0$ we have for μ a.e. x

$$\limsup_n \frac{1}{n} \log \sharp \{A^n \in P^n,\ A^n \cap B(x,n,\delta) \neq \emptyset\} \leq \mu([\partial P]^\delta) \log \sharp P,$$

  where $[\partial P]^\delta$ denotes the closed δ-neighborhood of the boundary of P.
- Build for any $\epsilon > 0$ a finite partition P_ϵ with diameter less than ϵ satisfying $\mu(\partial P_\epsilon) = 0$.
- Show that

$$\limsup_{\nu \to \mu} h(\nu) \leq h(\mu) + h^*(T).$$

Exercice 8. ***Robust tail entropy***.
Let X be a compact metric space and let $\mathcal{C}$ be a topological space of topological dynamical systems on X so that the topology on $\mathcal{C}$ is stronger than the usual C^0 topology. For any $T \in \mathcal{C}$ we let $h_C^*(T)$ be the robust tail entropy of T in $\mathcal{C}$

$$h_C^*(T) = \lim_{\epsilon \to 0} \limsup_{S \xrightarrow{\mathcal{C}} T} h^*(S, \epsilon).$$

Show that for any $\mu \in \mathcal{M}(X,T)$ we have

$$\limsup_{\nu \in \mathcal{M}(X,S) \to \mu} h_S(\nu) \leq h_T(\mu) + h_C^*(T).$$

Exercice 9. ***Examples***.
1. A continuous interval map f is said piecewise monotone, when there is a partition of $[0,1]$ given by a finite collection of intervals $\{I\}$ so that f is monotone on each I. Show that such maps are a. h-expansive (Hint: show that for any $0 < \epsilon < \inf_I |I|$ we have $h^*(f,\epsilon) \leq \log 2$). Assume moreover the map is piecewise expanding, i.e. $f|_I$ is C^1 and $|f'|_I| > \lambda > 1$ for any I. Prove then that f is also a. p-expansive.

2. Let $A \in GL_d(\mathbb{Z})$ and let f_A be the induced map on the d-torus. Such maps are called toral linear automorphisms maps. Prove these maps are h-expansive, i.e.

$$\exists \epsilon > 0 \ s.t. \ h^*(f,\epsilon) = 0.$$

3. Let X be a finite union of compact polygones in $\mathbb{R}^2$. We consider an homeomorphism of X which is affine on each polygone. Prove such systems are a. h-expansive.

4. We consider the geodesic flow $(g_t)_t$ on the unit tangent bundle of a compact surface with negative curvature. Show g_t is h-expansive for any t (Hint: We recall $\{y, \ \forall t \ d(g_t x, g_t y) < \epsilon\} = g_{]-\epsilon,\epsilon[}x$ for some positive ϵ. Use then Exercise 6).

3. Symbolic extensions and embeddings

Exercice 10. ***Symbolic extension of the identity***.
Let X be a compact metrizable space. Build a principal symbolic extension of the identity on X (Hint: Reduce to the case $X = \{0,1\}^{\mathbb{Z}}$. Then show there is a decreasing sequence $(Y_n)_n$ of subshifts with two letters, s.t. $h_{top}(Y_n) \xrightarrow{n} 0$ and Y_n is the disjoint union of 2^n subshifts).

Exercice 11. ***Principal symbolic extension and a.*** h***-expansiveness***.
Show that (X,T) admits a principal symbolic extension iff it is a. h-expansive.

Exercice 12. ***Non a.*** h***-expansive example***.
We consider a sequence of (a. h-)expansive topological dynamical systems $(X_n, T_n)_n$. Let (X,T) be the one-point compactification of the disjoint union of $(X_n, T_n)_n$, i.e.

- X is the disjoint union of the X_n's and a point $*$ and we consider the topology on X which admits as the basis of neighborhoods at $x \in X$ the open sets of X_n containing x if $x \in X_n$ and the collection

$\{\bigcup_{k>l} X_k,\ l\}$ if $x = *$ (one can check X is metrizable when so do the X_n's).
- $T = T_n$ on X_n for each n and $T* = *$.

Show $\mathsf{h}_{\mathsf{top}}(T) = \sup_n h_{top}(T_n)$ and $h^*(T) = \limsup_n \mathsf{h}_{\mathsf{top}}(T_n)$. When does T admit a measure of maximal entropy? Compute the tail entropy function, the symbolic extension entropy function and the order of accumulation of (X, T).

Exercice 13. ***Topological Rohlin towers***.
Let (X, T) be a zero-dimensional topological system. For any $\epsilon > 0$ and any integer n prove the existence of a clopen set U_n such that:

- the sets $T^k U_n$ are disjoint for $k = 0, ..., n$,
- $\bigcup_{|k|\le n} T^k U_n = X \setminus Per_n^\epsilon$ where Per_n^ϵ is an ϵ-neighborhood of the set Per_n of n-periodic points.

Exercice 14. ***Krieger's embedding Theorem***.
Let (X, T) be a zero-dimensional expansive system. Check that $\mathsf{h}_{\mathsf{top}}(T) < +\infty$. Prove there exists a topological embedding of (X, T) into the K-full shift with $K > \max\left(\mathsf{h}_{\mathsf{top}}(T), \sup_n \frac{\log \sharp Per_n(T)}{n}\right)$.

4. Entropy for C^r smooth systems

Exercice 15. ***Algebraic lemma in dimension*** 1.
Let $f :]0,1[\to \mathbb{R}$ be a semi-algebraic map. The functional version of the algebraic lemma in dimension 1 claims that for any positive integer r there exist C^r semi-algebraic maps $(\phi_i)_{i\in I}$ on $]0,1[$ s.t.

- $\bigcup_{i\in I} \phi_i(]0,1[) = f^{-1}(]0,1[)$,
- $\|\phi_i\|_r, \|f \circ \phi_i\|_r \le 1$,
- $\sharp I \le c(r)$.

(1) By considering the sets $\{|f'| \ge 1\}$ and $\{|f'| \le 1\}$ prove the reparametrization lemma for $r = 1$,
(2) Prove the lemma by induction on r (Hint: Consider intervals where $f^{(r+1)}$ is monotone and then use a quadratic parametrization $x \mapsto x^2$).

Exercice 16. ***Entropy conjecture for C^∞ maps***.
For a topological system (f, M) on a compact finite dimensional manifold we let $\rho(f_*)$ the maximal spectral radius of the induced map on the homology groups. The entropy conjecture states that for any C^1 dynamical system (f, M):

$$\mathsf{h}_{\mathsf{top}}(f) \ge \rho(f_*).$$

Assuming Yomdin's reparametrization lemma, prove the entropy conjecture for C^∞ maps.

Exercice 17. ***Positive local volume growth in finite smoothness***.
Consider a smooth system of the sphere $\mathbb{S}^{2l}$ with zero topological entropy and with a fixed point at which the map may be written in local coordinate as a diagonal matrix with $\lambda > 1$ on the first l terms of the diagonal and λ^{-1} on the l-last terms. Build a C^r smooth disc σ (as a graph of some C^r but not C^{r+1} function) such that the local volume growth of σ satisfies:

$$\frac{l \log \lambda}{r} \lesssim v^*(\sigma).$$

Exercice 18. ***u.s.c. of topological entropy for*** C^1 ***multimodal maps***.
Let $\mathcal{M}_k^1$ be the set C^1 piecewise monotone interval map with at most k intervals of monotonicity.

- Show that for any $\epsilon > 0$ and for any $g \in \mathcal{M}_k^1$,
$$h^*(g,\epsilon) \leq \frac{\log(k) \log^+(\|f'\|_\infty)}{|\log\left(w(g',\epsilon)\right)|},$$
where $w(g',\epsilon)$ denote the modulus of continuity of g', i.e.
$$w(g',\epsilon) := \sup_{|x-y|<\epsilon} |g'(x) - g'(y)|.$$
- Conclude the topological entropy is continuous on $\mathcal{M}_k^1$ (Hint: Use Exercise 8).

Exercice 19. ***Generic measures for*** C^r ***interval maps and surface diffeos with*** $r > 1$.
Let f be a C^r interval maps and surface diffeos with $r > 1$. Prove that for any $\epsilon > 0$ the set of measures with K-S entropy less than ϵ contains an open and dense set.

Exercice 20. ***Tail entropy bounds from superenvelopes***.
We recall $\frac{\chi^+}{r-1}$ is a superenveloppe for C^r interval maps or C^r surface diffeomorphisms with $r > 1$. In these settings deduce the following upperbound on the tail entropy function:

$$u(\mu) \leq \frac{\chi^+(\mu)}{r}.$$