# SYMBOLIC EXTENSIONS AND UNIFORM GENERATORS FOR TOPOLOGICAL REGULAR FLOWS.

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Abstract. Building on the theory of symbolic extensions and uniform generators for discrete transformations we develop a similar theory for topological regular flows. In this context a symbolic extension is given by a suspension flow over a subshift.

Contents



## 1. Introduction

Given an invertible dynamical system $(X, f)$ a *generator* is a finite partition $P$, which "generates" the system in the sense that the map from $(X, f)$ to $(P^{\mathbb{Z}}, \sigma)$, with $\sigma$ being the usual shift, which associates to any $x \in X$ its $P$-name $(P(f^k x))_{k\in\mathbb{Z}}$ defines an embedding (where $P(x)$ denotes the atom of $P$ containing $x \in X$). The nature of the embedding depends on the structure of the system, e.g. if we consider a measure preserving system (resp. Borel system, resp. topological system) we require the embedding to be measure theoretical (resp. Borel, resp. topological). A necessary set-theoretical condition for the existence of a generator is given by the cardinality of periodic points which has to be smaller than in a full shift over a finite alphabet, i.e. $\sup_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{n} \log \sharp\{x,\ f^n x = x\} < +\infty$. There are other conditions, of dynamical nature, namely entropic and expansive properties.

Generators in ergodic theory have a long history. As the entropy is preserved by isomorphism, an ergodic system with a generator has finite entropy. W.Krieger showed the converse in [19] : any ergodic system on a Lebesgue space with finite entropy admits a (finite) generator. For topological systems, expansiveness (which implies finite topological entropy) completely characterizes systems with a generator (see [18] and [20]). New developments appear recently for Borel systems. Namely M.Hochman has proved in [15, 16] that a Borel system admits a generator if and only if the entropy of its ergodic invariant measures is bounded from above.

When considering the time $t$-map $\phi_t$ of a (regular) flow $(X,\Phi)$ we are interrested in generators $P$ whose atoms are *towers* associated to a cross-section, i.e of the form $\{\phi_t(x),\ x\in A \text{ and } 0\leq t<t_S(x)\}$ with $A\subset S$ for a cross-section $S$ and its return time $t_S$. As a first step we aim to represent the system as a suspension flow or equivalently to build a (global) cross-section. For aperiodic ergodic flows it was achieved by W.Ambrose [2], whereas V.M.Wagh [33] obtained the analogous result in the Borel case. As shown by D.Rudolph [32] the roof function in this representation of ergodic flows may be always assumed to be two-valued ; moreover for finite entropy flows the two-partition consisting of the towers with constant return time is generating not only for the flow, but also for the ergodic time $t$-maps with $t$ less than the minimal return time. Recently K.Slutsky [30] built for Borel systems a Borel cross-section with a two-valued return map.

In the present paper we are interested in generators for topological systems. M.Boyle and T.Downarowicz [5] have developped a new theory of entropy revealing fine properties of expansiveness of discrete topological systems. They introduce new entropy invariants which allow in particular to know whether the system may be encoded with a finite alphabet or not. Formally such a code is given by a topological extension by a subshift over a finite alphabet, also called a *symbolic extension*. More recently T.Downarowicz and the author [10] have related the theory of symbolic extensions with a Krieger-like generators problem. For a discrete topological system $(X,T)$ they introduced *uniform generators* as Borel partitions $P$ of $X$ whose iterated partitions $P_T^{[-n,n]}:=\bigvee_{k=-n}^{n}T^kP$ have a diameter going to zero with $n$, in other terms $\sup_{y\in P_T^{[-n,n]}(x)}d(y,x)$ goes to zero uniformly in $x$ when $n$ goes to infinity (with $d$ being the distance on $X$). By Theorem 1 in [10] a uniform generator is given by a symbolic extension with a Borel embedding and vice versa. For aperiodic systems the existence of symbolic extensions is equivalent to the existence of uniform generators whereas the presence of periodic points generates other constraints to build uniform generators (see Theorem 55 in [10]).

We aim to develop such theories for topological flows. Regarding the discrete systems given by the time $t$-maps of a real flow, M.Boyle and T.Downarowicz have proved that for $t\neq 0$ the time $t$-map admits a symbolic extension if and only so does the time 1-map (Theorem 3.4 in [6]). Nevertheless we consider here the flow in its own, not only through its time-$t$ maps. We call *symbolic extension* of a topological flow any topological extension given by the suspension flow over a subshift with a positive continuous roof function, whereas a *uniform generator* for the flow is a symbolic extension of the flow with a Borel embedding.

For Axiom A flows R.Bowen [3] built a finite-to-one symbolic extension. Moreover the roof function may be chosen to be Hölder continuous and the subshift is of finite type. R.Bowen also introduced a notion of expansiveness for flows (satisfied in particular by Axiom A flows). The existence of symbolic extensions preserving entropy for expansive discrete dynamical systems is now well-known. For expansive flows, R.Bowen and P.Walters [4] built a symbolic extension but they ask whether this extension preserves the entropy. Their construction involves closed cross-sections and they wonder if one could choose carefully the closed cross-sections so that the associated symbolic extension has the same topological entropy. We will give a positive answer to this question for $C^2$ expansive flows.

A first step in the theory of symbolic extensions or uniform generators consists in reducing the problem to zero-dimensional systems. This is done by considering partitions with *small boundary*, i.e. with boundaries having zero measure for any invariant probability measure. The existence of partitions with small boundary and arbitrarily small diameter, known as the small boundary property, is related with the deep theory of mean dimension [25]. In the Section 2 we introduce a small boundary property for flows. This property is always satisfied for $C^2$ smooth flows when the set of periodic orbits with period less than $T$ is finite for any $T > 0$. For flows with the *small flow boundary* property we may build, by a similar construction as R.Bowen and P.Walters, a topological extension preserving entropy given by a suspension flow over a zero-dimensional discrete system. Then a symbolic extension of the flow may be built from a symbolic extension of this discrete system. Our main results may be stated as follows:

**Theorem 1.1.** *Let $(X, \Phi)$ be a (resp. aperiodic) regular topological flow with the small flow boundary property. It admits a symbolic extension (resp. a uniform generator) if and only if for some (any) $t \neq 0$, the time $t$-map admits a symbolic extension. Moreover this property is invariant under orbit equivalence.*

For a $C^\infty$ smooth aperiodic regular flow on a compact manifold, the time-$t$ maps is also $C^\infty$ smooth and thus admits a symbolic extension. Therefore the flow admits a uniform generator. In fact the symbolic extension (associated to the uniform generator) is in this case an isomorphic extension (see Section 2.3 for the definitions).

**Corollary 1.1.** *Any $C^\infty$ smooth aperiodic flow admits an isomorphic symbolic extension.*

For the time $t$-map $\phi_t$, $t \neq 0$, of a topological flow $(X, \Phi)$ we define a weak notion of uniform generators as follows. For $\alpha > 0$ a partition $P$ is said to be *an $\alpha$-uniform generator of $\phi_t$* when $\sup_{y \in P_{\phi_t}^{[-n,n]}(x)} d\left(y, \phi_{[-\alpha,\alpha]}(x)\right)$ with $\phi_{[-\alpha,\alpha]}(x) = \{\phi_s(x),\ |s| \leq \alpha\}$ goes to zero uniformly in $x \in X$.

**Theorem 1.2.** *Let $(X, \Phi)$ be a regular aperiodic flow admitting a uniform generator. Then for any $t \neq 0$ small enough (depending only on the topological entropy of $\Phi$) and for any $\alpha > 0$ there is a 3-partition of a Borel global cross-section such that the associated partition of $X$ in towers is an $\alpha$-uniform generator of $\phi_t$.*

## 2. Zero-dimensional suspension flows as models

Following R.Bowen and P.Walters we build from cross-sections an extension given by a suspension flow over a zero-dimensional system. When the cross-sections have small flow boundaries, this extension is isomorphic.

2.1. **Generalities on topological flows.** A pair $(X, \Phi)$ is called a *topological flow*, when $(X, d)$ is a compact metric space and $\Phi = X \times \mathbb{R} \to X$ is a continuous flow on $X$, i.e. $\Phi$ is continuous, $x \mapsto \Phi(x, 0)$ is the identity map on $X$ and $\Phi(\Phi(x, t), s) = \Phi(x, t + s)$ for all $t, s \in \mathbb{R}$, $x \in X$. For $t \in \mathbb{R}$ we let $\phi_t$ be the homeomorphism of $X$ given by $x \mapsto \Phi(x, t)$ and we will denote the flow by $\Phi = (\phi_t)_{t\in\mathbb{R}}$. The flow is said to be *singular* when there is (at least) a point $x \in X$ fixed by the flow, i.e. $\phi_t(x) = x$ for all $t$. Otherwise the flow is said to be *regular*. In the present paper the flow is always assumed to be regular.

2.1.1. *Cross-sections.* Following the pioneering works of Poincaré, we consider the return maps to cross-sections in order to study the flow.

**Definition 2.1.** *Let $(X, \Phi)$ be a topological flow. A cross-section $S$ of time $\eta > 0$ is a subset $S$ of $X$ such that the restriction of $\Phi : (x, t) \mapsto \phi_t(x)$ to $S \times [-\eta, \eta]$ is one-to-one.*

Any subset of a cross-section is itself a cross-section. The cross-section $S$ is *global* when there is $\xi > 0$ with $\Phi(S \times [-\xi, \xi]) = X$. Obviously any cross-section has an empty interior. Moreover any Borel cross-section has zero measure for any probability Borel measure invariant by the flow.

For an interval $I$ of $\mathbb{R}$ and a subset $E$ of $X$ we denote by $\phi_I(E)$ the subset $\Phi(E \times I) = \{\phi_t(x),\ t \in I \text{ and } x \in E\}$. Let $S$ be a cross-section of time $\eta$. For $0 < \zeta \leq \eta$ the set $S_\zeta := \phi_{[-\zeta,\zeta]}(S)$ is called the *$\zeta$-cylinder* associated to $S$. For a subset $E$ of $X$ we denote the interior of $E$ by $\mathsf{Int}(E)$, its closure by $\overline{E}$ and its boundary by $\partial E$. A cross-section $S$ of time $\eta$ is said *weakly extendable* when its closure $\overline{S}$ is itself a cross-section of time $\eta$. To shorten the notations we will write *w.e.c.* for weakly extendable cross-section. Again any subset of a w.e.c. is itself a w.e.c. and any closed cross-section is obviously a w.e.c.. We recall below a notion of interior adapted to w.e.c.'s (introduced in [4] for closed cross-sections).

**Definition 2.2.** *Let $S$ be a w.e.c. of time $\eta$ and let $0<\zeta\leq\eta$. The flow interior $\mathsf{Int}^\Phi(S)$ of $S$ is defined as follows*

$$\mathsf{Int}^\Phi(S):=\mathsf{Int}(S_\zeta)\cap S.$$

*We also define the flow boundary of $S$ as $\partial^\Phi S=\overline{S}\setminus\mathsf{Int}^\Phi(S)$ and the flow boundary of the cylinder $S_\zeta$ as $\partial^\Phi S_\zeta:=\phi_{[-\zeta,\zeta]}(\partial^\Phi S)$.*

The flow boundary $\partial^\Phi S$ of a w.e.c. is always closed since it may be written as $\partial^\Phi S=\overline{S}\setminus\mathsf{Int}(S_\zeta)$. The definition of $\mathsf{Int}^\Phi S$ does not depend on $0<\zeta\leq\eta$. Indeed if $x\in\mathsf{Int}(S_\eta)\setminus\mathsf{Int}(S_\zeta)$ for some $0<\zeta<\eta$ then there is a sequence $(x_n)_n$ in the complement of $S_\zeta$ converging to $x$. As $x$ belongs to $\mathsf{Int}(S_\eta)$ so does $x_n$ for $n$ large enough, in particular $x_n=\phi_{t_n}(y_n)$ for some $y_n\in S$ and $t_n\in]\zeta,\eta]$. By extracting subsequences there exist $y\in\overline{S}$ and $t\in[\zeta,\eta]$ with $x=\phi_t(y)$. Therefore, $x$ does not lie in $S$, because the closure $\overline{S}$ of $S$ is a cross-section of time $\eta$.

Consider a flow associated to a smooth nonvanishing vector field $\mathsf{X}$ on a compact $(d+1)$-manifold $M$. Then any embedded $d$-disc transverse to $\mathsf{X}$ defines a closed cross-section. We denote respectively by $\mathsf{B}_d$ and $\mathsf{S}_d$ the unit ball and the unit sphere in $\mathbb{R}^d$ for the Euclidean norm. If we let $h:\mathsf{B}_d\to M$ denote a smooth embedding satisfying $h(\mathsf{B}_d)=S$ then $\partial^\Phi S$ is the set $h(\mathsf{S}_d)$.

For a topological flow $(X,\Phi)$ any point belongs to the flow interior of a closed cross-section with arbitrarily small diameter (see [34] p. 270).

2.1.2. *Properties of the flow boundary.* The interior boundary of a w.e.c. may be characterized as follows :

**Lemma 2.1.** *Let $S$ be a w.e.c. of time $\eta$. The restriction of $\Phi$ to $\mathsf{Int}^\Phi(S)\times]-\eta,\eta[$ defines a homeomorphism onto $\mathsf{Int}(S_\eta)$. In particular $\mathsf{Int}^\Phi(\mathsf{Int}^\Phi(S))=\mathsf{Int}^\Phi(S)$. Moreover the boundary $\partial S_\zeta$ of $S_\zeta$ (in $X$) is the union of $\partial^\Phi S_\zeta$, $\phi_{-\zeta}(\overline{S})$ and $\phi_\zeta(\overline{S})$ for $0\leq\zeta\leq\eta$.*

*Proof.* As the restriction of $\Phi$ to $\overline{S}\times[-\eta,\eta]$ is a homeomorphism onto its image it is enough to check $\Phi(\mathsf{Int}^\Phi(S)\times]-\eta,\eta[)=\mathsf{Int}(S_\eta)$.

Take $x\in\mathsf{Int}(S_\eta)$. Let $y\in S$ with $\phi_\zeta(y)=x$ for some $\zeta$ with $|\zeta|\leq\eta$. Necessarily $|\zeta|<\eta$. If $\zeta=\eta$ we would have $\phi_t(y)\in\phi_{[-\eta,0]}S\subset S_\eta$ for $t>\eta$ close to $\eta$. Indeed for such $t$ we have $\phi_t(y)\notin\phi_{[0,\eta]}S$ by injectivity of $\Phi$ on $S\times[0,2\eta]$. Then any limit of $\phi_t(y)$ when $t$ goes to $\eta$ should belong to $\phi_{[-\eta,0]}\overline{S}$ but by continuity of the flow such a limit is necessarily equal to $\phi_\eta(y)$ contradicting the injectivity of $\Phi$ on $\overline{S}\times[-\eta,\eta]$. We argue similarly for $\zeta=-\eta$. It is thus enough to show $y\in\mathsf{Int}^\phi(S)$. Without loss of generality we can assume $\zeta<0$. Then we have $y\in S\cap\phi_{-\zeta}(\mathsf{Int}(S_\eta))=S\cap\mathsf{Int}(\phi_{-\zeta}(S_\eta))=S\cap\mathsf{Int}(S_{\eta+\zeta})$ where the last equality follows again from the injectivity of $\Phi$ on $S\times[0,2\eta]$.

Conversely let $y\in\mathsf{Int}^\Phi(S)$ and $\zeta\in]-\eta,\eta[$. We can assume $\zeta\geq 0$. Take $0<\xi<\eta-\zeta$. By definition of the flow interior the point $y$ belongs to $\mathsf{Int}(S_\xi)$ so that $x=\phi_\zeta(y)$ is in $\phi_\zeta(\mathsf{Int}(S_\xi))\subset\mathsf{Int}(S_\eta)$. Thus $\Phi(\mathsf{Int}^\Phi(S)\times]-\eta,\eta[)=\mathsf{Int}(S_\eta)$. In particular $\mathsf{Int}((\mathsf{Int}^\Phi S)_\eta)=\mathsf{Int}(S_\eta)$, and by taking the intersection with $\mathsf{Int}^\Phi(S)$ on both sides we get $\mathsf{Int}^\Phi(\mathsf{Int}^\Phi(S))=\mathsf{Int}^\Phi(S)$.

Finally we have for $0\leq\zeta\leq\eta$

$$\begin{aligned}\partial S_\zeta &= \overline{S_\zeta}\setminus\mathsf{Int}(S_\zeta),\\ &= \Phi\left(\left(\overline{S}\times[-\zeta,\zeta]\right)\setminus\left(\mathsf{Int}^\Phi(S)\times]-\zeta,\zeta[\right)\right),\\ &= \partial^\Phi S_\zeta\cup\phi_{-\zeta}(\overline{S})\cup\phi_\zeta(\overline{S}).\end{aligned}$$

□

We give below some topological properties of the flow boundary and the flow interior of a subset $A$ of a w.e.c. $B$ with respect to the induced topology on $B$.

**Lemma 2.2.** *Let $B$ be a w.e.c. of time $\eta$ and let $A\subset B$.*

*(1) $\mathsf{Int}^\Phi(A)$ is open in $B$,*

*(2)* $\partial_B A \subset \partial^\Phi A \subset \partial_B A \cup \partial^\Phi B$ *with* $\partial_B A$ *being the frontier of* $A$ *in* $B$.

*Proof.* (1) By definition we have $\mathsf{Int}^\Phi A = A \cap \mathsf{Int}(A_\eta)$. As $B$ is a cross-section of time $\eta$ containing $A$, then $\mathsf{Int}^\Phi(A)$ coincides with $B \cap \mathsf{Int}(A_\eta)$. The set $\mathsf{Int}(A_\eta)$ being open in $X$, the set $\mathsf{Int}^\Phi(A)$ is open in $B$ for the induced topology.

(2) The first inclusion follows directly from (1). We show the second one. Let $x \in \partial^\Phi A \backslash \partial^\Phi B \subset \mathsf{Int}^\Phi(B)$. If $x$ did not belong $\partial_B A$ then there would be an open subset $x \in O \subset \mathsf{Int}^\Phi(B)$ of $B$ with either $O \cap A = \emptyset$ or $O \subset A$. By Lemma 2.1 the set $\Phi_{]-\eta,\eta[}O$ is an open neighborhood of $x$ in $X$. In the first case this open neighborhood lies in the complement of $A$ contradicting $x \in \overline{A}$, whereas in the second case it lies in the interior of $A_\eta$ contradicting $x \notin \mathsf{Int}^\Phi(A)$. □

The flow boundary behaves with respect to intersection, union and complement in a similar way to the usual boundary.

**Lemma 2.3.** *Let* $A$ *and* $B$ *be w.e.c.'s of time* $\eta$.

*(1) When* $A \cup B$ *defines a w.e.c. of time* $\eta$*, we have*

$$\partial^\Phi(A \cup B) \subset \partial^\Phi A \cup \partial^\Phi B.$$

*If* $A$ *and* $B$ *are disjoint and closed, then the equality holds.*

*(2)*

$$\partial^\Phi(A \cap B) \subset \partial^\Phi A \cup \partial^\Phi B.$$

*(3)*

$$\partial^\Phi(B \setminus A) \subset \partial^\Phi B \cup \partial^\Phi A.$$

*Proof.* (1) The inclusion $\mathsf{Int}^\Phi(A) \cup \mathsf{Int}^\Phi(B) \subset \mathsf{Int}^\Phi(A \cup B)$ follows clearly from $\mathsf{Int}(A_\eta) \cup \mathsf{Int}(B_\eta) \subset \mathsf{Int}((A \cup B)_\eta)$. Then $\partial^\Phi(A \cup B) = \overline{A \cup B} \setminus \mathsf{Int}^\Phi(A \cup B) \subset \left(\overline{A} \setminus \mathsf{Int}^\Phi(A)\right) \cup \left(\overline{B} \setminus \mathsf{Int}^\Phi B\right)$. When $A$ and $B$ are closed and disjoint, the cylinder $(A \cup B)_\eta$ is the disjoint union of the closed cylinders $A_\eta$ and $B_\eta$. Thus we have $\mathsf{Int}((A \cup B)_\eta) = \mathsf{Int}(A_\eta) \cup \mathsf{Int}(B_\eta)$ and this easily implies the required equalities.

(2) Using cylinders as above we get easily $\mathsf{Int}^\Phi(A \cap B) \subset \mathsf{Int}^\Phi(A) \cap \mathsf{Int}^\Phi(B)$ so that $\partial^\Phi(A \cap B) = \overline{A \cap B} \setminus \mathsf{Int}^\Phi(A \cap B) \subset \left(\overline{A} \setminus \mathsf{Int}^\Phi(A)\right) \cap \left(\overline{B} \setminus \mathsf{Int}^\Phi(B)\right)$.

(3) Let $x \in \partial^\Phi(B \backslash A) \backslash \partial^\Phi B \subset \mathsf{Int}^\Phi(B)$. As $\mathsf{Int}^\Phi(A) \subset A$ is open in $\overline{B}$ then $\overline{B \setminus A} \cap \mathsf{Int}^\Phi(A) = \emptyset$ and $x \notin \mathsf{Int}^\Phi(A)$. If $x$ belongs to $\overline{A}$ then $x \in \partial^\Phi A$. If not, $x$ would belong to $\mathsf{Int}^\Phi(B) \setminus \overline{A}$ which is open in $\mathsf{Int}^\Phi(B)$. In particular $\phi_{]-\eta,\eta[}(\mathsf{Int}^\Phi(B) \setminus \overline{A})$ is open in $X$ according to Lemma 2.1. But this last open set contains $x$ and it is a subset of $(B \setminus A)_\eta$. Therefore $x$ should belong to $\mathsf{Int}^\Phi(B \setminus A)$. Contradiction. □

2.1.3. *Closed cross-sections.* We focus in this subsection on closed cross-sections and especially on global closed cross-sections.

**Lemma 2.4.** *Let* $S$ *be a closed cross-section of time* $\eta$.

*(1) The cylinder* $S_\eta$ *is closed, therefore* $\partial S_\eta \supset \partial^\Phi S_\eta$ *have an empty interior.*

*(2)* $\mathsf{Int}^\Phi(\partial^\Phi S) = \emptyset$*. In particular* $\partial^\Phi \partial^\Phi \partial^\Phi = \partial^\Phi \partial^\Phi$.

*(3) When* $S$ *is global, then* $\overline{\mathsf{Int}^\Phi(S)}$ *is also a global closed cross-section.*

*Proof.* (1) and (2) follow easily from the definitions. Let us check (3). By Lemma 2.1 the restriction of $\Phi$ to $\mathsf{Int}^\Phi(S) \times ]k\eta, (k+1)\eta[$ is an homeomorphism onto $\mathsf{Int}(\phi_{]k\eta,(k+1)\eta[}S)$ for any integer $k$. But $S$ being global we have $X = \Phi(S \times [-K\eta, K\eta])$ for some $K \in \mathbb{N}$. Thus the open set

$$\bigcup_{k=-K,\cdots,K-1} \Phi(\mathsf{Int}^\Phi(S) \times ]k\eta, (k+1)\eta[) = \bigcup_{k=-K,\cdots,K-1} \mathsf{Int}(\phi_{]k\eta,(k+1)\eta[}S)$$

is contained in $\Phi(\mathsf{Int}^\Phi(S)\times[-K\eta,K\eta])$ and is dense in $X$ by (1). Therefore $\Phi\left(\overline{\mathsf{Int}^\Phi(S)}\times[-K\eta,K\eta]\right)=\overline{\Phi(\mathsf{Int}^\Phi(S)\times[-K\eta,K\eta])}=X$. □

Now we consider a global closed cross-section $S$ of time $\eta$ and we let $\xi>0$ with $\Phi(S\times[-\xi,\xi])=X$. The first return time $t_S$ in $S$ defines a lower semicontinuous positive function as the cross-section $S$ is closed. Moreover $t_S$ is bounded from above by $2\xi$ and from below by $2\eta$. Let $\mathcal{C}_S\subset S$ be the (residual) subset of continuity points of $t_S$. The first return map in $S$, denoted by $T_S:S\to S$, $x\mapsto\phi_{t_S(x)}(x)$, is also continuous at any point of $\mathcal{C}_S$. In fact we may describe more precisely the continuity properties of $t_S$ and $T_S$.

**Lemma 2.5.** *The first return time $t_S$ in $S$ is a piecewise continuous map, i.e. there is a finite partition $(C_k)_k$ of $S$ into w.e.c.'s such that $t_S$ is uniformly continuous on each $C_k$. Moreover the boundaries in $S$ of the $C_k$'s have an empty interior in $S$.*

*Proof.* The set $Y$ defined by $Y:=\{(x,t)\in S\times\mathbb{R}^+,\ \phi_t(x)\in S\}$ is a closed subset of $S\times\mathbb{R}^+$. Let $\delta\in]0,\inf_{x\in S}t_S(x)[$. For any positive integer $k$ the closed intersection $Y\cap(S\times[k\delta,(k+1)\delta])$ is the graph of a continuous nonnegative function defined on a closed subset $B_k$ of $\mathfrak{S}$. Let denote this function by $f_k:B_k\to\mathbb{R}^+$. Then the return time $t_S$ coincides with $f_1$ on $C_1:=B_1$ (which may be the empty set) and with $f_k$ on $C_k:=B_k\setminus\left(\bigcup_{l<k}B_l\right)$ for every $k>1$. Moreover observe that $\bigcup_{1\leq k\leq K}C_k=\bigcup_{1\leq k\leq K}B_k=S$ with $K=\lceil\frac{2\xi}{\delta}\rceil$. The $B_k$'s being closed, the boundaries of the $C_k$'s in $S$ have an empty interior in $S$ (indeed the class of subsets, whose boundary has an empty interior, is closed under complement, finite unions and intersections and it contains the closed subsets). □

The set $\mathcal{C}_S$ of continuity points is not only residual, it contains the open and dense subset of $S$ given by the union of the interior sets in $S$ of the w.e.c.'s $C_k$ by Lemma 2.5. We relate below the set of discontinuity points of $t_S$ with the flow boundary of $S$.

**Lemma 2.6.** *Let $x\in S\setminus\mathcal{C}_S$. Then there exists $t\in[0,2\xi]$ with $\phi_t(x)\in\partial^\Phi S$.*

*Proof.* Assume by contradiction that there is no $t\in[0,2\xi]$ with $\phi_t(x)\in\partial^\Phi S$. Let us show $x$ belongs to $\mathcal{C}_S$. If not there would be a sequence $(x_n)_n$ of $S$ converging to $x$ with $2\xi\geq\lim_n t_S(x_n)>t_S(x)$. When $n$ is large enough, $\phi_{t_S(x)}(x_n)$ belongs to the complement of $\mathsf{Int}(S_\zeta)$ for some small $\zeta>0$ and thus so does $\phi_{t_S(x)}(x)$, in particular $\phi_{t_S(x)}(x)\in\partial^\Phi S$ contradicting our hypothesis. □

2.1.4. *Complete family of closed cross-sections.* A finite family $\mathcal{S}$ of disjoint closed cross-sections $S$ of time $\eta_{\mathcal{S}}>0$ is said *complete* when the cylinders $S_{\eta_{\mathcal{S}}/2}$ are covering $X$. In particular the set $\mathfrak{S}=\bigcup_{S\in\mathcal{S}}S$ defines a global closed cross-section and the first return time $t_{\mathfrak{S}}$ is bounded from above by $\eta_{\mathcal{S}}$. The *diameter* of such a family $\mathcal{S}$ is the maximum of the diameters of $S\in\mathcal{S}$. Any topological regular flow admits a complete family of closed cross-sections with arbitrarily small diameter (see Lemma 7 in [4]). Let us consider such a complete family $\mathcal{S}$ of cross-sections. For the closed global cross-section $\mathfrak{S}$, the conclusion of Lemma 2.5 holds for the partition $T_{\mathfrak{S}}^{-1}\mathcal{S}$ :

**Lemma 2.7.** *For every $S\in\mathcal{S}$ the first return time $t_{\mathfrak{S}}$ is uniformly continuous on the set $T_{\mathfrak{S}}^{-1}S$.*

*Proof.* We argue by contradiction. Let $(x_n)_n$ and $(y_n)_n$ be two sequences in $T_{\mathfrak{S}}^{-1}S$ with $\lim_n d(x_n,y_n)=0$ and $t_1:=\lim_n t_{\mathfrak{S}}(x_n)>t_2:=\lim_n t_{\mathfrak{S}}(y_n)>0$. By extracting subsequences we may assume $(x_n)_n$ (and thus $(y_n)_n$) is converging in $\mathfrak{S}$, say to $x$. Then $\phi_{t_1}(x)$ and $\phi_{t_2}(x)$ both belong to $S$. But we have also $t_1-t_2<\sup_{y\in\mathfrak{S}}t_{\mathfrak{S}}(y)\leq\eta_{\mathcal{S}}$. This contradicts the fact that $S$ is a cross-section of time $\eta_{\mathcal{S}}$. □

By Lemma 2.3 (1) we have $\partial^\Phi\mathfrak{S}=\bigcup_{S\in\mathcal{S}}\partial^\Phi S$. The flow boundary of $T_{\mathfrak{S}}^{-1}S$, for $S\in\mathcal{S}$, satisfies the following property :

**Lemma 2.8.** *For every $S\in\mathcal{S}$ we have*

$$\partial^\Phi(T_{\mathfrak{S}}^{-1}S)\subset\partial^\Phi\mathfrak{S}_{\eta_{\mathcal{S}}}\cup T_{\mathfrak{S}}^{-1}(\partial^\Phi S).$$

*Proof.* It is enough to show $\mathsf{Int}^\Phi(T_{\mathfrak{S}}^{-1}S) \supset T_{\mathfrak{S}}^{-1}(\mathsf{Int}^\Phi(S)) \cap \mathsf{Int}^\Phi(\mathfrak{S}) \cap \mathcal{C}_{\mathfrak{S}}$. Indeed this implies

$$\begin{aligned} \partial^\Phi(T_{\mathfrak{S}}^{-1}S) &= \overline{T_{\mathfrak{S}}^{-1}S} \setminus \mathsf{Int}^\Phi(T_{\mathfrak{S}}^{-1}S), \\ &\subset \left(\overline{T_{\mathfrak{S}}^{-1}S} \setminus T_{\mathfrak{S}}^{-1}S\right) \cup T_{\mathfrak{S}}^{-1}(\partial^\Phi S) \cup \partial^\Phi \mathfrak{S} \cup (\mathfrak{S} \setminus \mathcal{C}_{\mathfrak{S}}), \end{aligned}$$

but the set $\overline{T_{\mathfrak{S}}^{-1}S} \setminus T_{\mathfrak{S}}^{-1}S$ is contained in $\mathfrak{S} \setminus \mathcal{C}_{\mathfrak{S}}$, which by Lemma 2.6 is a subset of $\partial^\Phi \mathfrak{S}_{\eta_S}$.

Let $x \in T_{\mathfrak{S}}^{-1}(\mathsf{Int}^\Phi(S)) \cap \mathsf{Int}^\Phi(\mathfrak{S}) \cap \mathcal{C}_{\mathfrak{S}}$. Then $T_{\mathfrak{S}}(x) = \phi_{t_{\mathfrak{S}}(x)}(x)$ lies in $\mathsf{Int}(S_\zeta)$ for any small $\zeta > 0$. By continuity of the flow we have also $\phi_{t_{\mathfrak{S}}(x)}(y) \in \mathsf{Int}(S_\zeta)$ for $y \in \mathfrak{S}$ close enough to $x$. Therefore such points $y$ return in $S$ in a time close to $t_{\mathfrak{S}}(x)$. As $x$ belongs to $\mathcal{C}_{\mathfrak{S}}$ this correspond to their first return time. But $x$ also belongs to $\mathsf{Int}^\Phi(\mathfrak{S})$ so that there is an open subset $x \in O \subset \mathsf{Int}^\Phi(\mathfrak{S})$ of $\mathfrak{S}$ contained in $T_{\mathfrak{S}}^{-1}S$. Therefore $x$ belongs to $\mathsf{Int}^\Phi(T_{\mathfrak{S}}^{-1}S)$ by Lemma 2.1. □

2.1.5. *Suspension flows.* Let $(X, T)$ be a topological discrete system. Let $r : X \to \mathbb{R}^+$ be a positive continuous function. Consider the quotient space

$$X_r = \{(x, t) \,:\, 0 \le t \le r(x),\, x \in X \text{ and } (x, r(x)) \sim (Tx, 0)\}.$$

This quotient space $X_r$ is compact and metrizable (see Section 4 in [4]). The *suspension flow* over $(X, T)$ under the roof function $r$ is the flow $\Phi_r$ on $X_r$ induced by the time translation $T_t$ on $X \times \mathbb{R}$ defined by $T_t(x, s) = (x, s + t)$.

We call *Poincaré cross-section* of a topological flow $(X, \Phi)$ any closed cross-section $S$ such that the flow map $\Phi : (x, t) \mapsto \phi_t(x)$ is a surjective local homeomorphism from $S \times \mathbb{R}$ to $X$.

**Lemma 2.9.** *A cross-section is a Poincaré cross-section if and only if it is a global closed cross-section with empty flow boundary.*

*Proof.* Let $S$ be a Poincaré cross-section. By compactness of $X = \Phi(S \times \mathbb{R}) = \bigcup_{\zeta > 0} \Phi(S \times ]-\zeta, \zeta[)$ there is $\xi > 0$ with $\Phi(S \times [-\xi, \xi]) = X$ and $S$ is thus global. The sets $\phi_{]-\zeta,\zeta[}(S)$ for $\zeta > 0$ being open, we have $S = \mathsf{Int}^\Phi S$ and thus $\partial^\Phi S = \emptyset$ as $S$ is closed. Conversely, if $S$ is a closed cross-section with empty flow boundary, then from Lemma 2.1 the flow map $\Phi : S \times \mathbb{R} \to X$ defines a local homeomorphism onto its image. When the cross-section $S$ is moreover global, this map is then also surjective. □

As the roof function $r$ of the suspension flow $(X_r, \Phi_r)$ does not vanish, the flow is regular and the subset $X \times \{0\} \subset X_r$ defines a Poincaré cross-section of the suspension flow $X_r$. In fact any topological flow admits a Poincaré cross-section if and only if it is topologically conjugate to a suspension regular flow. When the flow space is one-dimensional, there always exists a Poincaré cross-section. Indeed any closed cross-section is zero-dimensional, so that any point belongs to a closed cross-section with empty flow boundary. Bowen-Walters construction then provides a complete family $\mathcal{S}$ of closed cross-sections with empty flow boundary. The union $\mathfrak{S} = \bigcup_{S \in \mathcal{S}} S$ defines therefore in this case a Poincaré cross-section. Here we consider topological suspension flows, but we may also define similarly a Borel (resp. ergodic) suspension flow over a discrete Borel (resp. ergodic) system with a bounded Borel (resp. integrable) roof function.

For the suspension flow $(X_r, \Phi_r)$ the $\Phi_r$-invariant measures are related with the $T$-invariant measures as follows. For a discrete topological system $(X, T)$ (resp. topological flow $(X, \Phi)$) we denote by $\mathcal{M}(X, T)$ (resp. $\mathcal{M}(X, \Phi)$) the set of $T$-invariant (resp. $\Phi$-invariant) Borel probability measures. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$. For any $\mu \in \mathcal{M}(X, T)$ the product measure $\mu \times \lambda$ induces a finite $\Phi_r$-invariant measure on $X_r$. This defines a homeomorphism between $\mathcal{M}(X_r, \Phi_r)$ and $\mathcal{M}(X, T)$. More precisely the map

$$\begin{aligned} \Theta : \quad \mathcal{M}(X, T) \to &\quad \mathcal{M}(X_r, \Phi_r), \\ \mu \mapsto &\quad \frac{(\mu \times \lambda)|_{X_r}}{\int r \, d\mu} \end{aligned}$$

is a homeomorphism (not affine in general), which preserves ergodicity.

2.1.6. $\Phi$*-invariant and* $\phi_t$*-invariant measures.* For any $t>0$ we let $i_t$ be the inclusion of $\mathcal{M}(X,\Phi)$ in $\mathcal{M}(X,\phi_t)$. In general this inclusion does not preserve ergodicity : for an ergodic $\mu\in\mathcal{M}(X,\Phi)$, the measure $i_t(\mu)$ need not be ergodic (however, by the spectral theory, for a fixed measure $\mu$, this may occur for at most countably many $t\in\mathbb{R}$).

For $t>0$ the map

$$\begin{aligned}\theta_t: \quad \mathcal{M}(X,\phi_t) \to & \quad \mathcal{M}(X,\Phi),\\ \mu \mapsto & \quad \frac{1}{t}\int_0^t \phi_s\mu\, ds\end{aligned}$$

defines a continuous affine map of $\mathcal{M}(X,\phi_t)$ onto $\mathcal{M}(X,\Phi)$, which is a retraction i.e. $\theta_t\circ i_t = \mathsf{Id}_{\mathcal{M}(X,\Phi)}$. As this last identity is immediate, we only check that $\theta_t(\mu)$ belongs to $\mathcal{M}(X,\Phi)$ for any $\mu\in\mathcal{M}(X,\phi_t)$. Clearly it is enough to show $\phi_u(\theta_t(\mu))=\theta_t(\mu)$ for any $u\in]0,t[$. This follows from the following equalities :

$$\begin{aligned}\phi_u\left(\frac{1}{t}\int_0^t\phi_s\mu\,ds\right) &= \frac{1}{t}\int_0^t \phi_{s+u}\mu\,ds,\\ &= \frac{1}{t}\int_u^{t+u}\phi_s\mu\,ds,\\ &= \frac{1}{t}\left(\int_u^t\phi_s\mu\,ds + \int_0^u\phi_{s+t}\mu\,ds\right) = \frac{1}{t}\int_0^t\phi_s\mu\,ds.\end{aligned}$$

2.1.7. *Orbit equivalence.* Two topological flows $(X,\Phi)$ and $(Y,\Psi)$ are *orbit equivalent* when there is a homeomorphism $\Lambda$ from $X$ onto $Y$ mapping $\Phi$-orbits to $\Psi$-orbits, preserving their orientation. Any flow obtained by a change of the time scale of a topological flow $(X,\Phi)$ is orbit equivalent to $(X,\Phi)$.

In the following we are interested in dynamical properties invariant under orbit equivalence. In general the topological entropy is not preserved by orbit equivalence. But for regular topological flows zero and infinite entropy are invariant [29].

## 2.2. **The small boundary property for flows.**

2.2.1. *Definitions.* For a topological discrete system $(X,T)$ (resp. topological flow $(X,\Phi)$), a subset $E$ has a *small boundary* when its boundary is a *null set*, i.e. it has zero measure for any $T$-invariant (resp. $\Phi$-invariant) Borel probability measure (similarly a Borel subset is said to be a *full set* when its complement is a null set).

We define now an adapted notion of small boundary for w.e.c.'s.

**Definition 2.3.** *Let* $(X,\Phi)$ *be a topological flow. A w.e.c.* $S$ *of time* $\eta$ *has a small flow boundary when* $\partial^\Phi S_\eta$ *is a null set.*

For a w.e.c. $S$, the closure $\overline{S}$ is also a cross-section and it is thus transverse to the flow, so that the subset $\phi_{-\eta}\overline{S}\cup\phi_\eta\overline{S}$ of $\partial S_\eta$ has zero measure for any $\Phi$-invariant Borel probability measure. Therefore in the above definition we may replace the flow boundary $\partial^\Phi S_\eta$ of $S_\eta$ by its usual boundary $\partial S_\eta$ according to Lemma 2.1. Moreover the small flow boundary property of $S$ does not depend of the time $\eta$.

**Lemma 2.10.** *Let* $(X,\Phi)$ *be a topological flow and let* $S$ *be a w.e.c. of time* $\eta>0$*. The following properties are equivalent :*

- *i)* $S$ *has a small flow boundary,*
- *ii)* $\lim_{\zeta\to 0}\frac{\mu(\partial^\Phi S_\zeta)}{\zeta}=0$ *for any* $\mu\in\mathcal{M}(X,\Phi)$*,*
- *iii)* $\frac{1}{\mathsf{T}}\sharp\{0<t<\mathsf{T},\ \phi_t(x)\in\partial^\Phi S\}\xrightarrow{T\to+\infty}0$ *uniformly in* $x\in X$*.*

*Proof.* We have trivially $i) \Rightarrow ii)$. Assume $ii)$ and let us prove $i)$. For a $\Phi$-invariant Borel probability measure $\mu$, we have for all positive integers $n$ and for all $\eta > 0$:

$$\begin{aligned}\mu(\partial^\Phi S_\eta) &= \mu\left(\phi_{[-\eta,\eta]}\partial^\Phi S\right),\\ &= \mu\left(\bigcup_{k=-n}^{n-1}\phi_{[k\eta/n,(k+1)\eta/n]}(\partial^\Phi S)\right),\\ &= 2n\times\mu(\partial^\Phi S_{\eta/2n}) = \eta\times\frac{\mu(\partial^\Phi S_{\eta/2n})}{\eta/2n}\xrightarrow{n} 0.\end{aligned}$$

Finally we show $ii)$ and $iii)$ are equivalent. We recall that $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$. By Birkhof ergodic theorem we have for $0 < \zeta < \eta/2$ and for any ergodic $\Phi$-invariant Borel probability measure $\mu$

$$\begin{aligned}\forall\mu \text{ a.e. } x, \quad \mu(\partial^\Phi S_\zeta) &= \lim_{\mathsf{T}\to+\infty}\frac{1}{\mathsf{T}}\lambda\left(\{0<t<\mathsf{T},\ \phi_t(x)\in\partial^\Phi S_\zeta\}\right),\\ &= \lim_{\mathsf{T}\to+\infty}\frac{2\zeta}{\mathsf{T}}\sharp\{0<t<\mathsf{T},\ \phi_t(x)\in\partial^\Phi S\},\\ \text{thus } \mu(\partial^\Phi S_\zeta) &\leq \limsup_{\mathsf{T}\to+\infty}\frac{2\zeta}{\mathsf{T}}\sup_{x\in X}\sharp\{0<t<\mathsf{T},\ \phi_t(x)\in\partial^\Phi S\}.\end{aligned}$$

By the ergodic decomposition this last inequality holds in fact for any $\Phi$-invariant Borel probability measure and therefore $iii) \Rightarrow ii)$.

We will use a Krylov-Bogolyubov's like argument to show $ii) \Rightarrow iii)$. For any $\mathsf{T} > 0$ take $x_\mathsf{T} \in X$ maximizing the function $x \mapsto \sharp\{0 < t < \mathsf{T},\ \phi_t(x) \in \partial^\Phi S\}$ on $X$. Let $\psi_\mathsf{T} : \mathbb{R} \to M$, $t \mapsto \phi_t(x_\mathsf{T})$ and let $\mu_\mathsf{T} := \psi_\mathsf{T}(\lambda_{[0,\mathsf{T}]})$ with $\lambda_{[0,\mathsf{T}]} := \frac{\lambda(\cdot\cap[0,\mathsf{T}])}{\mathsf{T}}$. For any $0 < \zeta < \eta/2$ we have

$$\begin{aligned}\mu_\mathsf{T}(\partial^\Phi S_\zeta) &= \frac{\lambda\left(\{0<t<\mathsf{T},\ \phi_t(x_\mathsf{T})\in\partial^\Phi S_\zeta\}\right)}{\mathsf{T}},\\ &\geq \frac{2\zeta}{\mathsf{T}}\left(\sharp\{0<t<\mathsf{T},\ \phi_t(x_T)\in\partial^\Phi S\}-1\right),\\ &\geq \frac{2\zeta}{\mathsf{T}}\left(\sup_{x\in X}\sharp\{0<t<\mathsf{T},\ \phi_t(x)\in\partial^\Phi S\}-1\right).\end{aligned}$$

As $\partial^\Phi S_\zeta$ is closed, any weak-$*$ limit $\mu$ of $(\mu_\mathsf{T})_\mathsf{T}$, when $\mathsf{T}$ goes to infinity, satisfies

$$\begin{aligned}\mu(\partial^\Phi S_\zeta) &\geq \limsup_{\mathsf{T}}\mu_\mathsf{T}(\partial^\Phi S_\zeta),\\ &\geq 2\zeta\times\limsup_{\mathsf{T}}\frac{\sup_{x\in X}\sharp\{0<t<\mathsf{T},\ \phi_t(x)\in\partial^\Phi S\}}{\mathsf{T}}.\end{aligned}$$

Thus if $\frac{\mu(\partial^\Phi S_\zeta)}{\zeta} \xrightarrow{\zeta\to 0} 0$ we get $iii)$. □

A discrete topological system (resp. topological flow) is said to have the *small boundary property* when there is a basis of neighborhoods with small boundary. We will consider the following corresponding notion for the flow boundary.

**Definition 2.4.** *A topological flow $(X, \Phi)$ is said to have the small flow boundary property when for any $x \in X$ and for any w.e.c. $S'$ with $x \in \mathsf{Int}^\Phi(S')$ there exists a subset $S$ of $S'$ with $x \in \mathsf{Int}^\Phi(S)$ such that the w.e.c. $S$ has a small flow boundary.*

In the above definition we may replace w.e.c. by closed cross-sections. For a w.e.c. with small flow boundary the associated cylinders have a small boundary, so that a topological flow with the small flow boundary property has in particular the small boundary property. Following the construction of R.Bowen and P.Walters any topological flow with the small flow boundary property admits a complete family of closed cross-sections with small flow boundary and with arbitrarily small diameter. Moreover we can assume that each cross-section in the family is contained in the flow interior of another closed cross-section (see also Lemma 2.4 in [17] for a similar construction).

2.2.2. *Essential and small boundary partitions.* For a discrete topological system $(X,T)$ (resp. topological flow $(X,\Phi)$) a partition $P$ of $X$ is said to have a small boundary when any atom in $P$ has a small boundary. Such a partition of $X$ is also called an *essential* partition.

**Lemma 2.11.** *For a topological system $(X,T)$ (resp. topological flow $(X,\Phi)$) a Borel partition $P$ of $X$ has a small boundary if and only if $\overline{A}\setminus A$ is a null set for every $A\in P$.*

*Proof.* The necessary condition is clear because the sets $\overline{A}\setminus A$ is contained in the boundary of $A\in P$. To prove the equivalence it is enough to see that $A\cap\partial A\subset\bigcup_{B\in P\setminus\{A\}}\overline{B}\setminus B$. Take $x\in A\cap\partial A$, in particular $x\notin \mathsf{Int}(A)$. Therefore there is a sequence $(x_n)_n$ in the complement set of $A$ going to $x$. By extracting a subsequence we may assume all $x_n$ are in $B$ for some $P\ni B\neq A$, so that $x$ belongs to $\overline{B}\setminus B$. □

The partition, generated by a finite cover of sets with small boundary, has itself a small boundary. Consequently a system with the small boundary property admits partitions with small boundary and arbitrarily small diameter. Similar properties also hold true for the small flow boundary property. A partition $P$ of a w.e.c. $S$ is said to have a *small flow boundary* when every atom in $P$ defines a w.e.c. with small flow boundary (in this case $S$ has itself a small flow boundary).

**Lemma 2.12.** *Let $(X,\Phi)$ be a topological flow with the small flow boundary property. Let $S\subset S'$ be w.e.c.'s with $\overline{S}\subset\mathsf{Int}^\Phi(S')$ such that $S$ has a small flow boundary. Then there are partitions of $S$ into w.e.c.'s with small flow boundary and arbitrarily small diameter.*

*Proof.* For all $x\in\overline{S}$ there is a w.e.c. $S_x\subset S'$ with small flow boundary and arbitrarily small diameter satisfying $x\in\mathsf{Int}^\Phi(S_x)\subset\mathsf{Int}^\Phi(S')$. The sets $(\overline{S}\cap\mathsf{Int}^\Phi S_x)_{x\in\overline{S}}$ define an open cover of $\overline{S}$. Let $E$ be a finite subset of $\overline{S}$ such that $(\overline{S}\cap\mathsf{Int}^\Phi(S_x))_{x\in E}$ is a finite open subcover. The partition of $S$ generated by the finite cover $(S\cap S_x)_{x\in E}$ of $S$ has a small flow boundary according to Lemma 2.3 (2). □

We define now the corresponding notion for global Borel cross-sections of a topological flow. Let $S$ be a global Borel cross-section of a topological flow $(X,\Phi)$. For $A\subset S$ we let $\mathsf{T}_A$ be the tower above $A$ defined as

$$\mathsf{T}_A:=\{\phi_t(x),\ x\in A \text{ and } 0\leq t<t_S(x)\}.$$

For a Borel partition $P$ of $S$, the towers $T_A$ for $A\in P$ define a Borel partition $\mathsf{T}_P$ of $X$.

**Definition 2.5.** *With the above notations, a Borel partition $P$ of $S$ is said essential when the associated partition $\mathsf{T}_P$ of $X$ in towers is essential.*

When $Q$ is a partition of $S$ and $P$ is an essential partition of $S$ finer than $Q$, then $Q$ is also essential.

**Lemma 2.13.** *Let $\mathcal{S}$ be a complete family of cross-sections with small flow boundary. Then the partition $\mathcal{S}$ of the global closed cross-section $\mathfrak{S}=\bigcup_{S\in\mathcal{S}}S$ is essential.*

*Proof.* It is enough to show the joined partition $P=\mathcal{S}\vee T_{\mathfrak{S}}^{-1}\mathcal{S}$ of $\mathfrak{S}$ is essential. For any $S\in\mathcal{S}$ the w.e.c. $T_{\mathfrak{S}}^{-1}(\partial^\Phi S)$ has a small flow boundary according to Lemma 2.10 :

$$\begin{aligned}\frac{1}{\mathsf{T}}\sharp\{0<t<\mathsf{T},\ \phi_t(x)\in T_{\mathfrak{S}}^{-1}(\partial^\Phi S)\} &= \frac{1}{\mathsf{T}}\sharp\{0<t<\mathsf{T},\ \phi_{t+t_{\mathfrak{S}}(\phi_t(x))}(x)\in\partial^\Phi S\},\\ &\leq \frac{1}{\mathsf{T}}\sharp\{0<t<\mathsf{T}+\eta_{\mathcal{S}},\ \phi_t(x)\in\partial^\Phi S\}\xrightarrow{\mathsf{T}\to+\infty}0 \text{ uniformly in } x\in X.\end{aligned}$$

Each $E\in P$ has therefore a small flow boundary by Lemma 2.8 and Lemma 2.3 (2). Moreover the restriction of $t_{\mathfrak{S}}$ to $E$ extends on $\overline{E}$ to a continuous function $t_{\mathfrak{S}}^E$ by Lemma 2.7. Therefore $\overline{T_E}\setminus T_E$ is contained in the union of $\overline{E}$, $\{\phi_{t_{\mathfrak{S}}^E(x)}(x),\ x\in\overline{E}\}$ and $\{\phi_t(x),\ 0\leq t\leq\sup t_{\mathfrak{S}}^E \text{ and } x\in\overline{E}\setminus E\}$. The first two sets are subsets of the global closed cross-section $\mathfrak{S}$ so that their measure is zero for any probability measure invariant by the flow. The last set is contained in $\partial^\Phi E_\zeta$ with $\zeta=\sup t_{\mathfrak{S}}$ and therefore it is also a null set because $E$ has a small flow boundary. □

**Remark 2.1.**
- *There is no statement similar to Lemma 2.11 for a complete family $\mathcal{S}$ of closed cross-sections with small flow boundary. Indeed, for every $S \in \mathcal{S}$ the set $\overline{S} \setminus S$ is empty but the cross-section $S$ has not necessarily a small flow boundary.*
- *For a Borel global cross-section, essential partitions are more general than partitions with small flow boundary, whose atoms are necessarily w.e.c.'s (the flow boundary makes only sense for a w.e.c.).*

2.2.3. *Small boundary for $\Phi$ and $\phi_t$.* The small flow boundary property for a closed cross-section may be related with the small boundary property of the associated cylinders for the discrete time-$t$ maps as follows.

**Lemma 2.14.** *Let $(X, \Phi)$ be a topological flow with a closed cross-section $S$ and let $t \neq 0$. The following properties are equivalent.*

*i) $S$ has a small flow boundary for the flow $\Phi$;*
*ii) for some $\eta > 0$ the flow boundary $\partial^\Phi S_\eta$ is a null set for the topological discrete system $(X, \phi_t)$.*

However there may exist $\phi_t$-invariant probability measures with $t \neq 0$ supported on $\phi_\eta S$ and thus on $\partial S_\eta$.

*Proof.* As any $\Phi$-invariant measure is $\phi_t$-invariant we have trivially $ii) \Rightarrow i)$. Assume $i)$ and let us prove $ii)$. Let $\mu$ be a $\phi_t$-invariant measure. The $\Phi$-invariant measure $\theta_t(\mu)$ satisfies $0 = \theta_t(\mu)(\partial S_\eta) = \frac{1}{t}\int_0^t \phi_s\mu(\partial S_\eta)\, ds$. Therefore we have $\phi_s\mu(\partial S_\eta) = \mu\left(\partial\phi_{[-\eta-s,\eta-s]}S\right) = 0$ for Lebesgue almost $s \in [0, t]$. This concludes the proof as we have $\partial^\Phi S_{\eta-s} \subset \partial\phi_{[-\eta-s,\eta-s]}S$ for $0 \leq s \leq \eta$. □

2.2.4. *Suspension flows.* Let $(X, T)$ be a topological discrete system. We consider the suspension flow $(X_r, \Phi_r)$ over the base $(X, T)$ under a positive continuous roof function $r : X \to \mathbb{R}^+$.

**Lemma 2.15.** *With the above notations, the flow $(X_r, \Phi_r)$ satisfies the small flow boundary property if and only if $(X, T)$ satisfies the small boundary property.*

*Proof.* Assume firstly $(X_r, \Phi_r)$ has the small flow boundary property. Let $(x, t) \in X_r$ with $0 \leq t < r(x)$. For any $R > 0$ the set $S' = \{y \in X,\ d(y, x) \leq R\} \times \{t\}$ is a closed cross-section containing $(x, t)$ in its flow interior (with $d$ being the distance on $X$). Therefore there exists another cross-section $S = U \times \{t\} \subset S'$ with a small flow boundary for the flow $\Phi_r$ and $(x, t) \in \mathsf{Int}^{\Phi_r}(S) = \mathsf{Int}(U) \times \{t\}$, i.e. $x \in \mathsf{Int}(U)$. For small enough $\zeta > 0$ we have $\Theta(\mu)(\partial^{\Phi_r} S_\zeta) = \frac{2\zeta\times\mu(\partial U)}{\int r\, d\mu} = 0$ for all $\mu \in \mathcal{M}(X, T)$. Thus $U \subset X$ is a neighborhood of $x$ with small boundary for $(X, T)$ and with diameter less than $R$. Therefore $(X, T)$ has the small boundary property.

Conversely we consider a topological system $(X, T)$ on the base with the small boundary property. Let $S'$ be a closed cross-section containing $(x, t)$ in its flow interior. Let $U$ be a closed neighborhood of $x$ with small boundary for $(X, T)$ and with diameter less than $R$. We let $S$ be the intersection of $S'$ with the $\xi$-cylinder of the closed cross-section $U \times \{t\}$. For $\xi > 0$ fixed and $\zeta$, $R$ small enough we have $\partial^{\Phi_r} S_\zeta \subset \partial U \times [-\xi + t, \xi + t]$. Therefore $(x, t)$ belongs to $\mathsf{Int}^{\Phi_r}(S)$ and $\Theta(\mu)(\partial^{\Phi_r} S_\zeta) \leq \frac{2\xi\times\mu(\partial U)}{\int r\, d\mu} = 0$ for all $\mu \in \mathcal{M}(X, T)$, thus $S$ has a small flow boundary. □

2.2.5. *Orbit equivalence.*

**Lemma 2.16.** *The small flow boundary property is preserved by orbit equivalence for regular flows.*

*Proof.* Let $(X, \Phi)$ and $(Y, \Psi)$ be two topological orbit equivalent flows, via a homeomorphism $\Lambda$ from $X$ onto $Y$. Clearly it is enough to show that the image by $\Lambda$ of a closed cross-section with small flow boundary is a closed cross-section with small flow boundary. By continuity of $\Lambda^{-1}$, we have

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall y \in Y,\ \psi_{[0,\delta]}(y) \subset \Lambda\left(\phi_{[0,\epsilon]}\left(\Lambda^{-1}y\right)\right).$$

If $S$ is a closed cross-section of $(X,\Phi)$ with time $\epsilon$, then $\Lambda(S)$ defines a closed cross-section of $(Y,\Psi)$ with time $\delta$. The homeomorphism $\Lambda^{-1}$ maps any $\Psi$-orbit of length $\mathsf{T}$ on a $\Phi$-orbit of length at most $\frac{\epsilon\mathsf{T}}{\delta}$ so that for any $y\in Y$

$$\begin{aligned}\frac{1}{\mathsf{T}}\sharp\{0<t<\mathsf{T},\ \psi_t(y)\in\partial^\Psi\Lambda(S)\} &\leq \frac{1}{\mathsf{T}}\sharp\{0<t<\mathsf{T},\ \psi_t(y)\in\Lambda(\partial^\Phi S)\},\\ &\leq \frac{1}{\mathsf{T}}\sharp\{0<t<\mathsf{T},\ \Lambda^{-1}(\psi_t(y))\in\partial^\Phi S\},\\ &\leq \frac{1}{\mathsf{T}}\sharp\{0<t<\frac{\epsilon\mathsf{T}}{\delta},\ \phi_t(\Lambda^{-1}y)\in\partial^\Phi S\}.\end{aligned}$$

It follows then from Lemma 2.10 iii), that if $S$ satisfies the small flow boundary property for $(X,\Phi)$ then so does $\Lambda(S)$ for $(Y,\Psi)$. □

2.2.6. *The case of $C^2$ smooth regular flows.* Building on works of E.Lindenstrauss and J.Kulesza we prove the small flow boundary property for $C^2$ smooth (regular) flows on compact manifolds (in fact our proof also applies to $C^1$ discrete systems). The closed cross-sections with small flow boundary obtained in our proof are given by smooth discs (in the previous works of E.Lindenstrauss and J.Kulesza we do not know if one can choose the neighborhoods with small boundary as topological balls). However we only deal with $C^2$ smooth flows on a compact manifold as we use differential transversality tools, whereas E.Lindenstrauss and J.Kulesza proved the small boundary property for homeomorphisms of a finite dimensional compact set.

A $C^2$ smooth flow is said to have the *smooth small flow boundary property* when Definition 2.4 holds true with closed cross-sections $S$ and $S'$ given by $C^1$ smooth discs transverse to the $C^1$ vector field generating the flow.

**Proposition 2.1.** *Let $(X,\Phi)$ be a $C^2$ smooth flow on a compact manifold $X$. We assume that for any $t>0$ the number of periodic orbits with period less than $t$ is finite. Then $(X,\Phi)$ has the smooth small flow boundary property.*

*Proof.* Let $d+1$ be the dimension of $X$. Fix $x\in X$ and let $S$ be a $C^1$ smooth embedded disc with $x\in\mathsf{Int}^\Phi(S)$. One easily builds a finite family of $C^1$ smooth embeddings $(h_i:\mathsf{B}_d\to X)_{i=0,\dots,N}$ with $h_0(0)=x$ and $h_0(B_d)\subset S$, such that the family $\mathcal{S}=(S_i)_i$ and $\mathcal{S}'=(S_i')_i$ with $S_i=h_i(\mathsf{B}_d/2)$ and $S_i'=h_i(\mathsf{B}_d)$ both define complete families of closed cross-sections with $\eta_{\mathcal{S}}=\eta_{\mathcal{S}'}$. For $0\leq i,j\leq N$ we let $t_{i,j}$ be the first hitting time from $S_i$ to $S_j$ :

$$\forall x\in S_i,\ t_{i,j}(x)=\min\{t>0,\ \phi_t(x)\in S_j\}.$$

We define similarly the first hitting time $t'_{i,j}$ from $S'_i$ to $S'_j$. Let $U_{i,j}=\{t_{i,j}\leq\eta_{\mathcal{S}}\}$ and $U'_{i,j}=\{t'_{i,j}<+\infty\}$. Finally we denote by $T_{i,j}:U_{i,j}\to S_j$ and $T'_{i,j}:U'_{i,j}\to S'_j$ the first associated hitting maps (see Figure 1). The following properties hold :

- the compact subset $h_i^{-1}(U_{i,j})$ of $\mathsf{B}_d/2$ is contained in $\mathsf{Int}\left(h_i^{-1}(U'_{i,j})\right)$,
- $h_j^{-1}\circ T'_{i,j}\circ h_i$ is a local $C^1$ diffeomorphism on $\mathsf{Int}\left(h_i^{-1}(U'_{i,j})\right)$, because the vector field $\mathsf{X}$ generating $\Phi$ is $C^1$ smooth,
- $T'_{i,j}=T_{i,j}$ on $U_{i,j}\subset U'_{i,j}$.

Let $\mathcal{C}_{i,j}$ be a finite collection of closed balls contained in $\mathsf{Int}\left(h_i^{-1}(U'_{i,j})\right)\subset\mathbb{R}^d$, which covers $h_i^{-1}U_{i,j}$, such that any $C\in\mathcal{C}_{i,j}$ is contained in an open ball, where $h_j^{-1}\circ T'_{i,j}\circ h_i$ is a $C^1$ diffeomorphism onto its image. Then for any $C\in\mathcal{C}_{i,j}$ the restriction of $h_j^{-1}\circ T'_{i,j}\circ h_i$ to $C$ extends $C^1$ smoothly to a diffeomorphism $T_C$ of $\mathbb{R}^d$. For all $(i_2,\dots,i_n)\in\{0,\dots,N\}^{n-1}$ and for all $C^n=(C_1,\dots,C_{n-1})\in\prod_{j=1}^{n-1}\mathcal{C}_{i_j,i_{j+1}}$ with $i_1=0$, we let $T^k_{C^n}:=T_{C_k}\circ\dots\circ T_{C_1}$ for $1\leq k\leq n-1$ (let also $T^{-k}_{C^n}=(T^k_{C^n})^{-1}$ for such $k$ and $T^0=\mathsf{Id}_{C_1}$). We denote by $I_{C^n}$ the subset consisting of 0 and the integers $k$ in $\{1,\dots,n-1\}$ with $i_{k+1}=0$. Finally we let $F_{C^n}$ be the closed subset of $C_1$ given by $F_{C^n}=\bigcap_{k=0}^{n-1}T^{-k}_{C^n}(C_{k+1})$.

We build a closed cross-section $\tilde{S}$ (given by a subdisc of $S_0$) with small flow boundary, arbitrarily small diameter and $x\in\mathsf{Int}^\Phi(\tilde{S})$. We let $E:=\{f\in C^1(\mathsf{S}_d,\mathbb{R}^+),\ \|f\|_\infty\leq 1\}$ endowed with the

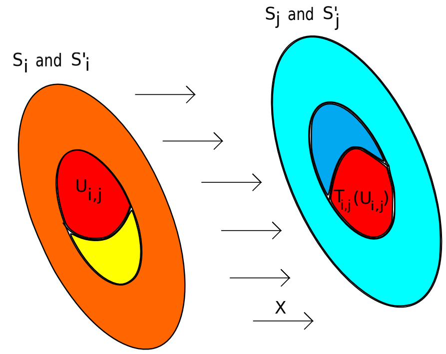


**Figure 1:** ***The set $U_{i,j} \subset S_i$ and its image by $T_{i,j}$ in red.***

usual $C^1$ topology (it is a Baire space). The cross-section $\tilde{S}$ will be of the form $\tilde{S} = h_0(S_f)$ with $S_f := \{rx,\ x \in \mathsf{S}_d,\ r \leq f(x)\} \subset \mathbb{R}^d$ for some positive $f \in E$. Observe that for such an $f$, the boundary $\partial S_f$ is a submanifold of dimension $d-1$ of $\mathbb{R}^d$. Moreover any $C^1$ small enough perturbation of this submanifold, i.e. any submanifold $H(\partial S_f)$ for a $C^1$ diffeomorphism $H$ of $\mathbb{R}^d$ close to the identity, is of the form $S_g$ for $g \in E$ close to $f$.

We will say that a map $f \in E$ is *$n$-transverse*, when for all $C^n$ as above, the submanifolds $\left(T_{C^n}^{-i}(\partial S_f)\right)_{i \in I_{C^n}}$ are mutually transverse on an open neighborhood of $F_{C^n}$. We recall that two submanifolds $M$ and $N$ of $\mathbb{R}^d$ are *mutually transverse* when either $M \cap N = \emptyset$ or $T_xM + T_xN = \mathbb{R}^d$ for all $x \in M \cap N$. In this last case the intersection $M \cap N$ is itself a submanifold with $\mathsf{codim}(M \cap N) = \mathsf{codim} M + \mathsf{codim} N \geq 0$. For $n > 2$ a family $(M_1, \cdots, M_n)$ of $n$ submanifolds of $\mathbb{R}^d$ is said *mutually transverse* when all proper subfamilies are mutually transverse and the submanifolds $M_i$ and $\bigcap_{j \neq i} M_j$ are transverse for any (some[1]) $i \in \{1, \cdots, n\}$. Then $\bigcap_{1=1,\cdots,n} M_i$ is either empty or a submanifold of codimension $\sum_{j=1,\cdots,n} \mathsf{codim}(M_j)$. Let $M_1, \cdots, M_{n-1}$ be mutually transverse submanifolds and let $M_n$ be an other submanifold. Then $M_1, \cdots, M_n$ are mutually transverse submanifolds if and only if $M_n$ and $\bigcap_{j \in J} M_j$ are mutually transverse for any $J \subset \{1, \cdots, n-1\}$. The submanifolds $(M_1, \cdots, M_n)$ are said mutually transverse on an open set $U$ when $(M_1 \cap U, \cdots, M_n \cap U)$ are mutually transverse.

**Claim 1.** *For any $n$ the subset $E_n$ of $E$ consisting of the $n$-transverse maps is open and dense.*

We postpone the proof of Claim 1. Let $f \in \bigcap_n E_n$. Any orbit of the flow hits at most $d-1$ times the set $\partial S_f$. Indeed for any $x \in S_f$ and any positive integer $n$ there is a $n$-uple $C^n$ such that (recall $T_{\mathfrak{S}}$ denotes the return map in the global cross-section $\mathfrak{S} = \bigcup_{S \in \mathcal{S}} S$) :

- $x \in F_{C^n}$,
- $\forall 0 \leq k \leq n,\ T_{\mathfrak{S}}^k(h_0(x)) = h_{i_{k+1}} \circ T_{C^n}^k(x)$.

The manifolds $\left(T_{C^n}^{-i}(\partial S_f)\right)_{i \in I_{C^n}}$ being mutually transverse submanifolds of dimension $d-1$ on an open neighborhood of $F_{C^n}$, any intersection of $d$'s of them with $F_{C^n}$ is empty. In particular $T_{C^n}^k(x) \in \partial S_f$ and therefore $T_{\mathfrak{S}}^k(h_0(x)) \in h_0(\partial S_f)$ for at most $d$ integers $k$ with $0 \leq k \leq n$. As it holds for any $n$, there at most $d$-many positive times $t$ with $\phi_t(h_0(x)) \in \partial S_f$. By Lemma 2.10 *iii*) the closed cross-section $h_0(S_f)$ has a small flow boundary. □

*Proof of the Claim 1.* By taking a finite intersection it is enough to consider a single $n$-uple $C^n = (C_1, ..., C_n)$ in the definition of $n$-transversality. To simplify the notations we then let $T^k = T_{C^n}^k$ for $0 \leq k \leq n-1$, $F_n = F_{C^n}$ and $I_n = I_{C^n}$. The tranversality being a $C^1$-stable property (see e.g. Proposition A.3.15 in [23]), the set $E_n$ is an open subset of $E$. We prove now by induction on $n$ the density of $E_n$ in $E$. Let $f \in E$. By induction hypothesis we may find a positive function $g \in E_{n-1}$ arbitrarily close to $f$. As the set of periodic orbits of $\Phi$ with period less than $t$ is finite

[1]Indeed this condition does not depend on $i$ because $\mathsf{dim}(T_xM_i + T_x(\bigcap_{j \neq i} M_j)) = \mathsf{dim}(T_xM_i) + d - \sum_{j \neq i} \mathsf{codim}(T_xM_j) - \mathsf{dim}(\bigcap_j T_xM_j)$ is equal to $d$ if and only if $\mathsf{codim}(\bigcap_j T_xM_j) = \sum_j \mathsf{codim}(T_xM_j)$

for all $t > 0$ we may assume that the spheres $(T^{-k}\partial S_g)_{k\in I_n}$ avoids the fixed points of $T^k$, $k \in I_n$ on $F_n$. Therefore there exist two finite families $(B_i)_{1\leq i\leq K}$ and $(B'_i)_{1\leq i\leq K}$ of open balls in $\mathbb{R}^d$ such that

- $\overline{B_i} \subset B'_i$ for all $i$,
- $B'_i, T^1 B'_i, ...T^{n-1}B'_i$ are pairwise disjoint for all $i$,
- $\bigcup_i B_i \supset F_n \cap \left(\bigcup_{k\in I_n} T^{-k}\partial S_g\right)$.

For $h \in E$ close enough to $g$ the property of the last item also holds true for $h$.

By induction on $j = 1, \cdots, K+1$ we produce a map $g^j$ arbitrarily close to $g$ in $E_{n-1}$ such that $T^{-k}\partial S_{g^j}$, $k \in I_n$, are mutually transverse on an open neighborhood $V_j$ of $F_n \cap \left(\bigcup_{i<j} \overline{B_i}\right)$. Finally we will get $g^{K+1} \in E_n$ for $g^{K+1}$ close enough to $g$ : if $O$ denotes an open neighborhood of $F_n \cap \left(\mathbb{R}^d \setminus (\bigcup_i B_i)\right)$ with $T^{-k}\partial S_{g^{K+1}} \cap O = \emptyset$ for all $k \in I_n$, then $(T^{-k}\partial S_{g^{K+1}})_{k\in I_n}$ are mutually transverse on the open neighborhood $O \cup V_{K+1}$ of $F_n$.

Take $g^1 = g$ and proceed to the inductive step by assuming $g^j \in E_{n-1}$ already built. In particular the submanifolds $(T^{-k}\partial S_{g^{j+1}})_{k\in I_n\setminus\{0\}}$ (resp. $(T^{-k}\partial S_{g^{j+1}})_{k\in I_n}$) are mutually transverse on an open neighborhood of $F_n$ (resp. of $F_n \cap \left(\bigcup_{i<j} \overline{B_i}\right)$) for $g^{j+1}$ $C^1$-close enough to $g^j$. It is therefore enough to find $g^{j+1}$ arbitrarily close to $g^j$ such that $T^{-k}\partial S_{g^{j+1}}$, $k \in I_n$, are mutually transverse on an open neighborhood of $F_n \cap \overline{B_j}$. When one only perturbs $\partial S_{g^j}$ on $B'_j$, it does not change the submanifolds $T^{-k}\partial S_{g^j}$, for $0 \neq k \in I_n$, on $B'_j$. By Theorem A.3.19 [23] there is a $C^1$ small perturbation $\partial S_{g^{j+1}}$ of $\partial S_{g^j}$, supported on $U_n^j \cap B'_j$ (i.e. $\partial S_{g^{j+1}} = H(\partial S_{g^j})$ for a diffeomorphism $H$ of $\mathbb{R}^d$ $C^1$-close to $\mathsf{Id}$ with $H = \mathsf{Id}$ apart from $U_n^j \cap B'_j$), such that $\partial S_{g^{j+1}}$ and $\bigcap_{k\in I'_n} T^{-k}\partial S_{g^j}$ are mutually transverse for any $I'_n \subset I_n \setminus \{0\}$ on a neighborhood of $F_n \cap \overline{B_j}$. But $\bigcap_{k\in I'_n} T^{-k}\partial S_{g^j}$ coincides with $\bigcap_{k\in I'_n} T^{-k}\partial S_{g^{j+1}}$ on $B'_j$, so that the submanifolds $\partial S_{g^{j+1}}$ and $\bigcap_{k\in I'_n} T^{-k}\partial S_{g^{j+1}}$ satisfy the same transversality property. As it holds for any $I'_n \subset I_n \setminus \{0\}$, the submanifolds $T^{-k}\partial S_{g^{j+1}}$, $k \in I_n$, are mutually transverse on an open neighborhood of $F_n \cap \overline{B_j}$. □

**Remark 2.2.** *We strongly believe the nonautonomous approach through the return maps developed above could be used to prove the small flow boundary property for general topological flows on finite-dimensional compact spaces by adapting the more sophisticated methods of E.Lindenstauss and J.Kulesza.*

**Remark 2.3.** *In* [14] *Proposition 4.1 it is was proved that a given finite partition of a smooth compact manifold, whose atoms have piecewise smooth boundaries, has a small boundary with respect to $C^r$ generic diffeomorphisms ($r \geq 1$). Here we use a"dual" approach by fixing the diffeomorphism (in fact the flow in our statement) and perturbing the boundaries of the partitions.*

2.3. **Representation by suspension flow.** Let $(X, \Phi = (\phi_t)_t)$ and $(Y, \Psi = (\psi_t)_t)$ be two continuous flows. We say $(Y, \Psi)$ is a topological extension of $(X, \Phi)$ when there is a continuous surjective map $\pi : Y \to X$ with $\pi \circ \psi_t = \phi_t \circ \pi$ for all $t$. The topological extension is said [2]:

- *principal* when it preserves the entropy of invariant measures, i.e. $h(\mu) = h(\pi\mu)$ for all $\mu \in \mathcal{M}(Y, \Psi)$,
- *isomorphic* when the map induced by $\pi$ on the sets of invariant Borel probability measures is bijective and $\pi : (Y, \Psi, \mu) \to (X, \Phi, \pi\mu)$ is a measure theoretical isomorphism for any $\mu \in \mathcal{M}(Y, \Psi)$,
- *strongly isomorphic* when there is a full set $E$ of $X$ such that the restriction of $\pi$ to $\pi^{-1}E$ is one-to-one.

Any strongly isomorphic extension is isomorphic and any isomorphic extension is principal.

For a discrete topological system $(X, T)$, to any nonincreasing sequence of partitions $(P_k)_{k\in\mathbb{N}}$ with $\mathsf{diam}(P_k) \xrightarrow{k} 0$ we may associate a zero dimensional extension $(Y, S)$ given by the closure $Y$ of $\{(P_k(T^n x))_{k,n}\,,\ x \in X\}$ in $\prod_k P_k^{\mathbb{Z}}$ (with the shift acting on each $k$-coordinate) mapping

[2] (Principal, isomorphic, strongly isomorphic) extensions are defined similarly for any group actions.

$(A_{k,n})_{k,n} \in Y$ to $\bigcap_{k,n} \overline{T^{-n}A_k^n} \in X$. When the partitions $P_k$ have small boundary then the extension is strongly isomorphic [11].

We build now in a similar way zero-dimensional extensions for a continuous flow $(X,\Phi)$. Let $\mathcal{S}$ be a complete family of closed cross-sections. As the first return map $T_{\mathfrak{S}}$ in $\mathfrak{S} = \bigcup_{S\in\mathcal{S}} S$ is a priori not continuous on $\mathfrak{S}$ we can not directly apply the previous construction to the system $(\mathfrak{S}, T_{\mathfrak{S}})$. We consider the skew product

$$\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S} := \left\{((A_n)_n, x) \in \mathcal{S}^{\mathbb{Z}} \times \mathfrak{S}, \forall n\ T_{\mathfrak{S}}^n x \in A_n\right\}.$$

As a consequence of Lemma 2.7 the map $\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S} \ni ((A_n)_n, x) \mapsto t_{\mathfrak{S}}(x)$ extends continuously on the closure of $\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}$ in $\mathcal{S}^{\mathbb{Z}} \times \mathfrak{S}$ (this extension will be again denoted by $t_{\mathfrak{S}}$). Moreover the map $T : \mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S} \circlearrowleft$, $((A_n)_n, x) \mapsto ((A_{n+1})_n, T_{\mathfrak{S}}x)$ extends to a homeomorphism of $\overline{\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}}$. We let $(\overline{\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}}, T)$ be the skew product system given by this extension.

**Lemma 2.17.** *The suspension flow over $(\overline{\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}}, T)$ with roof function given by $t_{\mathfrak{S}}$ is a topological extension of $(X,\Phi)$. Moreover when every $S \in \mathcal{S}$ has a small flow boundary, this extension is strongly isomorphic.*

*Proof.* Let $\left(\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}\right)_{t_{\mathfrak{S}}}$ be the invariant dense set above $\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}$ in the suspension flow $\left(\overline{\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}}\right)_{t_{\mathfrak{S}}}$. The map $\pi : \left(\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}\right)_{t_{\mathfrak{S}}} \to (X,\Phi)$ with $\pi\left(((A_n), x), t\right) = \phi_t(x)$ defines an equivariant surjective function continuously extendable on $\left(\overline{\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}}\right)_{t_{\mathfrak{S}}}$. Therefore its continuous extension, again denoted by $\pi$, defines a topological extension of $(X,\Phi)$.

The $\Phi$-orbit of a point with multiple $\pi$-preimages hits the set $\mathcal{C}_{\mathfrak{S}}$. Therefore by Lemma 2.6 the extension $\pi$ is one-to-one above the (residual) set of points whose $\Phi$-orbits do not visit the flow boundary of the closed cross-sections in $\mathcal{S}$. When these cross-sections have small flow boundaries this set has zero measure for any $\Phi$-invariant measure and thus the extension is strongly isomorphic. Indeed assume by contradiction there is $S \in \mathcal{S}$ with $\mu(\{x, \exists t \in [0,T]\ \phi_t(x) \in \partial^{\Phi} S\}) > 0$ for some $\mu \in \mathcal{M}(X,\Phi)$ and for some $T > 0$. By using the ergodic decomposition we may assume $\mu$ ergodic. From the ergodic theorem it follows then that the $\Phi$-orbit of $\mu$-almost every $x$ visits $\partial^{\Phi} S$ with a positive frequency contradicting item *iii*) of Lemma 2.10. □

In particular the skew product system $(\overline{\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}}, T)$ is a principal extension of $(X,\Phi)$ when the closed cross-sections in $\mathcal{S}$ have a small flow boundary. This answers in this case an open question of R.Bowen and P.Walters (aforementioned in the introduction).

A suspension flow over a zero-dimensional topological discrete system will be called a *zero-dimensional suspension flow* and a topological extension by a zero-dimensional suspension flow is said to be a *zero-dimensional extension.*

**Proposition 2.2.** *A topological flow with the small flow boundary property admits a zero-dimensional strongly isomorphic extension.*

*Proof.* Let $\mathcal{S}_0$ be a complete family of closed cross-sections with small flow boundary, such that each $S \in \mathcal{S}_0$ is contained in the flow interior of another closed cross-section. By Lemma 2.12 there is a nonincreasing sequence of partitions with small flow boundary $(\mathcal{S}_k)_{k\geq 1}$ of $\mathfrak{S} = \bigcup_{S\in\mathcal{S}_0} S$ finer than $\mathcal{S}_0$ satisfying $\mathsf{diam}(\mathcal{S}_k) \xrightarrow{k} 0$. Then we can follow the proof of Lemma 2.17, by replacing the skew product system $(\overline{\mathcal{S}^{\mathbb{Z}} \ltimes \mathfrak{S}}, T)$ by the closure $Y^{\mathcal{S}}$ in $\prod_{k\in\mathbb{N}} \mathcal{S}_k^{\mathbb{Z}}$ of $\left\{(\mathcal{S}_k(T_{\mathfrak{S}}^n x))_{k,n}\right\}$ with the shift acting on each $k$-coordinate, to get the desired zero-dimensional extension of $(X,\Phi)$ (the roof function is again given by $t_{\mathcal{S}_0}$). The extension is one-to-one above the set of points whose $\Phi$-orbits do not visit the flow boundary of the closed cross-sections in $\mathcal{S}_0$ and the boundary of $A$ in $S$ for every $(A,S) \in (\bigcup_{k\geq 1} \mathcal{S}_k, \mathcal{S}_0)$ with $A \subset S$. But this last boundary is a subset of the flow boundary of $A$ by Lemma 2.2. As the partitions $\mathcal{S}_k$, $k \geq 0$, have a small flow boundary, the extension is strongly isomorphic. □

For two orbit equivalent flows, the base of the suspension flows of the zero-dimensional extensions built in Proposition 2.2 may be chosen to be topologically conjugate. More precisely consider $(X, \Phi)$ and $(X', \Phi')$ two orbit equivalent topological flows via a homeomorphism $\Lambda : X \to X'$. Let $\mathcal{S} = (\mathcal{S}_k)_k$ be a sequence of complete families of closed cross-sections for $(X, \Phi)$ as in the proof of Proposition 2.2 and let $\Lambda(\mathcal{S}) = (\Lambda(\mathcal{S}_k))_k$ be the associated sequence for $(X', \Phi')$. Then the map

$$(\mathcal{S}_k(T^n_{\mathfrak{S}} x))_{k,n} \mapsto \left(\Lambda(\mathcal{S}_k)(T^n_{\Lambda(\mathfrak{S})}(\Lambda x))\right)_{k,n}$$

extends to a topological conjugacy of $Y^{\mathcal{S}}$ and $Y^{\Lambda(\mathcal{S})}$ (with the notations of the above proof).

Any topological system $(X, T)$ has a principal zero-dimensional extension [13]. By Proposition 2.2 any topological flow with the small flow boundary property admits a strongly isomorphic, therefore principal, zero-dimensional extension.

**Question 2.1.** *Does a topological flow always admits a principal zero-dimensional extension?*

## 3. Symbolic extensions and uniform generators for flows

In this section we develop a theory of symbolic extensions and uniform generators for flows. From now on we only consider topological flows and discrete systems with finite topological entropy. We recall that the topological entropy of a flow is given by the topological entropy of its time 1-map.

### 3.1. Definitions.

For a topological system $(X, T)$ a *symbolic extension* is a topological extension $\pi : (Y, S) \to (X, T)$ where $(Y, S)$ is a subshift over a finite alphabet. *A symbolic extension with an embedding* is a symbolic extension $\pi : (Y, S) \to (X, T)$ endowed with a Borel embedding $\psi : (X, T) \to (Y, S)$ satisfying $\pi \circ \psi = \mathsf{Id}_X$. A *uniform generator* is a Borel partition $P$ of $X$ such that the diameter of $\bigvee_{k=-n}^{n} T^{-k} P$ goes to zero when $n$ goes to infinity. The following statement follows from Theorem 1.2 in [10]. The characterization for clopen uniform generators was first proved in [18].

**Proposition 3.1.** *A topological system $(X, T)$ admits a uniform generator (resp. essential, resp. clopen) if and only if it admits a symbolic extension with an embedding (resp. a strongly isomorphic symbolic extension, resp. is topologically conjugate to a subshift).*

For a partition $P$ of $X$ we let $\psi^T_P : (X, T) \to (P^{\mathbb{Z}}, \sigma)$ be the equivariant map which associates to $x$ its $P$-name, i.e. $\psi^T_P : x \mapsto (P(T^k x))_{k \in \mathbb{Z}}$. Given a uniform generator $P$ we may in fact build explicitly a symbolic extension with an embedding. Indeed in this case the map $\psi^T_P$ is a Borel embedding and $\pi^T_P : \left(\overline{\psi^T_P(X)}, \sigma\right) \to (X, T)$, $(A_k)_k \mapsto \bigcap_{n \in \mathbb{N}} \overline{\bigcap_{|k| \le n} T^{-k} A_k}$ is a symbolic extension satisfying $\pi^T_P \circ \psi^T_P = \mathsf{Id}_X$. When moreover the uniform generator $P$ has a small boundary (resp. is clopen), then the symbolic extension $\pi^T_P$ is a strongly isomorphic extension (resp. a topological conjugacy).

Conversely when $\pi : (Y, S) \to (X, T)$ is a symbolic extension with an embedding $\psi$ then the partition $\psi^{-1} Q$ with $Q$ being the zero-coordinate partition of $(Y, S)$ defines a uniform generator. When this symbolic extension is a strongly isomorphic extension (resp. a topological conjugacy) then the corresponding uniform generator has a small boundary (resp. is clopen). Let us check this last point which did not appear in [10].

**Lemma 3.1.** *Let $\pi : (Y, S) \to (X, T)$ be a strongly isomorphic symbolic extension. Then the partition $\psi^{-1} Q$ with $Q$ being the zero-coordinate partition of $(Y, S)$ defines an essential uniform generator.*

*Proof.* Let $E$ be a full set of $X$ such that $\pi$ is one-to-one on $\pi^{-1} E$. Let $A \in Q$. A point $x \in \partial \psi^{-1} A$ is a limit of a sequence $(x_n)_n$ in $\psi^{-1} B$ for some $B \neq A \in Q$. Let $y_n = \psi(x_n)$ for all $n$. As $B$ is closed one can assume by extracting a subsequence that $(y_n)_n$ is converging to $y \in B$. Then $\pi(B) \ni \pi(y) = \lim_n \pi(y_n) = \lim_n \pi \circ \psi(x_n) = \lim_n x_n = x$. But we may also write $x$ as the limit of a sequence in $\psi^{-1} A$, therefore $x \in \pi(A)$. Thus $x$ has at least two preimages under $\pi$. In particular $\partial \psi^{-1} A \subset X \setminus E$ is a null set. $\square$

We consider now similar notions for a topological regular flow $(X,\Phi)$. A *symbolic extension* $\pi:(Y_r,\Phi_r)\to(X,\Phi)$ of $(X,\Phi)$ is a topological extension, where $(Y_r,\Phi_r)$ is a suspension flow over a subshift $(Y,S)$ with a positive continuous roof function $r$. *A symbolic extension with an embedding* of $(X,\Phi)$ is a symbolic extension $\pi:(Y_r,\Phi_r)\to(X,\Phi)$ endowed with a Borel embedding $\psi:(X,\Phi)\to(Y_r,\Phi_r)$ satisfying $\pi\circ\psi=\mathsf{Id}_X$. *A uniform generator of* $(X,\Phi)$ is a Borel global cross-section $S$ together a Borel partition $P$ of $S$ such that $\sup_{y\in P_{T_S}^{[-n,n]}(x)} d(y,x)$ and $\sup_{y\in P_{T_S}^{[-n,n]}(x)} |t_S(y)-t_S(x)|$ both go to zero uniformly in $x\in X$ when $n$ goes to infinity. We will say that this uniform generator $P$ :

- is *essential*, when the partition $P$ is essential,
- is *clopen*, when any atom in $P$ is closed with empty flow boundary.[3]

**Proposition 3.2.** *A topological flow $(X,\Phi)$ admits a uniform generator (resp. essential, resp. clopen) if and only if it admits a symbolic extension with an embedding (resp. a strongly isomorphic symbolic extension, resp. is topologically conjugate to a suspension flow over a subshift).*

*Proof.* Here again we make explicit the symbolic extension with an embedding for a given uniform generator, thus proving the necessary condition. For a Borel global cross-section $S$ and and a Borel partition $P$ of $S$ we let $\psi_P^\Phi:X\to P^{\mathbb{Z}}\times\mathbb{R}$ be the function which maps $x\in X$ to $\left(\psi_P^{T_S}(Tx),t(x)\right)$ with $t(x)\in\mathbb{R}^+$ and $T(x)\in S$ being respectively the last hitting time and the last hitting of $x$ in $S$ (in particular we have $x=\phi_{t(x)}(Tx)$). Assume $P$ defines a uniform generator. For any $(A_k)_k=\psi_P^{T_S}(x)$ with $x\in S$ we let $r\left((A_k)_k\right)=t_S(x)$. As the diameter of $t_S\left(P_{T_S}^{[-n,n]}(x)\right)$ goes to zero uniformly in $x$, the map $r$ extends continuously on $\overline{\psi_P^{T_S}(X)}$. The closure $\overline{\psi_P^{\Phi}(X)}$ is then just the suspension flow over $\left(\overline{\psi_P^{T_S}(X)},\sigma\right)$ with roof function $r$. The map $\psi_P^\Phi$ is then a Borel embedding in this flow and $\pi_P^\Phi:\overline{\psi_P^{\Phi}(X)}\to X,\ ((A_k)_k,t)\mapsto\phi_t\left(\bigcap_{n\in\mathbb{N}}\overline{\bigcap_{|k|\leq n}T_S^{-k}A_k}\right)$ is a symbolic extension satisfying $\pi_P^\Phi\circ\psi_P^\Phi=\mathsf{Id}_X$. When moreover the uniform generator $P$ is clopen then the maps $t$, $T$, $t_S$ and $T_S$ are continuous and therefore the Borel embedding $\psi_P^\Phi$ is a topological embedding. Assume now $P$ is essential, i.e. the partition $\mathsf{T}_P$ of $X$ in towers is essential. The extension $\pi_P^\Phi$ is one-to-one above points, whose orbit by the flow only lies in $S\cup\bigcup_{A\in P}\mathsf{Int}(\mathsf{T}_A)$. The complement of these points being a null set, the extension is strongly isomorphic.

Conversely, let $\pi:(Y_r,\Phi_r)$ be a symbolic extension of $(X,\Phi)$ with an embedding $\psi$, given by the suspension flow over the subshift $(Y,\sigma)$ with a positive continuous roof function $r$. Let $S$ be the global Borel cross-section given by $S=\psi^{-1}(Y\times\{0\})$. We denote by $Q$ the zero-coordinate partition of $Y$. We show now that the partition $P=\psi^{-1}(Q\times\{0\})$ of $S$ defines a uniform generator of $(X,\Phi)$. Firstly we have $P_{T_S}^{[-n,n]}(x)=\psi^{-1}(Q_\sigma^{[-n,n]}(\psi(x)))\subset\pi(Q_\sigma^{[-n,n]}(\psi(x)))$ for all $n\in\mathbb{N}$ and $x\in X$, so that $\mathsf{diam}(P_{T_S}^{[-n,n]})\xrightarrow{n}0$ by (uniform) continuity of $\pi$. Then for $x\in X$ we have $t_S^\Phi(x)=t_{Y\times\{0\}}^{\Phi_r}(\psi(x))=r(\psi(x))$. Therefore $\sup_{y\in P_{T_S}^{[-n,n]}(x)}|t_S^\Phi(y)-t_S^\Phi(x)|\leq\sup_{z\in Q_\sigma^{[-n,n]}(\psi(x))}|r(z)-r(\psi(x))|$ goes to zero uniformly in $x\in X$ when $n$ goes to infinity by (uniform) continuity of $r$. When $\psi$ is a topological embedding, the cross-section $S$ is a Poincaré cross-section and $(S,T_S)$ is topologically conjugate to $(Y,\sigma)$ through $\psi$. Therefore $(X,\Phi)$ is topologically conjugate to a suspension flow over a subshift in this case. When the symbolic extension $\pi$ is strongly isomorphic, by imitating the proof of Lemma 3.1 any point in the boundary of $\mathsf{T}_P$ lies in $\pi(\mathsf{T}_A)\cap\pi(\mathsf{T}_B)$ for some $A\neq B\in Q$ and thus does not belong to the full set $E$ for which $\pi:\pi^{-1}E\to E$ is one-to-one. The uniform generator $P$ is thus essential. □

3.2. **Entropy structure of topological flows.** We investigate now the theory of entropy structures for (regular) topological flows. To relate the entropy structure of a discrete system with the entropy structure of an associated suspension flow, we need to work with an a priori non monotone sequence. In order to deal with this case we generalize below the abstract theory of convergence developed by T.Downarowicz.

[3] In this case the cross-section $S$ is a Poincaré cross-section.

3.2.1. *Abstract theory of convergence.* Existence of symbolic extensions and uniform generators for discrete systems are related with some subtle properties of convergence of the entropy of measures computed at finer and finer scales. In [11] T.Downarowicz introduced an abstract framework to study the pointwise convergence of a nondecreasing sequence of functions. We recall now this theory with a slight generalization.

For a compact metric space $\mathfrak{X}$ we consider the set $\mathfrak{F}_{\mathfrak{X}}$ of all bounded sequences of real functions on $\mathfrak{X}$, i.e. the set of all sequences $\mathcal{H} = (h_k : \mathfrak{X} \to \mathbb{R})_{k\in\mathbb{N}}$ with $-\infty < \inf_k \inf_\mu h_k(\mu) \leq \sup_k \sup_\mu h_k(\mu) < +\infty$. Let $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ and $\mathcal{G} = (g_k)_{k\in\mathbb{N}}$ be two sequences in $\mathfrak{F}_{\mathfrak{X}}$. Following [11] we say $\mathcal{G}$ *uniformly dominates* $\mathcal{H}$ and we write $\mathcal{G} \succ \mathcal{H}$ when

$$\limsup_k \limsup_l \sup_{\mu\in\mathfrak{X}} (h_k - g_l)(\mu) \leq 0.$$

Similarly, for $\Gamma = (\gamma_k)_k$ and $\Theta = (\theta_k)_k$ in $\mathfrak{F}_{\mathfrak{X}}$ we say $\Gamma$ *uniformly yields* $\Theta$ and we write $\Gamma \prec \Theta$ when

$$\limsup_k \limsup_l \sup_{\mu\in\mathfrak{X}} (\gamma_l - \theta_k)(\mu) \leq 0.$$

The relation $\succ$ (idem for $\prec$) is preserved by translation : when $f : \mathfrak{X} \to \mathbb{R}$ is bounded and $(g_k)_k \succ (h_k)_k$ then $(g_k + f)_k \succ (h_k + f)_k$. Moreover $(g_k)_k \succ (h_k)_k$ if and only if $(-g_k)_k \prec (-h_k)_k$, but in this case $(-h_k)_k \succ (-g_k)_k$ does not hold true in general.

The binary relation $\succ$ is transitive on $\mathfrak{F}_{\mathfrak{X}}$. In particular when $\mathcal{H} \succ \mathcal{G}$ and $\mathcal{G} \succ \mathcal{H}$ then $\mathcal{H} \succ \mathcal{H}$. Thus the transitive relation $\succ$ induces an equivalence relation $\sim^{\succ}$ on $\mathfrak{G}_{\mathfrak{X}}^{\succ} := \{\mathcal{H} \in \mathfrak{F}_{\mathfrak{X}},\ \mathcal{H} \succ \mathcal{H}\}$ by letting

$$[\mathcal{G} \sim^{\succ} \mathcal{H}] \Leftrightarrow [\mathcal{G} \succ \mathcal{H} \text{ and } \mathcal{H} \succ \mathcal{G}].$$

In an obvious way we define similarly $\mathfrak{G}_{\mathfrak{X}}^{\prec}$ and $\sim^{\prec}$.

**Lemma 3.2.**
- *Any nonincreasing (resp. nondecreasing) sequence in $\mathfrak{F}_{\mathfrak{X}}$ belongs to $\mathfrak{G}_{\mathfrak{X}}^{\succ}$ (resp. $\mathfrak{G}_{\mathfrak{X}}^{\prec}$);*
- *Any sequence $\mathcal{H}$ in $\mathfrak{G}_{\mathfrak{X}}^{\succ}$ (resp. $\mathfrak{G}_{\mathfrak{X}}^{\prec}$) is converging pointwisely to a limit function* $\lim \mathcal{H}$.
- *Let $\mathcal{H} = (h_k)_k \in \mathfrak{G}_{\mathfrak{X}}^{\succ}$ and $\mathcal{G} = (g_k)_k \in \mathfrak{F}_{\mathfrak{X}}$ with $\lim_k \sup_{\mu\in\mathfrak{X}} |h_k - g_k|(\mu) = 0$ then $\mathcal{G}$ belongs to $\mathfrak{G}_{\mathfrak{X}}^{\succ}$ and $\mathcal{H} \sim^{\succ} \mathcal{G}$.*

*Proof.* The first point follows directly from the definitions. Let us check any sequence $(h_k)_k$ in $\mathfrak{G}_{\mathfrak{X}}^{\succ}$ is converging pointwisely. It is enough to check $\limsup_k h_k(\mu) = \liminf_l h_l(\mu)$ for all $\mu \in \mathfrak{X}$ :

$$\limsup_k h_k(\mu) - \liminf_l h_l(\mu) \quad = \quad \limsup_k \limsup_l (h_k - h_l)(\mu) \leq 0.$$

Finally we consider $\mathcal{H} = (h_k)_k \in \mathfrak{G}_{\mathfrak{X}}^{\succ}$ and $\mathcal{G} = (g_k)_k \in \mathfrak{F}_{\mathfrak{X}}$ with $\lim_k \sup_{\mu\in\mathfrak{X}} |h_k - g_k|(\mu) = 0$. Then we have $\mathcal{G} \succ \mathcal{H}$ :

$$\begin{aligned}
\limsup_k \limsup_l \sup_{\mu\in\mathfrak{X}} (h_k - g_l)(\mu) &\leq \limsup_k \limsup_l \sup_{\mu\in\mathfrak{X}} \left((h_k - h_l)(\mu) + (h_l - g_l)(\mu)\right), \\
&\leq \limsup_k \limsup_l \sup_{\mu\in\mathfrak{X}} (h_k - h_l)(\mu) + \limsup_l \sup_{\mu\in\mathfrak{X}} (h_l - g_l)(\mu), \\
&\leq \limsup_k \limsup_l \sup_{\mu\in\mathfrak{X}} (h_k - h_l)(\mu) \leq 0,
\end{aligned}$$

and one proves similarly $\mathcal{H} \succ \mathcal{G}$.

□

For a real function $f : \mathfrak{X} \to \mathbb{R}$ we let $\tilde{f}$ denote the upper semicontinuous envelope, i.e. $\tilde{f} = \inf\{g,\ g \geq f \text{ and } g \text{ is upper semicontinuous}\}$ if $f$ is bounded from above and $\tilde{f}$ is the constant function equal to $+\infty$ if not.

**Lemma 3.3.** *Let $\Theta = (\theta_k)_k \in \mathfrak{G}_{\mathfrak{X}}^{\prec}$. Then the following identity holds true :*

$$\limsup_k \sup_\mu \theta_k(\mu) = \sup_\mu \limsup_k \tilde{\theta_k}(\mu).$$

*Proof.* The inequality $\limsup_k \sup_\mu \theta_k(\mu) \geq \sup_\mu \limsup_k \tilde{\theta_k}(\mu)$ is trivial. Let $M$ be the supremum of $\limsup_k \tilde{\theta_k}$. Argue by contradiction by assuming $\limsup_k \sup_\mu \theta_k(\mu) > M$. Therefore we have for infinitely many $l$ and for some $\mu_l$

$$\theta_l(\mu_l) > M.$$

Then for any fixed $l' \leq l$ we have

$$\begin{aligned}\theta_{l'}(\mu_l) + (\theta_l - \theta_{l'})(\mu_l) &> M,\\ \theta_{l'}(\mu_l) + \sup_\mu(\theta_l - \theta_{l'})(\mu) &> M.\end{aligned}$$

We get for some weak-$*$ limit $\mu_\infty$ of $(\mu_l)_l$ after taking the limsup in $l$

$$\tilde{\theta_{l'}}(\mu_\infty) + \limsup_l \sup_\mu(\theta_l - \theta_{l'})(\mu) > M.$$

We let now $l'$ go to infinity. By using $\Theta \in \mathfrak{G}_{\mathfrak{X}}^{\prec}$ we obtain the following contradiction

$$\limsup_k \tilde{\theta_k}(\mu_\infty) > M = \sup_\mu \limsup_k \tilde{\theta_k}(\mu).$$

□

**Lemma 3.4.** *Let $\mathcal{H} = (h_k)_k \in \mathfrak{F}_{\mathfrak{X}}$ and $\mathcal{G} = (g_l)_l \in \mathfrak{G}_{\mathfrak{X}}^{\succ}$. If $\limsup_l \widetilde{(h_k - g_l)}(\mu) \leq 0$ for all $k \in \mathbb{N}$ and for all $\mu \in \mathfrak{X}$, then $\mathcal{G}$ uniformly dominates $\mathcal{H}$.*

*Proof.* For all $k$ the sequence $(h_k - g_l)_l$ belongs to $\mathfrak{G}_{\mathfrak{X}}^{\prec}$. By Lemma 3.3 we get for all $k$

$$\begin{aligned}\limsup_l \sup_\mu(h_k - g_l)(\mu) &= \sup_\mu \limsup_l \widetilde{(h_k - g_l)}(\mu),\\ &\leq 0.\end{aligned}$$

By taking the limsup in $k$ we conclude $\mathcal{G} \succ \mathcal{H}$. □

A *superenvelope* of $\mathcal{H} = (h_k)_k \in \mathfrak{G}_{\mathfrak{X}}^{\succ}$ is an upper semicontinuous function $E : \mathfrak{X} \to \mathbb{R}^+ \cup \{+\infty\}$ satisfying

$$\lim_k \widetilde{(E - h_k)} = E - \lim \mathcal{H}. \tag{1}$$

Theorem 2.3.2 in [11] may be stated in our slightly general context as below. The proof follows the same lines.

**Lemma 3.5.** *Let $\mathcal{H} = (h_k)_k, \mathcal{G} = (g_k)_k \in \mathfrak{G}_{\mathfrak{X}}^{\succ}$ with $\mathcal{H} \sim^{\succ} \mathcal{G}$, then*

- $\lim \mathcal{H} = \lim \mathcal{G}$,
- *$\mathcal{H}$ is uniformly convergent if and only so is $\mathcal{G}$,*
- $\limsup_k \widetilde{(\lim \mathcal{H} - h_k)} = \limsup_k \widetilde{(\lim \mathcal{G} - g_k)}$,
- *$\mathcal{H}$ and $\mathcal{G}$ have the same superenvelopes.*

3.2.2. *Entropy structure for flows, definition.* For a discrete topological system $(X, T)$, we let $\mathfrak{G}_T$ be the subset of $\mathfrak{G}_{\mathcal{M}(X,T)}^{\succ}$ consisting of sequences $\mathcal{H}$ with $\lim \mathcal{H}$ equal to the metric entropy function $h_T = h$ on $\mathcal{M}(X, T)$. The *entropy structure of* $(X, T)$ is the equivalence class for the equivalence relation $\sim^{\succ}$ on $\mathfrak{G}_T$ of the sequence $\mathcal{H}^{Leb} \in \mathfrak{G}_T$ defined below. By abuse of language any representative of this class is called an entropy structure of $(X, T)$. The product of $(X, T)$ with a circle rotation $(\mathsf{R}_\alpha, \mathbb{S}^1)$ by an angle $\alpha \notin \mathbb{Q}$ has the small boundary property (see Theorem 6.2 in [25]). Fix a nonincreasing sequence $(R_k)_k$ of partitions with small boundary satisfying $\mathsf{diam}(R_k) \xrightarrow{k} 0$. Let $\lambda$ be the Lebesgue measure on the circle $\mathbb{S}^1$. Then we define the sequence $\mathcal{H}_{Leb} = (h_k)_k$ by $h_k : \mu \mapsto h_{T \times \mathsf{R}_\alpha}(\mu \times \lambda, R_k)$ for all $k \in \mathbb{N}$.

For a topological flow $(X, \Phi)$ we define the entropy structure similarly. First we recall that the metric entropy $h(\mu)$ of $\mu \in \mathcal{M}(X, \Phi)$ is the entropy of $i_1(\mu)$ for the time 1-map $\phi_1$ of $\Phi$. We denote by $\mathfrak{G}_\Phi$ the subset of $\mathfrak{G}_{\mathcal{M}(X,\Phi)}^{\succ}$ consisting of sequences $\mathcal{H}$ with $\lim \mathcal{H}$ equal to the metric entropy function $h$ on $\mathcal{M}(X, \Phi)$. Then we define the *entropy structure of* $(X, \Phi)$ as the equivalence class for $\sim^{\succ}$ (denoted simply by $\sim$ in the following) on $\mathfrak{G}_\Phi$ of the sequence $\mathcal{H}^\Phi_{BK}$ defined below. For $x \in X$, $\epsilon > 0$ and $\tau > 0$ we let $B_\Phi(x,\epsilon,\tau)$ be the $\Phi$-dynamical ball:

$$B_\Phi(x,\epsilon,\tau) := \{y \in X,\ d(\phi_s(x),\phi_s(y)) < \epsilon\ \forall 0 \leq s \leq \tau\}.$$

For a $\Phi$-invariant measure $\mu$ we let for all $x \in M$

$$h^\Phi(\mu,\epsilon,x) := \limsup_{\tau\to+\infty} -\frac{1}{\tau}\log\mu\left(B_\Phi(x,\epsilon,\tau)\right),$$

and then

$$h^\Phi(\mu,\epsilon) := \int h^\Phi(\mu,\epsilon,x)\,d\mu(x).$$

M.Brin and A.Katok [7] have shown this quantity is converging to $h(\mu)$ when $\epsilon$ goes to zero. For a fixed decreasing sequence $(\epsilon_k)_k$ with $\lim_k \epsilon_k = 0$ we let $\mathcal{H}^\Phi_{BK} := \left(h^\Phi(\cdot,\epsilon_k)\right)_k$.

3.2.3. *Relations with the entropy structure of the time t-maps.* For any $t > 0$ we recall that the map $\theta_t : \mathcal{M}(X,\phi_t) \to \mathcal{M}(X,\Phi)$ defined by $\mu \mapsto \frac{1}{t}\int_0^t \phi_s\mu\,ds$ is a (affine) retraction, i.e. we have $\theta_t \circ i_t = \mathsf{Id}_{\mathcal{M}(X,\Phi)}$ with $i_t$ being the inclusion of $\mathcal{M}(X,\Phi)$ in $\mathcal{M}(X,\phi_t)$.

For a map $\pi : \mathcal{N} \to \mathcal{M}$ and a sequence $\mathcal{H} = (h_k)_{k\in\mathbb{N}}$ of real functions on $\mathcal{M}$, we let $\mathcal{H}\circ\pi$ be the sequence $(h_k\circ\pi)_k$ on $\mathcal{N}$.

**Lemma 3.6.** *i) If $\mathcal{H}^{\phi_t}$ is an entropy structure of $(X,\phi_t)$ with $t > 0$ then $\frac{1}{t}\mathcal{H}^{\phi_t}\circ i_t$ is an entropy structure of $\mathcal{M}(X,\Phi)$,*

*ii) If $\mathcal{H}^\Phi$ is an entropy structure of $(X,\Phi)$ then $t\mathcal{H}^\Phi\circ\theta_t$ is an entropy structure of $\mathcal{M}(X,\phi_t)$ for $t > 0$.*

*Proof.* i) For a fixed $t > 0$ we may define the Brin-Katok entropy of $\phi_t$ by considering the $\phi_t$-dynamical ball

$$B_{\phi_t}(x,\epsilon_k,n) := \{y \in X,\ d(\phi_{lt}(x),\phi_{lt}(y)) < \epsilon_k\ \forall 0 \leq l < n\}.$$

More precisely we let for all $x \in X$, for all $\mu \in \mathcal{M}(X,\phi_t)$ and for all $k \in \mathbb{N}$

$$h^{\phi_t}(\mu,\epsilon_k,x) := \limsup_{n\to+\infty} -\frac{1}{n}\log\mu\left(B_{\phi_t}(x,\epsilon_k,n)\right),$$

and then

$$h^{\phi_t}(\mu,\epsilon_k) := \int h^{\phi_t}(\mu,\epsilon_k,x)\,d\mu(x).$$

This defines an entropy structure $\mathcal{H}^{\phi_t}_{BK} = \left(h^{\phi_t}(\cdot,\epsilon_k)\right)_k$ of the discrete system $(X,\phi_t)$ (see Appendix A). For all $\epsilon > 0$ there exists $\tilde{\epsilon} > 0$ such that for all $\tau > 0$ we have

$$\forall x \in X,\ B_{\phi_t}(x,\tilde{\epsilon},[\tau/t]) \quad\subset\quad B_\Phi(x,\epsilon,\tau) \subset B_{\phi_t}(x,\epsilon,[\tau/t]).$$

Let $\mu$ be a $\Phi$-invariant measure and let $\epsilon > 0$. From the above inclusions we get :

$$\forall x \in X,\ \frac{1}{t}h^{\phi_t}(\mu,\epsilon,x) \leq h^\Phi(\mu,\epsilon,x) \leq \frac{1}{t}h^{\phi_t}(\mu,\tilde{\epsilon},x),$$

and then by integrating with respect to $\mu$

$$\frac{1}{t}h^{\phi_t}(\mu,\epsilon) \leq h^\Phi(\mu,\epsilon) \leq \frac{1}{t}h^{\phi_t}(\mu,\tilde{\epsilon}).$$

In particular $\frac{1}{t}\mathcal{H}^{\phi_t}_{BK}\circ i_t$ and $\mathcal{H}^\Phi_{BK}$ are equivalent.

ii) According to the first item we can assume the entropy structure $\mathcal{H}^\Phi$ is $\frac{1}{t}\mathcal{H}^{\phi_t}_{BK}\circ i_t$. As the functions in $\mathcal{H}^{\phi_t}_{BK}$ are harmonic[4], the sequence $t\mathcal{H}^\Phi\circ\theta_t = \mathcal{H}^{\phi_t}_{BK}\circ\theta_t$ is just the sequence of functions $\mu \mapsto \frac{1}{t}\int_0^t h^{\phi_t}_{BK}(\phi_s\mu,\epsilon_k)\,ds$, $k \in \mathbb{N}$.

[4]A real function $f$ defined on the Choquet simplex of probability invariant measures is said *harmonic* when for any invariant probability measure $\mu$ we have $f(\mu) = \int f(\nu_x)d\mu(x)$ with $\mu = \int \nu_x\,d\mu(x)$ being the ergodic decomposition of $\mu$. In particular, harmonic functions are affine.

For all $\epsilon > 0$ there exists $g(\epsilon) \leq \epsilon$ such that $d(x,y) < g(\epsilon) \Rightarrow d(\phi_s(x), \phi_s(y)) < \epsilon$ for any $s$ with $|s| \leq t$. In particular we have for all $x \in X$, $n \in \mathbb{N}$ and $\epsilon > 0$ :

$$\phi_{-s} B_{\phi_t}(x, g(\epsilon), n) \subset B_{\phi_t}(\phi_{-s}(x), \epsilon, n)$$

and for all $\mu \in \mathcal{M}(X, \phi_t)$,

$$\begin{aligned} h^{\phi_t}(\phi_s\mu, g(\epsilon), x) &= \limsup_{n\to+\infty} -\frac{1}{n} \log \mu\left(\phi_{-s} B_{\phi_t}(x, g(\epsilon), n)\right), \\ &\geq \limsup_{n\to+\infty} -\frac{1}{n} \log \mu\left(B_{\phi_t}(\phi_{-s}x, \epsilon, n)\right), \\ &\geq h^{\phi_t}(\mu, \epsilon, \phi_{-s}x). \end{aligned}$$

By integrating this last inequality with respect to $\phi_s\mu$ we get for all $\mu \in \mathcal{M}(X, \phi_t)$ and for all $|s| \leq t$ :

$$h_{BK}^{\phi_t}\left(\phi_s\mu, g(\epsilon)\right) \geq h_{BK}^{\phi_t}\left(\mu, \epsilon\right).$$

Consequently the sequence $t\mathcal{H}^\Phi \circ \theta_t = \left(\frac{1}{t}\int_0^t h_{BK}^{\phi_t}\left(\phi_s\cdot, \epsilon_k\right) ds\right)_k$ is equivalent to the Brin-Katok entropy structure $\mathcal{H}_{BK}^{\phi_t} = \left(h^{\phi_t}(\cdot, \epsilon_k)\right)_k$.

□

For discrete dynamical systems entropy structures are preserved by principal extensions (see Theorem 5.0.3 (2) in [11]). A principal extension between two topological flows induces a principal extension between their time $t$-maps. Indeed the entropy function being harmonic we have $h_\Phi(\theta_t(\mu)) = h_{\phi_t}(\mu)$ for any $\mu \in \mathcal{M}(X, \phi_t)$ for a topological flow $(X, \Phi)$. Then for a principal extension $\pi : (Y, \Psi) \to (X, \Phi)$ we get for any $\mu \in \mathcal{M}(Y, \psi_t)$ :

$$\begin{aligned} h_{\phi_t}(\mu) &= h_\Phi(\theta_t(\mu)), \\ &= h_\Psi(\pi\theta_t(\mu)), \\ &= h_\Psi(\theta_t(\pi\mu)) = h_{\psi_t}(\pi\mu). \end{aligned}$$

As a consequence of Lemma 3.6 we obtain then the result analogous to Theorem 5.0.3 (2) [11] for topological flows :

**Corollary 3.1.** *Entropy structures of flows are preserved by principal extensions, i.e. if* $\pi : (Y, \Psi) \to (X, \Phi)$ *is a principal extension then* $\mathcal{H} \circ \pi$ *is an entropy structure of* $(Y, \Psi)$ *if and only if* $\mathcal{H}$ *is an entropy structure of* $(X, \Phi)$.

For a discrete topological system the entropy with respect to a nonincreasing sequence of partitions with small boundary defines an entropy structure. In our context we have :

**Corollary 3.2.** *Let* $(X, \Phi)$ *be a topological flow. If* $\mathcal{P} = (P_k)_k$ *is a sequence of partitions of* $X$ *with small boundary such that* $\mathcal{H}_\mathcal{P} := (h(\cdot, P_k))_k$ *belongs to* $\mathfrak{G}_\Phi$*, then* $\mathcal{H}_\mathcal{P}$ *defines an entropy structure of* $(X, \Phi)$.

*Proof.* Let $(R_k)_k$ be a nonincreasing sequence of partitions of $X$ defining the entropy structure $\mathcal{H}_{Leb}$ for $T = \phi_1$ (see Subsection 3.2.2). Then we have for any $\mu \in \mathcal{M}(X, \phi_1)$

$$\begin{aligned} h(\mu, P_k) - h(\mu\times\lambda, R_l) &\leq h\left(\mu\times\lambda, (P_k\times\mathbb{S}^1)\vee R_l | R_l\right), \\ h(\mu\times\lambda, R_l) - h(\mu, P_k) &\leq h\left(\mu\times\lambda, (P_k\times\mathbb{S}^1)\vee R_l | P_k\times\mathbb{S}^1)\right). \end{aligned}$$

For $k$ (resp. $l$) fixed the first (resp. second) right member defines an upper semicontinuous function on $\mathcal{M}(X, \Phi)$ going pointwisely to zero when $l$ (resp. $k$) goes to infinity. The sequence $\mathcal{H}_\mathcal{P}$ and the restriction of $\mathcal{H}_{Leb}$ to $\mathcal{M}(X, \Phi)$ being both in $\mathfrak{G}_\Phi$ they are equivalent by Lemma 3.4. By Lemma 3.6 this restriction is an entropy structure of the flow. Therefore $\mathcal{H}_\mathcal{P}$ is also an entropy structure of the flow. □

3.2.4. *Entropy of suspension flows.* In this paragraph we consider a suspension flow $(X_r, \Phi_r = (\phi_t)_t)$ over a topological system $(X,T)$ with a continuous positive roof function $r$. To simplify the notations we will denote by $\nu_\mu$ the $\Phi_r$-invariant measure $\Theta(\mu)$ associated to the $T$-invariant measure $\mu$ and $\mu_\nu$ the $T$-invariant measure $\Theta^{-1}(\nu)$ associated to the $\Phi_r$-invariant measure $\nu$, where $\Theta$ is the homeomorphism defined in Subsection 2.1.5. The entropy of $\mu$ and $\nu_\mu$ are related by the following formula due to L.M.Abramov [1]:

$$h(\nu_\mu) = \frac{h(\mu)}{\int r\, d\mu}.$$

Abramov formula holds for any measurable suspension flow over a measurable system. It follows from the formula for the entropy of an induced system. We recall below the corresponding formula for the entropy with respect to a given partition.

**Lemma 3.7.** *Let $(Y, f, \mathcal{B}, \nu)$ be a measure preserving system and let $A \subset Y$ with $\nu(A) > 0$. Then for any finite Borel partition $P$ of $A$ we have*

$$\nu(A)h(\nu_A, f_A, P \vee R_A) = h(\nu, f, \overline{P}),$$

*where $\overline{P}$ is the partition of $Y$ given by $\overline{P} := \{Y \setminus A,\ B\ :\ B \in P\}$ and $R_A := \{\tau_A = k,\ k \in \mathbb{N}\setminus\{0\}\}$ is the partition of $A$ with respect to the first return time $\tau_A$ in $A$.*

*Proof.* As both terms in the above equality is preserved by the ergodic decomposition[5], one can assume without loss of generality the ergodicity of $\nu$. The induced measure $\nu_A$ on $A$ is then also ergodic. From the Birkhof ergodic theorem, we get

$$\forall \nu-\text{a.e. } x,\ \frac{1}{n}\sum_{k=0}^{n-1}\tau_A(f_A^k(x)) \quad \xrightarrow{n} \quad \int \tau_A\, d\nu_A = \frac{1}{\nu(A)}. \tag{2}$$

Then by Shanon-McMillan-Breiman formula we have:

$$\forall \nu_A-\text{a.e. } x,\ h(\nu_A, f_A, P \vee R_A) \quad = \quad \lim_n -\frac{1}{n}\log \nu_A\left((P \vee R_A)^n_{f_A}(x)\right), \tag{3}$$

$$\forall \nu-\text{a.e. } x,\ h(\nu, f, \overline{P}) \quad = \lim_{n'} -\tfrac{1}{n'}\log \nu\left(\overline{P}^{n'}_f(x)\right), \tag{4}$$

where $(P\vee R_A)^n_{f_A}(x)$ and $\overline{P}^{n'}_f(x)$ denote respectively the atom of the iterated partition $\bigvee_{k=0}^{n-1} f_A^{-k}(P\vee R_A)$ and $\bigvee_{k=0}^{n'-1} f^{-k}\overline{P}$ containing $x$. But we have $(P\vee R_A)^n_{f_A}(x) = \overline{P}^{n'_x}_f(x)$ with $n'_x := \sum_{k=0}^{n-1}\tau_A(f_A^k(x))$. By taking a point $x$ satisfying the three above properties (2), (3), (4), we get :

$$\begin{aligned}
h(\nu_A, f_A, P \vee R_A) &= \lim_n -\frac{1}{n}\log \nu_A\left((P \vee R_A)^n_{f_A}(x)\right), \\
&= \lim_n -\frac{n'_x}{n}\frac{1}{n'_x}\log \nu_A\left(\overline{P}^{n'}_f(x)\right), \\
h(\nu_A, f_A, P \vee R_A) &= \frac{h(\nu, f, \overline{P})}{\nu(A)}.
\end{aligned}$$

□

We return now to our suspension flow $(X_r, \Phi_r)$. Let $\underline{r} := \inf_{x\in X} r(x) > 0$. We may deduce from Lemma 3.7 the following inequalities for the entropy of suspension flows.

**Lemma 3.8.** *Let $P$ be a Borel partition of $X$, then for all $\delta \in ]0, \underline{r}[$ and for all $\mu \in \mathcal{M}(X,T)$*

$$\frac{h(\nu_\mu, \phi_\delta, \overline{P_\delta})}{\delta} \geq \frac{h(\mu, T, P)}{\int r\, d\mu}, \tag{5}$$

[5]If the ergodic decomposition of $\nu$ is given by $\nu = \int_Y \nu^x\, d\nu(x)$, then $\nu_A = \int_A \nu^x(A)\nu_A^x\, d\nu_A(x)$ is the ergodic decomposition of $\nu_A$.

*where* $\overline{P_\delta}$ *is the partition of* $X_r$ *given by* $\overline{P_\delta} := \{X_r \setminus (X \times [0,\delta[),\ B \times [0,\delta[ :\ B \in P\}$. *When we moreover assume* $|r(x) - r(y)| < \delta$ *for all* $x, y$ *in the same atom of* $P$, *then*

$$\frac{h(\nu_\mu, \phi_\delta, \overline{P_\delta})}{\delta} \leq \frac{h(\mu, T, P) + \log 3}{\int r \, d\mu},$$

*Proof.* Let $0 < \delta < \underline{r}$. Let $A_\delta$ be the subset of $X_r$ given by $A_\delta = X \times [0, \delta[$. For the partition $P$ of $X$ we first denote by $P_\delta = \{B \times [0,\delta[,\ B \in P\}$ the partition induced on $A_\delta$. We also let $R_\delta = R_{A_\delta}$ be the partition with respect to the first return time in $A_\delta$. By applying Lemma 3.7 to $\nu_\mu$, $\phi_\delta$, $X_r$ and $P_\delta$ we get

$$\begin{aligned} h\left((\nu_\mu)_{A_\delta}, (\phi_\delta)_{A_\delta}, P_\delta \vee R_\delta\right) &= \frac{h\left(\nu_\mu, \phi_\delta, \overline{P_\delta}\right)}{\nu_\mu(A_\delta)}, \\ &= \frac{\int r \, d\mu}{\delta} h\left(\nu_\mu, \phi_\delta, \overline{P_\delta}\right). \end{aligned} \tag{6}$$

But the partition $\bigvee_{k=0}^{n-1} (\phi_\delta)_{A_\delta}^{-k} P_\delta$ of $A_\delta$ is just the partition $\bigvee_{k=0}^{n-1} T^{-k} P \times [0, \delta[$ and therefore we get, with $H_\mu(Q) = \sum_{C \in Q} -\mu(C) \log \mu(C)$ :

$$\begin{aligned} h\left((\nu_\mu)_{A_\delta}, (\phi_\delta)_{A_\delta}, P_\delta\right) &= \lim_n \frac{1}{n} H_{(\nu_\mu)_{A_\delta}} \left( \bigvee_{k=0}^{n-1} T^{-k} P \times [0,\delta[ \right), \\ &= \lim_n \frac{1}{n} \sum_{C \in \bigvee_{k=0}^{n-1} T^{-k} P \times [0,\delta[} -(\nu_\mu)_{A_\delta}(C) \log (\nu_\mu)_{A_\delta}(C). \end{aligned}$$

For any $B \in \bigvee_{k=0}^{n-1} T^{-k} P$ and $C = B \times [0, \delta[$ we have $(\nu_\mu)_{A_\delta}(C) = \frac{\nu_\mu(C)}{\nu_\mu(A_\delta)} = \mu(B)$. Therefore we obtain finally :

$$h\left((\nu_\mu)_{A_\delta}, (\phi_\delta)_{A_\delta}, P_\delta\right) = h(\mu, T, P),$$

which implies the first inequality.

Then by using again Equality (6) we get

$$\begin{aligned} \left| \frac{1}{\delta} h\left(\nu_\mu, \phi_\delta, \overline{P_\delta}\right) - \frac{h(\mu, T, P)}{\int r d\mu} \right| &= \frac{1}{\int r d\mu} \left| h\left((\nu_\mu)_{A_\delta}, (\phi_\delta)_{A_\delta}, P_\delta \vee R_\delta\right) - h\left((\nu_\mu)_{A_\delta}, (\phi_\delta)_{A_\delta}, P_\delta\right) \right|, \\ &= \frac{1}{\int r d\mu} h\left((\nu_\mu)_{A_\delta}, (\phi_\delta)_{A_\delta}, P_\delta \vee R_\delta | P_\delta\right). \end{aligned}$$

Under the additional assumption of small oscillation of $r$, any element of $P_\delta$ has a non empty intersection with at most 3 elements of $R_\delta$ so that the conditional entropy $h\left((\nu_\mu)_{A_\delta}, (\phi_\delta)_{A_\delta}, P_\delta \vee R_\delta | P_\delta\right)$ is bounded from above by $\log 3$. □

3.2.5. *Relations with the entropy structure of the base dynamics for a suspension flow.* Let $(X_r, \Phi_r)$ be a suspension flow over a zero-dimensional discrete system $(X, T)$. We relate the entropy structure of the flow $(X_r, \Phi_r)$ with the entropy structure of $(X, T)$. To any sequence $\mathcal{H} = (h_k)_k \in \mathfrak{G}_T$ we associate the sequence $\mathcal{H}_r \in \mathfrak{G}_\Phi$ defined by $\mathcal{H}_r = \left(\nu \mapsto \frac{h_k(\mu_\nu)}{\int r \, d\mu_\nu}\right)_k$.

**Lemma 3.9.** *The map* $\mathcal{H} \mapsto \mathcal{H}_r$ *is well-defined and compatible with the equivalence relation* $\sim$, *i.e.* $[\mathcal{H} \sim \mathcal{G}] \Leftrightarrow [\mathcal{H}_r \sim \mathcal{G}_r]$.

The proof follows from the continuity of the map given by $\mathcal{M}(X,T) \ni \mu \mapsto \frac{1}{\int r \, d\mu}$ and the continuity of $\Theta$ and $\Theta^{-1}$. The details are left to the reader.

**Lemma 3.10.** *Assume* $(X, T)$ *is an aperiodic zero-dimensional system. There exist a nonincreasing sequence* $(Q_k)_k$ *of clopen partitions of* $X$ *with* $\mathsf{diam}(Q_k) \xrightarrow{k \to +\infty} 0$ *and and a sequence of partitions* $(P_k)_k$ *of* $X_r$ *with small boundary (for the flow* $\Phi_r$*) such that*

$$\sup_{\mu \in \mathcal{M}(X,T)} \left| h(\nu_\mu, P_k) - \frac{h(\mu, Q_k)}{\int r \, d\mu} \right| \quad \xrightarrow{k \to +\infty} \quad 0.$$

*Proof.* We consider a sequence $(U_k)_{k\in\mathbb{N}}$ of nested topological Rohlin towers (see Lemma 8.5.4 [12]):

- $U_k$ is a clopen set for every $k$,
- $U_{k+1}\subset U_k$ for every $k$,
- $X=\bigcup_{n\in\mathbb{N}}T^nU_k$,
- $\underline{\tau_{U_k}}:=\min_{x\in U_k}\tau_{U_k}(x)\xrightarrow{k\to+\infty}+\infty$.

Let $k\in\mathbb{N}$. By Kac's formula we have $\mu(U_k)\leq\frac{1}{\underline{\tau_{U_k}}}$. The flow $(X_r,\phi_r)$ may be represented as a suspension flow over $U_k$ with roof function $r_k:=\sum_{0\leq l<\tau_{U_k}}r\circ T^l$. Note that $\int r_k\,d\mu_{U_k}=\frac{\int r\,d\mu}{\mu(U_k)}$. A clopen partition $R_k$ of $U_k$ finer than $R_{U_k}:=\{\{\tau_{U_k}=l\}\mid l\in\mathbb{N}\setminus\{0\}\}$ induces a clopen partition $Q_k$ of $X$ by letting $Q_k=\{T^m(\{\tau_{U_k}=l\}\cap A)\mid A\in R_k,\ l\in\mathbb{N}\setminus\{0\},\ 0\leq m<l\}$. Observe that $h(\mu,T,\overline{R_k})=h(\mu,T,Q_k)$ where $\overline{R_k}$ denotes the partition of $X$ given by $\overline{R_k}:=\{X\setminus U_k,B\mid B\in R_k\}$ (indeed for any positive integer $n$ the partition $Q_k^{n+M}$ is finer than $\overline{R_k}^{n+M}$, which is itself finer than $Q_k^n$, with $M$ being a fixed integer larger than $\max_{x\in U_k}\tau_{U_k}(x)$).

We may choose such a sequence $(R_k)_k$ that the induced partitions $(Q_k)_k$ satisfy $\mathsf{diam}(Q_k)<1/k$ and $Q_{k+1}$ finer than $Q_k$ for all $k$. Moreover we may assume the diameter of $R_k$ so small that $|r_k(x)-r_k(y)|<\delta$ for any points $x$ and $y$ in the same atom of $R_k$. Since the partition $R_k$ is finer than $R_{U_k}$ we get according to Lemma 3.7 :

$$\mu(U_k)h(\mu_{U_k},T_{U_k},R_k)=h(\mu,T,\overline{R_k})=h(\mu,T,Q_k).$$

Fix $p\in\mathbb{N}^*$ with $\delta:=1/p<\underline{r}$. By applying Lemma 3.8 for $\delta$ to the suspension flow over $U_k$ we get :

$$\begin{aligned}\frac{h(\mu_{U_k},T_{U_k},R_k)}{\int r_k\,d\mu_{U_k}}&\leq\tfrac{1}{\delta}h(\nu_{\mu_{U_k}},\phi_\delta,\overline{(R_k)_\delta})&&\leq\frac{h(\mu_{U_k},T_{U_k},R_k)+\log 3}{\int r_k\,d\mu_{U_k}},\\ \frac{h(\mu,T,Q_k)}{\int r\,d\mu}&\leq\tfrac{1}{\delta}h(\nu_\mu,\phi_\delta,\widetilde{\overline{(R_k)_\delta}})&&\leq\frac{h(\mu,T,Q_k)+\mu(U_k)\log 3}{\int r\,d\mu},\end{aligned}$$

where $\widetilde{\overline{(R_k)_\delta}}$ is the partition of $X_r$ obtained from the partition $\overline{(R_k)_\delta}$ of $X_{r_k}$ through the natural topological conjugacy between the two suspension flows. We let $P_k$ be the partition given by $P_k:=\bigvee_{l=0}^{p-1}\phi_{l/p}^{-1}\left(\widetilde{\overline{(R_k)_\delta}}\right)$ so that we have (recall $\delta=1/p$) :

$$\frac{h(\mu,T,Q_k)}{\int r\,d\mu}\leq h(\nu_\mu,\phi_1,P_k)\leq\frac{h(\mu,T,Q_k)+\mu(U_k)\log 3}{\int r\,d\mu}.$$

The partitition $R_k$ of $U_k$ being clopen, the sets $B\times[0,\delta[$ for $B\in R_k$ have a small boundary for $\Phi_r$. Consequently $\overline{(R_k)_\delta}$, and then $P_k$, is a partition of $(X_r,\Phi_r)$ with small boundary. □

The sequence $(P_k)_k$ built in the above lemma is a priori not nonincreasing. That is why we have generalized the theory of entropy structures in Subsection .

**Corollary 3.3.** *With the above notations the following assertions are equivalent:*

*(1) $\mathcal{H}$ is an entropy structure of $(X,T)$,*
*(2) $\mathcal{H}_r$ is an entropy structure of $(X_r,\Phi_r)$.*

*Proof.* We first prove (1) ⇒ (2) for an aperiodic zero-dimensional system $(X,T)$. Let $\mathcal{P}:=(P_k)_k$ and $\mathcal{Q}=(Q_k)_k$ be as in Lemma 3.10. The sequence $\mathcal{H}_{\mathcal{Q}}$ is an entropy structure of $(X,T)$ (see [12]). Then if $\mathcal{H}$ is an entropy structure of $(X,T)$, we have $\mathcal{H}_{\mathcal{Q}}\sim\mathcal{H}$ and therefore $(\mathcal{H}_{\mathcal{Q}})_r\sim\mathcal{H}_r$ by Lemma 3.9. By the last item of Lemma 3.2, the sequence $\mathcal{H}_{\mathcal{P}}$ belongs to $\mathfrak{G}_{\Phi_r}$ and $(\mathcal{H}_{\mathcal{Q}})_r\sim\mathcal{H}_{\mathcal{P}}$. But the sequence $\mathcal{H}_{\mathcal{P}}$ defines also an entropy structure of $(X,\Phi)$ according to Corollary 3.2. Thus $\mathcal{H}_r$ is an entropy structure of $(X,\Phi)$.

We deal now with the general case. Consider an aperiodic principal zero-dimensional extension $\pi:(Y,S)\to(X,T)$. Let $\mathcal{H}=(h_k)_k$ be an entropy structure of $(X,T)$. As entropy structures are preserved by principal extensions, the sequence $\mathcal{H}\circ\pi$ is an entropy structure of $(Y,S)$. Let $(Y_{r'},\Phi_{r'})$ be the suspension flow of $(Y,S)$ under the roof function $r'=r\circ\pi$. The map $\pi'$ :

$(Y_{r'}, \Phi_{r'}) \to (X_r, \Phi_r)$, $(y,t) \mapsto (\pi(y), t)$, defines a principal extension. From the aperiodic case the sequence $(\mathcal{H} \circ \pi)_{r'} = \mathcal{H}_r \circ \pi'$ defines an entropy structure of $(Y_{r'}, \Phi_{r'})$. But if $\mathcal{F} = (f_k)_k$ is an entropy structure of $(X_r, \Phi_r)$ then $\mathcal{F} \circ \pi'$ is also an entropy structure of $(Y_{r'}, \Phi_{r'})$ by Corollary 3.1. Thus $\mathcal{F} \circ \pi'$ is equivalent to $\mathcal{H}_r \circ \pi'$. Therefore $\mathcal{H}_r \sim \mathcal{F}$ is an entropy structure of $(X_r, \Phi_r)$.

The other implication (2) $\Rightarrow$ (1) follows easily from (1) $\Rightarrow$ (2). Indeed let $\mathcal{G}$ be an entropy structure of $(X, T)$. Then $\mathcal{G}_r$ is an entropy structure of $(X_r, \Phi_r)$. Let $\mathcal{H}$ be a sequence in $\mathfrak{G}_T$ such that $\mathcal{H}_r$ is an entropy structure of $(X_r, \Phi_r)$. We have $\mathcal{H}_r \sim \mathcal{G}_r$, therefore $\mathcal{H} \sim \mathcal{G}$ and $\mathcal{H}$ is also an entropy structure of $(X, T)$. □

3.2.6. *Superenvelope of flows.* For a discrete dynamical system (or a topological flow), a *superenvelope of the entropy structure* (or simply a *superenvelope*) is a superenvelope of a given entropy structure (seen as a representative sequence in the equivalence class). This definition does not depend on the choice of the representative sequence by Lemma 3.5.

The statement below follows easily from Corollary 3.1 and the definition of superenvelopes (see Lemma 8.4.8 in [12] for the analogous result for discrete systems) :

**Lemma 3.11.** *Superenvelopes of flows are preserved by principal extensions, i.e. if* $\pi : (Y, \Psi) \to (X, \Phi)$ *is a principal extension then* $E \circ \pi : \mathcal{M}(Y, \Psi) \to \mathbb{R}^+ \cup \{+\infty\}$ *is a superenvelope of* $(Y, \Psi)$ *if and only if* $E : \mathcal{M}(X, \Phi) \to \mathbb{R}^+ \cup \{+\infty\}$ *is a superenvelope of* $(X, \Phi)$.

We now relate the superenvelopes of the flow with those of its time-$t$ map for $t > 0$.

**Lemma 3.12.** *For any* $t > 0$ *the map* $E \mapsto tE \circ \theta_t$ *defines an injective map from the set of (affine) superenvelopes for the flow to the corresponding set for the time* $t$*-map* $\phi_t$.

*Proof.* The injectivity follows from the retraction property of $\theta_t$ (the map $E \mapsto \frac{1}{t} E \circ i_t$ defines a right inverse). It remains to check the image of a superenvelope is a superenvelope. Let $\mathcal{H}^\Phi = (h_k^\Phi)_k$ be an entropy structure of the flow. By Lemma 3.6 the sequence $t\mathcal{H}^\Phi \circ \theta_t$ is an entropy structure of $\phi_t$. Then $t\left(E \circ \theta_t - h_k^\Phi \circ \theta_t\right) = t\left(E - h_k^\Phi\right) \circ \theta_t$ and by continuity of $\theta_t$ we get

$$\lim_k \left(t\left(E - h_k^\Phi\right) \circ \theta_t\right)^{\tilde{}} = \lim_k t\left(E - h_k^\Phi\right)^{\tilde{}} \circ \theta_t = t\left(E - h^\Phi\right) \circ \theta_t.$$

□

We consider now superenvelopes of suspension flows.

**Lemma 3.13.** *Let* $(X_r, \Phi_r)$ *be a zero-dimensional flow given by a suspension flow over a zero-dimensional system* $(X, T)$ *with a positive continuous roof function* $r : X \to \mathbb{R}^+$. *The map*

$$\Gamma : E \mapsto E_r := \frac{E(\mu_\nu)}{\int r \; d\mu_\nu}$$

*is a bijection between the (affine) superenvelopes of* $(X, T)$ *and the (affine) superenvelopes of* $(X_r, \Phi)$.

*Proof.* Let $\mathcal{H} = (h_k)_k$ be an entropy structure of $(X, T)$ and let $\mathcal{H}_r = (g_k)_k$. By continuity of $\mathcal{M}(X, \Phi) \ni \nu \mapsto \frac{1}{\int r \, d\mu_\nu}$ we have for all $\nu \in \mathcal{M}(X, \Phi)$:

$$(E_r - g_k)\tilde{}(\nu) = \frac{(E - h_k)\tilde{}(\mu_\nu)}{\int r \; d\mu_\nu},$$

$$\lim_k (E_r - g_k)\tilde{}(\nu) = \frac{\lim_k (E - h_k)\tilde{}(\mu_\nu)}{\int r \; d\mu_\nu}.$$

Thus $E$ is a superenvelope of $(X, T)$ if and only if $E_r$ is a superenvelope of $(X_r, \Phi_r)$. Note finally that the map $\Gamma$ is invertible with $\Gamma^{-1}(E_r) : \mu \mapsto \int r \; d\mu \times E_r(\nu_\mu)$ for any superenvelope $E_r$ of $(X_r, \Phi_r)$.

Assume now $E$ is affine. Let us denote by $E$ its affine extension on the set $\mathcal{N}(X, T)$ of $T$-invariant positive finite measures (not necessarily probability ones). Similarly we denote by $\mathcal{N}(X_r, \Phi_r)$ the set of $\Phi_r$-invariant positive finite measures. The map $\nu \mapsto \frac{\mu_\nu}{\int r \, d\mu_\nu}$ being an affine bijection from $\mathcal{N}(X_r, \Phi_r)$ into $\mathcal{N}(X, T)$ (the inverse is given by $\mu \mapsto \mu \times \lambda$), the map $E_r$ defines an affine function on $\mathcal{N}(X_r, \Phi_r)$ and thus on the simplex $\mathcal{M}(X_r, \Phi_r)$ by restriction. Similarly $E$ is affine when $E_r$ is affine.

□

3.3. **Periodic structure.** For a topological flow $(X, \Phi)$ we let $Per(\Phi)$ be the set of $\Phi$-periodic orbits. We denote by $t(\gamma)$ the minimal period of $\gamma \in Per(\Phi)$. We define the *global periodic growth* $p(\Phi)$ of $(X, \Phi)$ as follows :

$$p(\Phi) = \sup_{t>0} \frac{1}{t} \log \sharp\{\gamma \in Per(\Phi),\ t(\gamma) \leq t\}.$$

To estimate the local exponential growth of periodic orbits, we introduce *the periodic structure* as the equivalence class for $\sim^{\prec}$ on $\mathcal{M}(X, \Phi)$ of the following nonincreasing sequence $\mathcal{P} = (p_k)_k$ of nonnegative functions on $\mathcal{M}(X, \Phi)$ (again we call periodic structure any representative in this class). Recall we have fixed a nonincreasing sequence $(\epsilon_k)_k$ with $\lim_k \epsilon_k = 0$. We let $D = D^X$ be a convex distance on the set $\mathcal{M}(X)$ of Borel probability measures on $X$ inducing the weak-$*$ topology, e.g. with a dense countable family $(f_n)_n$ of real continuous nonzero functions on $X$

$$\forall \mu, \nu \in \mathcal{M}(X, \Phi), \quad D(\mu, \nu) = \sum_n \frac{\left|\int f_n\, d\mu - \int f_n\, d\nu\right|}{2^n \sup_x |f_n(x)|}.$$

We let $\nu_\gamma$ be the periodic measure associated to $\gamma \in Per(\Phi)$. Then we let for all $k$

$$p_k^{\Phi}(\nu_\gamma) = \frac{1}{t(\gamma)} \log \sharp\{\gamma' \in Per(\Phi),\ D(\nu_\gamma, \nu'_\gamma) < \epsilon_k \text{ and } t(\gamma') \leq t(\gamma)\}.$$

The functions $p_k^{\Phi}$ are then extended harmonically on the simplex $\mathcal{M}(X, \Phi)$ by letting $p_k^{\Phi}(\nu) = 0$ for any aperiodic measure $\nu$. We get in this way a nonincreasing sequence $\mathcal{P} = (p_k^{\Phi})_k$ of nonnegative functions on $\mathcal{M}(X, \Phi)$. *The tail periodic function* $u_1^{\Phi}$ is then defined as

$$u_1^{\Phi} = \lim_k \widetilde{p_k^{\Phi}}.$$

When the global periodic growth $p(\Phi)$ is finite, the sequence $(p_k^{\Phi})_k$ is converging pointwisely to zero and

$$u_1^{\Phi} \leq p(\Phi) < +\infty.$$

The equivalence class of $\mathcal{P}$ (and thus $u_1^{\Phi}$ by Lemma 3.5) depend neither on the choice of the sequence $(\epsilon_k)_k$ nor on the distance $D$.

Similarly we define $u_1^T$ for a discrete system $(X, T)$ as $u_1^T = \lim_k \widetilde{p_k^T}$ with $p_k^T$ harmonic, vanishing on aperiodic measures and $p_k^T(\mu_x) = \frac{1}{n} \log \sharp \left\{\mu_{x'},\ D(\mu'_x, \mu_x) < \epsilon_k \text{ and } T^n x' = x'\right\}$ for any periodic point $x \in X$ with minimal period $n$ (where $\mu_x$ denotes here the periodic measure associated to $x$). A similar quantity $\mathfrak{u}_1^T$ was first defined in [10] by letting $\mathfrak{u}_1^T = \lim_k \widetilde{\mathfrak{p}_k^T}$ with $\mathfrak{p}_k^T$ harmonic satisfying $\mathfrak{p}_k^T(\mu_x) = \frac{1}{n} \log \sharp \left\{x',\ x' \in B(x, \epsilon_k, n) \text{ and } T^n x' = x',\ T^k x' \neq x' \text{ for } k < n\right\}$ for any periodic point $x \in X$ with minimal period $n$. Obviously we have $\mathfrak{u}_1^T \leq u_1^T$. For a subshift $(X, T)$ we clearly have $\mathfrak{u}_1^T = 0$. In this case $u_1^T = 0$ also holds true (see Lemma B.1 in Appendix C). For a topological discrete system $(X, T)$ we also let $p(T)$ be *the global periodic growth* $p(T) = \sup_{n>0} \frac{1}{n} \log \sharp\{x \in X,\ T^n x = x\}$.

**Lemma 3.14.** *Let* $\pi : (Y, \Psi) \to (X, \Phi)$ *be an isomorphic extension then*

$$u_1^{\Phi} \circ \pi = u_1^{\Psi}.$$

*Proof.* The proof follows directly from the fact, that the induced map $\pi : \mathcal{M}(Y, \Psi) \to \mathcal{M}(X, \Phi)$ is a homeomorphism preserving the periodic measures and their periods. □

For a topological flow $(X, \Phi)$ one checks easily that the time $t$-map, $t \neq 0$, satisfies $u_1^{\phi_t}(\mu) = 0$ for $\mu \notin \mathcal{M}(X, \Phi)$ and $\frac{u_1^{\phi_t}}{t}(\mu) \leq u_1^{\Phi}(\mu)$ for $\mu \in \mathcal{M}(X, \Phi)$. However this last inequality may be strict. We investigate now the behaviour of $u_1$ under suspensions.

**Lemma 3.15.** *Let $(X_r, \Phi_r)$ be a zero-dimensional flow given by a suspension flow over a zero-dimensional system $(X, T)$ with a positive continuous roof function $r$. Then we have for all $\nu \in \mathcal{M}(X_r, \Phi_r)$:*

$$u_1^{\Phi_r}(\nu) = \frac{u_1^T(\mu_\nu)}{\int r\, d\mu_\nu}.$$

*In particular $u_1^{\Phi_r} = 0$ when $(X, T)$ is a subshift.*

*Proof.* Let us show $u_1^{\Phi_r}(\nu_\mu) \leq \frac{u_1^T(\mu)}{\int r\, d\mu}$, the other inequality being proved similarly by reversing the roles of $T$ and $\Phi_r$. For all $\mu \in \mathcal{M}(X, T)$ we let $\overline{p_k^T}(\mu) = (1 + \epsilon_k) \sup\{p_k^T(\mu'),\ D(\mu, \mu') \leq \epsilon_k\}$. From Lemma 5.4 in [10] it follows that $u_1^T = \lim_k \overline{p_k^T}$. As the functions $\mu \mapsto \overline{p_k^T}(\mu)$ are upper semicontinuous it is enough to show that for any fixed $l$ there is $k$ with

$$\forall \nu \in \mathcal{M}(X_r, \Phi_r), \quad p_k^{\Phi_r}(\nu) \int r\, d\mu_\nu \quad \leq \overline{p_l^T}(\mu_\nu). \tag{7}$$

The distance $D^X$ being convex and the function $p_l^T$ being affine, the function $\overline{p_k^T}$ is concave and therefore superharmonic (as a concave upper semicontinuous function). We may extend $p_k^{\Phi_r}$ on $\mathcal{N}(X_r, \Phi_r)$ so that $\mu \mapsto p_k^{\Phi_r}(\nu_\mu) \int r\, d\mu = p_k^{\Phi_r}(\mu \times \lambda)$ is harmonic. Therefore to prove the inequality (7) we can assume $\mu_\nu$ (therefore $\nu$) to be periodic without loss of generality. Let $\gamma$ be the associated periodic orbit of the flow. We let $k$ be so large that for any $\mu, \mu' \in \mathcal{M}(X, T)$ with $D^{X_r}(\nu_\mu, \nu_{\mu'}) < \epsilon_k$ we have $D^X(\mu, \mu') < \epsilon_l/2$ and $\frac{\int r\, d\mu}{\int r\, d\mu'} < 1 + \epsilon_l$.

To simplify the notations we let $\mu_\gamma$ be the $T$-periodic measure $\mu_{\nu_\gamma}$ for a $\Phi_r$-periodic orbit $\gamma$. We pick up a periodic orbit $\overline{\gamma'}$ in the set of periodic orbits $\gamma'$ with $t(\gamma') \leq t(\gamma)$ and $D^{X_r}(\nu_{\gamma'}, \nu_\gamma) < \epsilon_k$ such that the period $n_{\mu_{\overline{\gamma'}}}$ of $\mu_{\overline{\gamma'}}$ maximizes the period of the $\mu_{\gamma'}$'s. The homeomorphism $\Theta : \mathcal{M}(X, T) \to \mathcal{M}(X_r, \Phi_r)$ maps the $T$-periodic measures $\mu_x$ of period $n$ to the $\Phi_r$-periodic measures $\mu_\gamma$ of period $n \int r\, d\mu_x$. Therefore we get

$$\begin{aligned} p_k^\Phi(\nu_\gamma) t(\gamma) &\leq p_l^T(\mu_{\overline{\gamma'}}) n_{\overline{\gamma'}}, \\ &\leq p_l^T(\mu_{\overline{\gamma'}}) \frac{t(\overline{\gamma'})}{\int r\, d\mu_{\overline{\gamma'}}}, \\ &\leq p_l^T(\mu_{\overline{\gamma'}}) \frac{t(\gamma)}{\int r\, d\mu_{\overline{\gamma'}}}, \\ \text{and thus } p_k^\Phi(\nu_\gamma) \int r\, d\mu_\gamma &\leq (1 + \epsilon_l) p_l^T(\mu_{\overline{\gamma'}}) \leq \overline{p_l^T}(\mu_\gamma). \end{aligned}$$

□

**Remark 3.1.** *One can show $p(\Phi) \leq h_{top}(\Phi) + \sup_\mu u_1^\Phi(\mu)$. We did not provide a proof as this inequality will not be used in the following. We refer to Section 6 in [10] for the analogous inequality in the discrete case.*

3.4. **Expansiveness and asymptotical expansiveness.** Following R.Bowen and P.Walters a topological flow $(X, \Phi)$ is said *expansive* when $\forall \epsilon > 0\ \exists \delta > 0$ such that if $d(\phi_t(x), \phi_{s(t)}(y)) < \delta$ for all $t \in \mathbb{R}$ and for a continuous map $s : \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$, then $y = \phi_t(x)$ with $|t| < \epsilon$. Recall that a discrete dynamical system is expansive whenever there is $\epsilon > 0$ with $\bigcap_{n \in \mathbb{Z}} T^{-k} B(T^k x, \epsilon) = \{x\}$ for all $x \in X$. The expansiveness property is invariant under topological conjugacy. The global periodic growth and the topological entropy of an expansive discrete system (resp. expansive flow) is finite (see Theorem 5 in [4]).

R.Mañé has proved that expansive dynamical systems only act on finite dimensional metric spaces [27]. This result was extended to flows by H.R.Keynes and M.Sears [17]. By Proposition 2.1 any $C^2$ smooth expansive flow satisfies the smooth small flow boundary property. In this case Lemma 2.17 gives a positive answer to the aforementioned open question of R.Bowen and P.Walters.

Furthermore expansiveness is preserved by suspension :

**Theorem 3.1.** *(Theorem 6 in [4]) Let $(X_r, \Phi_r)$ be a zero-dimensional flow given by a suspension flow over a zero-dimensional system $(X,T)$ with a positive continuous roof function $r$. The flow $(X_r, \Phi_r)$ is expansive if and only if $(X,T)$ is expansive.*

M.Misiurewicz introduced in [28] the asymptotic $h$-expansiveness property for topological systems. A topological system $(X,T)$ is *asymptotically $h$-expansive* when

$$\lim_{\epsilon \to 0} \sup_{x \in X} h_{top}\left(B_T(x, \epsilon, \infty)\right) = 0.$$

Asymptotical $h$-expansiveness is invariant under topological conjugacy [28], even under principal extensions [24]. The metric entropy of an asymptotical $h$-expansive system is upper semicontinuous. In particular such a system always admits a measure of maximal entropy. T.Downarowicz characterizes asymptotical $h$-expansiveness in terms of entropy structure as follows :

**Theorem 3.2.** *(Theorem 9.0.2 in [11]) A topological system is asymptotical $h$-expansiveness if and only if any (some) entropy structure of $(X,T)$ is converging uniformly to the entropy function $h$.*

For a topological flow $(X, \Phi)$ it is easily seen that $(X, \phi_1)$ is asymptotically $h$-expansive if and only if so does $(X, \phi_t)$ for any $t \neq 0$. In this case the flow will be said *asymptotically $h$-expansive.*[6]

A topological system $(X,T)$ (resp. flow $(X, \Phi)$) is said to be *asymptotically expansive* when it is asymptotically $h$-expansive and $u_1^T = 0$ (resp. $u_1^\Phi = 0$). From the definitions one easily checked that $u_1^T = 0$ (resp. $u_1^\Phi = 0$) if and only if the periodic structure $\mathcal{P} = (p_k)_k$ of $(X,T)$ (resp. $(X,\Phi)$) is converging uniformly to zero. As for discrete systems, asymptotical $h$-expansiveness may be also characterized by the uniform convergence of entropy for flows with the small flow boundary property.

**Lemma 3.16.** *Let $(X, \Phi)$ be a topological flow with the small flow boundary property. The following properties are equivalent:*

*i) $(X, \Phi)$ is asymptotically $h$-expansive (resp. asymptotically expansive),*
*ii) any (some) entropy structure $\mathcal{H} = (h_k)_k$ is converging uniformly to $h$ (resp. and any (some) periodic structure is converging uniformly to zero).*

*Proof.* It is enough to deal with the asymptotical $h$-expansiveness because we already observed that $u_1^\Phi$ is equal to zero if and only if any (some) periodic structure is converging uniformly to zero. The proof then follows from the following equivalences :

$$\begin{gathered}
(X, \Phi) \text{ is asymptotically } h\text{-expansive,} \\
\Longleftrightarrow, \\
(X, \phi_1) \text{ is asymptotically } h\text{-expansive,} \\
\Longleftrightarrow \\
\text{the entropy structure of } (X, \phi_1) \text{ is converging uniformly to } h, \\
\overset{\text{Lemma 3.6}}{\Longleftrightarrow} \\
\text{the entropy structure of } (X, \Phi) \text{ is converging uniformly to } h.
\end{gathered}$$

□

We show now that asymptotical expansiveness is also preserved by suspension.

**Lemma 3.17.** *Let $(X_r, \Phi_r)$ be a suspension flow over a topological system $(X,T)$ under a positive continuous roof function $r$. Then the following properties are equivalent :*

*i) $(X_r, \Phi_r)$ is asymptotically $h$-expansive (resp. asymptotically expansive),*
*ii) $(X,T)$ is asymptotically $h$-expansive (resp. asymptotically expansive).*

*Proof.* The proof follows from the equivalences :

[6] For topological flows, R.F.Thomas has defined and studied another notion of $h$-expansiveness in [31].

$(X,T)$ is asymptotically $h$-expansive (resp. asymptotically expansive),

$$\overset{\text{Theorem 3.2}}{\Longleftrightarrow}$$

any (some) entropy structure (resp. and any (some) periodic structure) of $(X,T)$ is converging uniformly,

$$\overset{\text{Corollary 3.3 (resp. and Lemma 3.15)}}{\Longleftrightarrow}$$

any (some) entropy structure (resp. and any (some) periodic structure) of $(X_r,\Phi_r)$ is converging uniformly,

$$\overset{\text{Lemma 3.16}}{\Longleftrightarrow}$$

$(X_r,\Phi_r)$ is asymptotically $h$-expansive (resp. asymptotically expansive).

□

### 3.5. **Relating symbolic extensions and uniform generators with expansiveness properties.**

3.5.1. *The case of expansive systems.* Krieger's embedding theorem characterizes systems with a clopen uniform generator, or equivalently by Proposition 3.1 systems topologically conjugate to a subshift :

**Theorem 3.3.** *(Krieger's topological embedding theorem [20]) A discrete topological system $(X,T)$ is topologically conjugate to a subshift if and only if the following properties hold:*

- *$X$ is zero-dimensional,*
- *$(X,T)$ is expansive.*

We recall that the expansiveness property implies the finiteness of the topological entropy and of the global periodic growth, which are both invariant under topological conjugacy. We state and show now the analogous result for topological flows.

**Theorem 3.4.** *A topological flow $(X,\Phi)$ is topologically conjugate to a suspension flow over a subshift if and only if the following properties hold:*

- *$X$ is one-dimensional,*
- *$(X,\Phi)$ is expansive.*

*Proof.* The necessary conditions are clear. Conversely we assume that $X$ is one-dimensional and $(X,\Phi)$ is expansive. As already mentioned such a flow is conjugate to a suspension flow $(Z_r,\Phi_r)$ over a zero-dimensional system $(Z,R)$ under a positive continuous roof function $r$. By Theorem 3.1 this zero-dimensional discrete system is expansive. The system $(Z,R)$ is therefore topologically conjugate to a subshift according to Theorem 3.3. □

3.5.2. *Symbolic extensions of a suspension flow.* We are in position to express the existence of symbolic extensions and uniform generators for a topological flow in terms of superenvelopes.

For a topological extension $\pi:(Y,\Psi)\to(X,\Phi)$ and a function $g:\mathcal{M}(Y,\Psi)\to\mathbb{R}$ we let

$$\begin{aligned} g^\pi:\mathcal{M}(X,\Phi) &\to \mathbb{R}^+,\\ \mu &\mapsto \sup_{\nu,\ \pi\nu=\mu} g(\nu). \end{aligned}$$

This notation was introduced earlier for a topological extension between discrete topological systems (see e.g. [5]).

**Lemma 3.18.** *Let $(X_r,\Phi_r)$ be a zero-dimensional flow given by a suspension flow over a zero-dimensional system $(X,T)$ with a positive continuous roof function $r$.*

*For any symbolic extension (resp. with an embedding) $\pi:(Y,S)\to(X,T)$ of $(X,T)$ the suspension flow $(Y_{r'},\Phi_{r'})$ over $(Y,S)$ under $r':=r\circ\pi$ defines a symbolic extension $\pi'$ (resp. with an embedding) of $(X_r,\Phi_r)$ with $\pi'(y,t)=(\pi(x),t)$ for all $(y,t)\in Y_{r'}$ satisfying*

$$\forall\nu\in\mathcal{M}(X_r,\Phi_r),\ h^{\pi'}(\nu)=\frac{h^\pi(\mu_\nu)}{\int r\,d\mu_\nu}.$$

*Moreover for any symbolic extension $\tau : (Z, \Psi) \to (X_r, \Phi_r)$ (resp. with an embedding) there is a symbolic extension $(Y, S)$ of $(X, T)$ (resp. with an embedding) such that the suspension flow $(Y_{r'}, \Phi_{r'})$, as defined above, is topologically conjugate to $(Z, \Psi)$.*

*Proof.* As the first part of the statement is easily checked, we focus on the last part. Let $\tau : (Z, \Psi) \to (X, \Phi)$ be a topological extension between two flows. If $S$ is a Poincaré cross-section of $(X, \Phi)$ then $S' = \tau^{-1}S$ is also a Poincaré cross-section of $(Z, \Psi = (\psi_t)_t)$ by Lemma 2.9. Indeed $S'$ is firstly a closed global cross-section (with a continuous return time $t_{S'} = t_S \circ \tau$). Then, for $\zeta > 0$, the set $\phi_{]-\zeta,\zeta[}S$ is open because $S$ has an empty flow boundary. Therefore $\psi_{]-\zeta,\zeta[}(S') = \tau^{-1}(\phi_{]-\zeta,\zeta[}S)$ is open by continuity of $\tau$ and the cross-section $S'$ has an empty flow boundary. Consequently $(Z, \Psi)$ is topologically conjugate to the suspension flow over $(S', T_{S'})$ under the continuous positive roof function $t_{S'}$.

In our context when $\tau : (Z, \Psi) \to (X_r, \Phi_r)$ is a symbolic extension (resp. with an embedding $\psi$) we let $S = X \times \{0\}$. According to Theorem 3.1 the system $(S', T_{S'})$ is expansive (with $S' = \tau^{-1}S$). Then by Theorem 3.3 this system is (topologically conjugate to) a subshift, which defines a topological extension of $(X, T)$ (resp. with an embedding) via the projection map $\pi' = \pi|_{S'}$ (resp. and the embedding $\psi|_S$). □

3.5.3. *Characterization of Symbolic Extensions.* The main result in the entropy theory of symbolic extensions, known as the Symbolic Extension Entropy Theorem, may be stated as follows :

**Theorem 3.5.** *(Theorem 5.5 in [5]) Let $(X, T)$ be a topological system. The systems admits a symbolic extension (resp. principal) if and only if there exists a finite superenvelope $E$ (resp. $(X, T)$ is asymptotically $h$-expansive).*

*More precisely a function $E$ on $\mathcal{M}(X, T)$ equals $h^\pi$ for some symbolic extension $\pi$ if and only if $E$ is an affine superenvelope of the entropy structure of $(X, T)$.*

We show now the corresponding statement for topological flows :

**Theorem 3.6.** *Let $(X, \Phi)$ be a topological flow with the small flow boundary property. The flow admits a symbolic extension (resp. principal) if and only if there exists a finite superenvelope $E$ (resp. $(X, \Phi)$ is asymptotically $h$-expansive).*

*More precisely a function $E$ on $\mathcal{M}(X, \Phi)$ equals $h^\pi$ for some symbolic extension $\pi$ if and only if $E$ is an affine superenvelope of the entropy structure of $(X, \Phi)$.*

*Proof.* We first consider the case of a suspension flow $(X_r, \Phi_r)$ over a zero-dimensional system $(X, T)$ under a positive continuous roof function $r$. By Lemma 3.13 and Lemma 3.18 the map

$$f \mapsto \left[\nu \mapsto \frac{f(\mu_\nu)}{\int r \, d\mu_\nu}\right]$$

defines a bijection between affine superenvelopes on one hand and the entropy functions in symbolic extensions $h^\pi$ on the other hand, for $(X, T)$ and $(X_r, \Phi_r)$. But according to Theorem 3.5 the affine superenvelopes are exactly the functions $h^\pi$ for the discrete system $(X, T)$. Therefore the same holds for the suspension flow $(X_r, \Phi_r)$.

We deal now with the general case. By Proposition 2.2 any topological flow $(X, \Phi)$ with the small flow boundary property admits a principal extension $\pi$ by a flow $(X_r, \Phi_r)$ of the previous form. Then if $E$ is an affine superenvelope of $(X, \Phi)$, it follows from Lemma 3.11 that $E \circ \pi$ is also a superenvelope of $(X_r, \Phi_r)$. According to the previous case there exists a symbolic extension $\pi' : (Y, \Psi) \to (X_r, \Phi_r)$ with $h^{\pi'} = E \circ \pi$. Then $\pi' \circ \pi$ is a symbolic extension of $(X, \Phi)$ with $h^{\pi' \circ \pi} = E$. Conversely, from Theorem 7.5 in [5] (which applies to topological flows with the same proof), for any symbolic extension $\pi'$ of $(X, \Phi)$ there exists a symbolic extension $\pi''$ of $(X_r, \Phi_r)$ with the same entropy function, i.e. $h^{\pi''} = h^{\pi'} \circ \pi$. Since $h^{\pi''}$ is an affine superenvelope of $(X_r, \Phi_r)$, the entropy function $h^{\pi'}$ is an affine superenvelope of $(X, \Phi)$. □

Together with Lemma 3.12 we get :

**Lemma 3.19.** *Let $(X, \Phi)$ be a topological flow with the small flow boundary property. The flow admits a symbolic extension (resp. principal) if and only if so does its times $t$-map for some (any) $t \neq 0$.*

The fact, that $\phi_t$ admits a symbolic extensions does not depend on $t \neq 0$, was first proved by T.Downarowicz and M.Boyle in Theorem 3.4 of [6]. For rational $t$ is was done by just considering a standard power rule for the entropy whereas for an irrational $t$ they build explicitly a symbolic extension of $\phi_t$ from a symbolic extension of $\phi_1$ by using the coding of irrational rotations via Sturmian sequences.

3.5.4. *Characterization of Uniform generators.* In [10] T.Downarowicz and the author also characterize the entropy function in a symbolic extension with an embedding. The case of strongly isomorphic symbolic extension follows from [9] as detailed in the Appendix C, whereas the characterization of systems topologically conjugate to a subshift first appeared in [20].

**Theorem 3.7.** *(Theorem 55 in [10], Main Theorem in [9], Krieger's topological embedding Theorem [20]) Let $(X,T)$ be a topological system with the small boundary property. The system admits a uniform generator (resp. essential, resp. clopen) if and only if there exists a finite superenvelope $E$ and $p(T) < +\infty$ (resp. $(X,T)$ is asymptotically expansive, resp. $T$ is expansive and $X$ is zero-dimensional).*

*More precisely a function $E$ on $\mathcal{M}(X,T)$ equals $h^\pi$ for some symbolic extension $\pi$ with an embedding if and only if $E$ is an affine superenvelope of the entropy structure of $(X,T)$ with $E \geq h^\pi + \mathfrak{u}_1^T$.*

By following the proof of Theorem 3.6 with making use of Lemma 3.15 we get :

**Theorem 3.8.** *Let $(X,\Phi)$ be a topological flow with the small flow boundary property. The flow admits a uniform generator (resp. essential) if and only if there exists a finite superenvelope $E$ and $p(\Phi) < +\infty$ (resp. the flow is asymptotically expansive).*

*More precisely a function $E$ on $\mathcal{M}(X,\Phi)$ equals $h^\pi$ for some symbolic extension $\pi$ with an embedding if and only if $E$ is a superenvelope of the entropy structure of $(X,\Phi)$ and $E \geq h + u_1^\Phi$.*

We are now in position to prove Theorem 1.1 stated in the Introduction.

*Proof of Theorem 1.1.* Fix $t > 0$. By Lemma 3.12 the system $(X,\Phi_t)$ admits a (finite affine) superenvelope (resp. the entropy function $h$ is a super envelope) if and only if so does the flow $(X,\Phi)$. Then, by Theorem 3.5 and Theorem 3.6, the time $t$-map admits a symbolic extension if and only if so does the flow. The corresponding statement for uniform generators follows from Theorem 3.7 and Theorem C.1. The invariance of these properties under orbit equivalence is proved below in Theorem 3.9.

□

In general there is no relation between uniform generators for the flow and uniform generators for the time $t$-maps. Indeed consider the standard suspension (i.e. with roof function $r = 1$) of the identity of a compact metrizable space $X$. Then the flow $\Phi = (\phi_t)_t$ admits a uniform generator if and only if the base $X$ of the suspension is a finite set, whereas $\phi_t$ admits a uniform generator if and only if $t$ is irrational. Indeed when $X$ is infinite the flow $\Phi$ (resp. the times $t$-map $\phi_t$ with $t = \frac{p}{q} \in \mathbb{Q}$) has infinitely many periodic orbits with period 1 (resp. $q$) and thus can not be embedded in a symbolic flow (resp. subshift). When $t$ is irrational, then $\phi_t$ has the small boundary property by the aforementioned result of E.Lindenstrauss (Theorem 6.2 in [25]). Also $\phi_t$ is clearly aperiodic and asymtotically $h$-expansive. By Theorem 31 in [10] it admits an (strongly isomorphic) uniform generator.

3.6. **Invariance by orbit equivalence.** R.Bowen and P.Walters have proved that expansiveness is invariant under orbit equivalence for topological flows. Here we show :

**Theorem 3.9.** *The asymptotic (h-)expansiveness, the existence of symbolic extensions and the existence of uniforms generators are also dynamical properties invariant by orbit equivalence for topological flows with the small flow boundary property.*

*Proof.* Let us consider two orbit equivalent topological flows via a homeomorphism $\Lambda$. As already mentioned the orbit equivalence induces a topological conjugacy of the base systems $Y^{\mathcal{S}}$ and $Y^{\Lambda(\mathcal{S})}$ of the zero-dimensional strongly isomorphic extensions built in Proposition 2.2 (but the roof functions may differ). For discrete systems, the existence of (principal) symbolic extensions (with an embedding), is invariant under topological conjugacy. Therefore, both zero-dimensional flows admit such symbolic extensions or not by Lemma 3.17. □

**Remark 3.2.** *As the topological entropy, the infimum of the entropy of symbolic extensions and the minimal cardinality of uniform generators may be modified by a change of the time scale. In particular these quantities are not invariant under orbit equivalence.*

## 4. Representation of symbolic flows

After Ambrose's representation theorem, D.Rudolph showed that a suspension flow over an ergodic transformation is always isomorphic to another one where the new roof function takes only two values (in general one can not hope the roof to be constant as such suspension flows are not mixing).

In the same spirit we wonder what is the "simplest model" for the roof function of a *symbolic flow*, i.e. a suspension flow $(Y_r, \Phi_r)$ over a subshift $(Y, \sigma)$ with a positive continuous roof function $r$. In this section we will only consider aperiodic flows. In this case there is a very nice topological version of Rokhlin towers for the subshift $(Y, S)$ :

**Lemma 4.1.** *(Lemma 7.5.4 in [12]) Let $(X, T)$ be an aperiodic zero-dimensional system. For any integer $n > 0$ there exists a clopen set $U_n$ such that :*

- $\bigcup_{k=0}^{n+1} T^k U_n = X$,
- $U_n, TU_n, ..., T^{n-1}U_n$ *are pairwise disjoint.*

Such a clopen set $U_n$ will be called a *n-marker* of $(X, T)$. We first show the roof function may be chosen almost constant.

**Lemma 4.2.** *Any aperiodic symbolic flow $(Y_r, \Phi_r)$ is topologically conjugate to a symbolic flow over a subshift of $\{0, 1\}^{\mathbb{Z}}$ under a roof function arbitrarily close to $\frac{\log 2}{h_{top}(\Phi_r)}$.*

Equivalently $(Y_r, \Phi_r)$ admits a Poincaré cross-section $S$ with return time $t_S$ arbitrarily close to $\frac{\log 2}{h_{top}(\Phi_r)}$ and with $h_{top}(T_S) \leq \log 2$.

*Proof.* Fix $1/2 > \epsilon > 0$. Let $a = \frac{\log 2}{h_{top}(\Phi_r)} + \epsilon$ and let $N$ be an integer larger then $3/\epsilon$. We take $N' > N$ so large that any integer larger than $N'$ belongs to $[Na]\mathbb{N} + ([Na] + 1)\mathbb{N}$. Let $U_n$ be a $n$-marker of $(Y, S)$ with $Nn\underline{r} > N'$. The set $V_n := U_n \times \{0\} \subset Y_r$ defines a Poincaré cross-section with return time $t_{V_n}$ larger than $n\underline{r}$. In particular for any $u \in V_n$ there are positive integers $k, l$ such that $|Nt_{V_n}(u) - k[Na] - l([Na]+1)| < 1/2$ and therefore $|t_{V_n}(u) - k[Na]/N - l([Na]+1)/N| < \epsilon/6$. There is a partition $P$ of $U_n$ in clopen sets such that for any two points $x$ and $y$ in the same atom of the induced partition of $V_n$ we have $|t_{V_n}(x) - t_{V_n}(y)| < \epsilon/6$. In particular we may choose the above integers $k$ and $l$ independently of $u \in A$ for $A \in P$, i.e. there are nonnegative integers $k_A$ and $l_A$ such that $|t_{V_n}(u) - k_A[Na]/N - l_A([Na]+1)/N| < \epsilon/3$ for any $u \in A$. Finally we let $S$ be the union of $\phi_t A$ over $A \in P$ and $t \in \{k'[Na]/N,\ 0 \leq k' < k_A\} \cup \{k_A[Na]/N + l'([Na]+1)/N,\ 0 \leq l' < l_A\}$. The set $S$ is a Poincaré cross-section with return time $|t_S - a| < 1/N + \epsilon/3 < 2\epsilon/3$ and therefore $\frac{\log 2}{h_{top}(\Phi_r)} + \epsilon/3 < t_S < \frac{\log 2}{h_{top}(\Phi_r)} + 2\epsilon$. By Abramov entropy formula the topological entropy of the first return map $T_S$ in $S$ is less than $\log 2$. As it is an (aperiodic) subshift it may be topologically embedded in the full shift with two symbols by Krieger's topological embedding theorem [20]. □

For a discrete topological system $(X, T)$ the orbit capacity $\mathsf{ocap}(E)$ of a subset $E$ of $X$ is defined as follows:

$$\mathsf{ocap}^T(E) = \lim_{n \to +\infty} \frac{1}{n} \sup_{x \in X} \sharp\{0 \leq k \leq n,\ T^k x \in E\}.$$

Similarly for a topological flow $(X, \Phi)$ we let

$$\mathsf{ocap}^\Phi(E) = \lim_{\tau \to +\infty} \frac{1}{\tau} \sup_{x \in X} \lambda\left(\{t \in [0, \tau],\ \phi_t(x) \in E\}\right).$$

Note that the limits are well defined by Fekete and Hille subadditive Lemma. When $E$ is a closed subset of $X$ we have $\mathsf{ocap}(E) = \sup_\mu \mu(E)$ where the supremum holds over all invariant probability measures $\mu$. By Lemma 2.10 a closed cross-section $S$ of time $\eta$ has a small flow boundary if and only if $\mathsf{ocap}^\Phi(\partial^\Phi S_\eta) = 0$.

We may refine Lemma 4.2 under the following form (similar to the representation in Rudolph's theorem). For a subshift $Y$ over a finite alphabet $\mathcal{A}$ and for $a_{-k}, \cdots, a_0 \in \mathcal{A}$, we let $[a_{-k} \cdots a_0]$ be the cylinder set

$$[a_{-k} \cdots a_0] := \{(y_n)_n \in Y,\ y_{-l} = a_{-l} \text{ for } l = 0, \cdots, k\}.$$

Moreover for $a \in \mathcal{A}$ and $l \in \mathbb{N} \setminus \{0\}$, we let $a^l$ be the subword given by $a^l := \underbrace{a \cdots a}_{l \text{ times}}$.

**Lemma 4.3.** *Let $(Y_r, \Phi_r)$ be an aperiodic symbolic flow over a subshift $(Y, \sigma)$. Then for any rationally independent positive real numbers $p$ and $q$ with $h_{top}(\Phi_r) < \frac{2\log 2}{p+q}$, for any $\epsilon > 0$ and for any $\delta \in ]0, \min(p,q)[$ the flow is topologically conjugate to a symbolic flow over a subshift $(Z, T)$ [7] of $\{0,1,2\}^{\mathbb{Z}}$ over a roof function $r'$ satisfying for $z = (z_n)_n$:*

- $\{r' = p\} = [0]$ *and* $\mathsf{ocap}^T([0]) \leq \frac{1}{2}$,
- $\{r' = p\} = [1]$ *and* $\mathsf{ocap}^T([1]) \leq \frac{1}{2} + \epsilon$,
- $\{0 < r' < \delta\} = [2]$ *and* $\mathsf{ocap}^T([2]) < \epsilon$.

Equivalently $(Y_r, \Phi_r)$ admits a Poincaré cross-section $S$ together a clopen uniform generator $\{\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}\}$ of $(S, T_S)$ with $t_S = p$ on $\mathfrak{P}$, $t_S = q$ on $\mathfrak{Q}$ and $t_S < \delta$ on $\mathfrak{R}$ and with $\mathsf{ocap}^{T_S}(\mathfrak{P}) \leq \frac{1}{2}$, $\mathsf{ocap}^{T_S}(\mathfrak{Q}) \leq \frac{1}{2} + \epsilon$ and $\mathsf{ocap}^{T_S}(\mathfrak{R}) < \epsilon$. The condition on the orbit capacity implies in particular that $\frac{1}{2} - 2\epsilon \leq \mu([i]) \leq \frac{1}{2} + \epsilon$ for any $\mu \in \mathcal{M}(Z, T)$ and for $i = 0, 1$.

*Proof.* Fix $\delta > 0$, $\epsilon > 0$ and rationally independent positive real numbers $p$ and $q$ with $h_{top}(\Phi_r) < \frac{2\log 2}{p+q}$. By Lemma 4.2 we may assume $h_{top}(\sigma) < \log 2$ and $r \simeq \frac{\log 2}{h_{top}(\Phi_r)}$.

For any $x \in \mathbb{R}$ we let $D(x) = \min\{x - (kp + lq) \geq 0 \ : \ k, l \in \mathbb{N} \text{ with } \frac{1}{1+\epsilon} \leq k/l \leq 1\}$. As $p$ and $q$ are rationally independent we have $\lim_{x \to +\infty} D(x) = 0$. Fix $\epsilon \in ]0, \frac{\log 2 - h_{top}(\sigma)}{2}]$ small and take $N$ with $D(x) < \delta/2$ for $x > N$. We argue then as in the proof of Lemma 4.2. Let $U_n$ be a $n$- marker of $(Y, \sigma)$ with $n > \max(N/\underline{r}, 1/\epsilon)$. The set $V_n = U_n \times \{0\} \subset Y_r$ defines a Poincaré cross-section with return time $t_{V_n}$ larger than $n\underline{r}$. In particular for any $u \in V_n$ there are positive integers $k, l$ such that $|t_{V_n}(u) - kp - lq| < \delta/2$. There is a partition $P$ of $U_n$ in clopen sets such that for any two points $x$ and $y$ in the same atom of the induced partition of $V_n$ we have $|t_{V_n}(x) - t_{V_n}(y)| < \delta/2$. In particular we may choose the above integers $k$ and $l$ independently of $u \in A$ for $A \in P$, i.e. there are nonnegative integers $k_A$ and $l_A$ such that $0 < t_{V_n}(u) - k_A p - l_A q < \delta$ for any $u \in A$. We may assume $P$ is finer than $Q^n$ with $Q$ being the zero coordinate of $Y$. Moreover we choose $\epsilon > 0$ small and $n$ large enough so that the cardinality of the set of $n$-words in $Y$ is less than $e^{n(h_{top}(\sigma)+\epsilon)} \leq \binom{k_A}{2k_A}$ for any $A \in P$. Indeed $\binom{m}{2m} \sim \frac{2^{2m}}{\sqrt{\pi m}}$ and we have for small $\epsilon > 0$

$$\begin{aligned} k_A(p+q) &\simeq k_A p + l_A q, \\ &\simeq nr, \\ &\simeq n\frac{\log 2}{h_{top}(\Phi_r)}, \\ &> n(p+q)/2. \end{aligned}$$

Following D.Rudolph we may now encode the system $(Y, \sigma)$ by ordering the subdivision of $[0, k_A p + l_A q]$ into $k_A$ intervals of length $p$ and $l_A$ intervals of length $q$. For any $k_A$ we fix a bijection between the set of $n$-words of $Y$ and the $2k_A$-uple of 0 and 1 with exactly $k_A$ terms equal to 0 and 1. To any $A \in P$ we let $w^A = (w^A_{k''})_{k''}$ be the $2k_A$-uple associated to the element of $Q^n$ containing $A$. For any $k' \leq 2k_A$ we let[8] $t_{k'}(A) = \sum_{k'' \leq k'} \left(p\delta_1(w^A_{k''}) + q\delta_0(w^A_{k''})\right)$ and for any

[7] the shift map on $Z$ is denoted here by $T$ to avoid any confusion with $(Y, \sigma)$.

[8] For $i = 0, 1$ we let $\delta_i(x) = 1$ if $x = i$ and $0$ if not.

$2k_A < k' \leq k_A + l_A$ we let $t_{k'}(A) = t_{2k}(A) + (k' - 2k_A)q$. Finally we let $S$ be the union of $\phi_t A$ over $A \in P$ and $t \in \{t_{k'}(A),\ k' \leq k_A + l_A\}$. The set $S$ is a Poincaré cross-section with return time $t_S \in \{p, q\} \cup ]0, \delta[$. By construction the associated clopen partition of $S$ consisting of $\mathfrak{P} = \{t_S = p\}$, $\mathfrak{Q} = \{t_S = q\}$ and $\mathfrak{R} = \{t_S \in ]0, \delta[\}$ is generating. Moreover $\mathsf{ocap}(T_S \in \mathfrak{R}) \leq \min_A \frac{1}{2k_A} < \frac{1}{n} < \epsilon$. As $\frac{1}{1+\epsilon} \leq k_A / l_A \leq 1$ we also have $\mathsf{ocap}(T_S \in \mathfrak{P}) \leq 1/2$ and $\mathsf{ocap}(T_S \in \mathfrak{Q}) \leq \frac{1}{2} + \epsilon$. □

**Remark 4.1.** *We would like to remove the remaining set $\mathfrak{R}$ by using a multiscale approach for a sequence of nested $n$-markers as D.Rudolph did for the ergodic case. One can follow this procedure. In this way one gets a Borel section $S$ with return time $t_S$ in $\{p, q\}$, such that the partition $\{t_S = p\}$, $\{t_S = q\}$ of $S$ is a generator for the induced Borel system on $S$. Unfortunately the obtained generator is not uniform. Indeed to approach the base of the $k^{th}$ tower with an error term of size $\epsilon_k$ one needs to reencode a piece of orbit of length $l_k$ with $l_k \to +\infty$ when $\epsilon_k \to 0$, so that the limit map does not admit a priori a continuous inverse.*

**Question 4.1.** *Does an aperiodic symbolic flow admit a symbolic extension with an embedding given by a suspension flow over a subshift of $\{0,1\}^{\mathbb{Z}}$ with a roof function constant on the two atoms of the zero-coordinate partition? Note that if $(X, \Phi)$ has periodic orbits one can not always ensure the roof function is two-valued. Indeed any period should then belong to $\mathbb{N}p + \mathbb{N}q$ where $p$ and $q$ are the values of the roof function.*

**Lemma 4.4.** *Let $(Y_r, \Phi_r)$ be an aperiodic symbolic flow over a subshift $(Y, \sigma)$. Then for any rationally independent positive real numbers $p < q$ with $h_{top}(\Phi_r) < \frac{2 \log 2}{p+q}$, for any integer $M \geq 2$ and for any $\delta > 0$ the flow is topologically conjugate to a symbolic flow over a subshift $(Z, T)$ of $\{0,1\}^{\mathbb{Z}}$ over a roof function $r'$ satisfying for some positive integer $K$ :*

- $\{r' = p\} = [1]$,
- $\{r' \in [q, q + \delta]\} = [0]$,
- $\{r' > q\} = T([0^{M+K}10^K 1])$,

*Moreover we have*

$$Z = \bigcup_{0 \leq k < +\infty} T^k \{r' > q\}.$$

*Proof.* We only have to slightly modify the construction in Lemma 4.3 as follows (we keep the notations of that proof). When encoding the $2k_A$-uple $w^A$ associated to $A \in P$ we may always start and finish with the letter 1, avoid a sequence of $K$ consecutive 0's for a large enough integer $K$ and take only $k_A - 1$ (not $k_A$) terms equal to 1. Moreover, we can also assume $l_A \geq k_A + M + K + 2$. All these requirements may be established by taking $n$ large enough. Then we extend $w^A$ to a $(k_A + l_A - 1)$-uple $v^A = (v^A_{k''})_{k''}$ by adding to $w^A$ a suffix of the form $0^L 1 0^K$ with $L \geq M + K$.

Then we consider the Poincaré cross-section $S$ defined by the union $\phi_t A$ over $A \in P$ and $t \in \{t_{k'}(A),\ k' < k_A + l_A\}$ with $t_{k'}(A) = \sum_{k'' \leq k'} \left(p\delta_1(v^A_{k''}) + q\delta_0(v^A_{k''})\right)$. The return time in $S$ is now either equal to $p$ or in $[q, q + \delta]$. Moreover it is larger than $q$ if and only if the first return in $S$ lies in $U_n \times \{0\}$. Finally the partition $\{t_S = p\}$, $\{t_S \geq q\}$ of $S$ defines again a clopen generator of $(S, T_S)$. □

For the time $t$-map of a topological flow we define the following weaker notion of uniform generators.

**Definition 4.1.** *Let $(X, \Phi = (\phi_t)_t)$ be a topological flow. For $\alpha > 0$ and $t \neq 0$ a partition $P$ is said to be* an $\alpha$-uniform generator of $\phi_t$ *when $\sup_{y \in P^{[-n,n]}_{\phi_t}(x)} d\left(y, \phi_{[-\alpha,\alpha]}(x)\right)$ goes to zero uniformly in $x \in X$.*

The above definition does not depend on the choice of the metric, but only on the topology of $X$ (the same holds for uniform generators). In particular $\alpha$-uniform generators are preserved by topological conjugacy. Clearly any uniform generator of $\phi_t$ is an $\alpha$-uniform generator of $\phi_t$ for all $\alpha$.

**Lemma 4.5.** *Let $(Y_r, \Phi_r)$ be an aperiodic symbolic flow over a subshift $(Y, \sigma)$. Then for any $t \in ]0, \frac{\log 2}{h_{top}(\Phi_r)}[$ and for any $\alpha > 0$, the time $t$-map $\phi_t$ admits an $\alpha$-uniform generator given by the towers associated to a clopen 3-partition of a Poincaré cross-section.*

*Proof.* We let $p = t \in ]0, \frac{\log 2}{h_{top}(\Phi_r)}[$ and we take $q$ rationally independent from $p$ with $\alpha = q - p \in ]0, p[$ so small that we have $h_{top}(\Phi_r) < \frac{2\log 2}{p+q}$. Without loss of generality we may assume $(Y_r, \Phi_r)$ is the model $(Z_{r'}, \Phi_{r'})$ given by Lemma 4.4 with respect to $p, q, \epsilon$ and $\delta = \alpha = q - p$. Let $M \geq 2$ be so large that for all $s \in [0, q[$ there exists $0 \leq u < M$ and $0 \leq v \leq u + 1$ with $s + up = vq + \beta$ for $0 \leq \beta < \alpha$. We let $\mathcal{T}$ be the 2-partition $\{\mathfrak{P}, \mathfrak{Q}\}$ of the Poincaré cross-section $Z \times \{0\}$ given by $\mathfrak{P} = [0] \times \{0\}$ and $\mathfrak{Q} = [1] \times \{0\}$. We let $\overline{r} = \sup_{y \in Y} r(y)$. We consider the compact space $\tilde{Y} = Y \times [0, \overline{r}]$ endowed with the metric $d_{\tilde{Y}}$ given by $d_{\tilde{Y}}((x,t),(y,s)) = d_Y(x,y) + |t-s|$. We also let $\pi_r : \tilde{Y} \to Y_r$ be the (uniformly) continuous map which associates to any $(y,t) \in \tilde{Y}$ the point $\phi_t^r(y,0)$ in $Y_r$ (with $(y,0) \in Y_r$ and $\Phi_r = (\phi_t^r)_t$). Fix some metric $d_{Y_r}$ on $Y_r$, for example the Bowen-Walters metric (see [4]). Finally we let $w : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\epsilon \to 0} w(\epsilon) = 0$ be a modulus of uniform equicontinuity of $(\phi_\beta)_{|\beta| \leq \alpha}$ and $\pi_r$, i.e.

$$\forall \beta \in [-\alpha, \alpha]\ \forall \mathcal{Z}, \mathcal{Z}' \in Y_r, \quad d_{Y_r}(\phi_\beta(\mathcal{Z}), \phi_\beta(\mathcal{Z}')) < w(d_{Y_r}(\mathcal{Z}, \mathcal{Z}'))$$

and

$$\forall \tilde{u}, \tilde{v} \in \tilde{Y}, \quad d_{Y_r}(\pi_r(\tilde{u}), \pi_r(\tilde{v})) < w(d_{\tilde{Y}}(\tilde{u}, \tilde{v})).$$

The 3-partition $\mathsf{T}_{\mathcal{R}}$ in towers of $Y_r$ associated to the partition $\mathcal{R} := \{\mathfrak{P}, \mathfrak{Q}, \phi_\alpha \mathfrak{Q}\}$ of the Poincaré cross-section $S' = S \cup \phi_\alpha \mathfrak{Q}$ with $S = Y \times \{0\}$ is an $\alpha$-uniform generator of $\phi_t$ (recall $\alpha = q - p$). Indeed we claim that, for any positive integer $n$, for any $\mathcal{X} = (x, s) \in Y_r$ with $0 \leq s < r(x)$ and for any $\mathcal{Y} \in \mathsf{T}_{\mathfrak{R}}^{[-2n,2n]}(\mathcal{X})$, there exists $\beta$ with $|\beta| \leq \alpha$ such that $\phi_\beta(\mathcal{Y}) = \pi_r(y, s)$ with $y \in Q^{[-n,n]}(x)$, where $Q$ denotes the zero-coordinate partition of $Y$. Then we have

$$\begin{aligned} d_{Y_r}(\mathcal{Y}, \phi_{[-\alpha,\alpha]}\mathcal{X}) &\leq w(d_{Y_r}(\phi_\beta(\mathcal{Y}), \mathcal{X}), \\ &\leq w \circ w\left(d_{\tilde{Y}}((y,s),(x,s))\right), \\ &\leq w \circ w\left(\mathsf{diam}\left(Q^{[-n,n]}(x)\right)\right) \xrightarrow{n \to +\infty} 0 \text{ uniformly in } x \in Y, \text{ thus in } \mathcal{X} \in Y_r. \end{aligned}$$

We show now the above claim. Let $\mathcal{N}^{\phi_t}$ be the $\mathsf{T}_{\mathcal{R}}$-name of $\mathcal{X}$ with respect to $\phi_t$, i.e. $\mathcal{N}^{\phi_t} = (\mathsf{T}_{\mathcal{R}}(\phi_{kt}\mathcal{X}))_{k \in [-2n,2n]}$. Any letter $T_{\mathfrak{Q}}$ is followed by $T_{\phi_\alpha \mathfrak{Q}}$ in $\mathcal{N}^{\phi_t}$ but both correspond to the same return in $S$. Then a subword of $\mathcal{N}^{\phi_t}$ of the form $T_{\mathfrak{P}}\mathsf{T}_1 ... \mathsf{T}_{K''}\mathsf{T}_{\mathfrak{P}}$, with $\mathsf{T}_i = \mathsf{T}_{\mathfrak{Q}}$ or $\mathsf{T}_i = \mathsf{T}_{\phi_\alpha \mathfrak{Q}}$ for $i = 1, \cdots, K''$, and $\sharp\{i, \mathsf{T}_i = \mathsf{T}_{\phi_\alpha \mathfrak{Q}}\} \in \{K, K+1\}$ indicates the return times in $\{r' > q\}$. These subwords are called the marking subwords. Then any subword $\mathsf{T}_{\mathfrak{P}}^L$ of $\mathcal{N}^{\phi_t}$ between two such consecutive marking subwords correspond to exactly $L$ consecutive returns of $T_S$ in $\mathfrak{P}$ because the associated return times in $S'$ are equal to $t = p$. This is also the case of the subwords $\mathsf{T}_{\phi_\alpha \mathfrak{Q}}^L$ of $\mathcal{N}^{\phi_t}$, whose last letter is not the penultimate letter of a marking subword. In this sole case, the return time in $S'$ from $\phi_\alpha \mathfrak{Q}$ may differ from $p$, but we know the subword in $Y$ associated to a marking subword is given by $10^K 1$. Combining these facts, the $Q$-name of $x$ is obtained from $\mathcal{N}^{\phi_t}$ by first replacing the marking subwords by $10^K 1$, then by deleting the letters $\mathsf{T}_{\mathfrak{Q}}$ and finally by replacing the remaining letters $T_{\mathfrak{P}}$ and $\mathsf{T}_{\phi_\alpha \mathfrak{Q}}$ respectively by 0 and 1. For example, if $\mathcal{N}^{\phi_t}$ is the following sequence (where we write the marking subwords in blue, the zero-coordinate in green and the large block of 0's before the marking subword in orange)

$$\cdots \mathsf{T}_{\phi_\alpha \mathfrak{Q}} \cdots \mathsf{T}_{\phi_\alpha \mathfrak{Q}}\ \mathsf{T}_{\mathfrak{P}} \mathsf{T}_{\mathfrak{Q}} \mathsf{T}_{\phi_\alpha \mathfrak{Q}} \cdots \mathsf{T}_{\phi_\alpha \mathfrak{Q}} \mathsf{T}_{\mathfrak{P}}\ \mathsf{T}_{\phi_\alpha \mathfrak{Q}} \mathsf{T}_{\mathfrak{P}} \mathsf{T}_{\mathfrak{Q}} T_{\phi_\alpha \mathfrak{Q}} \cdots \mathsf{T}_{\mathfrak{P}}\ \mathsf{T}_{\phi_\alpha \mathfrak{Q}} \cdots \mathsf{T}_{\phi_\alpha \mathfrak{Q}}\ \mathsf{T}_{\mathfrak{P}} \mathsf{T}_{\phi_\alpha \mathfrak{Q}} \mathsf{T}_{\phi_\alpha \mathfrak{Q}} \cdots \mathsf{T}_{\phi_\alpha \mathfrak{Q}} \mathsf{T}_{\mathfrak{P}} \cdots$$

we obtain the subword of $Y$ given by $\cdots 0^L\ 10^K 1\ 010 \cdots 1\ 0^{L'}\ 10^K 1 \cdots$ for some $L, L' \geq K + M \geq K + 2$. As we delete at most one letter in two, it contains $Q_T^{[-n,n]}(x)$ as a subword. Thus for any $\mathcal{Y} \in \mathsf{T}_{\mathfrak{R}}^{[-2n,2n]}(\mathcal{X})$ there is $(y, u) \in \tilde{Y}$ with $0 \leq u < r(y)$ and $y \in Q_T^{[-n,n]}(x)$. Let $N$ be a positive integer with $Y = \bigcup_{0 \leq k < N} \sigma^k \{r > q\}$. To conclude the proof of the claim we show that there exists $\beta$ with $|\beta| \leq \alpha$ and $s = u + \beta$, whenever $n$ is larger than $N$. It is enough to see that any orbit of $\phi_t$ visits at least one time the tower $\mathsf{T}_{\mathfrak{Q}} = \phi_{[0,\alpha[}\mathfrak{Q}$ of height $\alpha$ between two return times of the flow in $\{r > q\}$. But this follows easily from the choice of $M$ and the presence of more than

$M$ consecutive 0's before any return in $\{r>q\}$ (which corresponds to the blocks in orange in the above example). $\square$

We are now in position to prove Theorem 1.2 stated in the introduction.

*Proof of Theorem 1.2.* Let $(X,\Phi=(\phi_t)_t)$ be an aperiodic flow with a uniform generator given by a symbolic extension $\pi:(Y_r,\Phi_r=(\phi^r_t)_t)\to(X,\Phi)$ with an embedding $\psi$. We recall that it means $\psi:(X,\Phi)\to(Y_r,\Phi_r)$ is a Borel equivariant injective map with $\pi\circ\psi=\mathsf{Id}_X$. Note that the flow $(Y_r,\Phi_r)$ is necessarily also aperiodic. Fix $\alpha>0$. By Lemma 4.5 for $t$ small enough the time $t$-map $\phi^r_t$ of this symbolic flow admits an $\alpha$-uniform generator $\mathsf{T}_{\mathcal{R}}$ given by the towers above the atoms of a clopen 3-partition $\mathcal{R}$ of a Poincaré cross-section $S'$. Then $\psi^{-1}\mathsf{T}_{\mathcal{R}}$ is an $\alpha$-uniform generator of $\phi_t$ given by the towers of the partition $\psi^{-1}\mathcal{R}$ of the global Borel section $\psi^{-1}S'$. Indeed we have with $d=d_X$ :

$$\begin{aligned}\sup_{y\in(\psi^{-1}\mathsf{T}_{\mathcal{R}})^{[-n,n]}_{\phi_t}(x)} d\left(y,\phi_{[-\alpha,\alpha]}(x)\right) &= \sup_{y\in\psi^{-1}\left((\mathsf{T}_{\mathcal{R}})^{[-n,n]}_{\phi^r_t}(\psi(x))\right)} d\left(y,\phi_{[-\alpha,\alpha]}(x)\right),\\ &\leq \sup_{z\in(T_{\mathcal{R}})^{[-n,n]}_{\phi^r_t}(\psi(x))} d\left(\pi(z),\pi\left(\phi^r_{[-\alpha,\alpha]}(\psi(x))\right)\right).\end{aligned}$$

and this last right member goes to zero uniformly in $x$ with $n$ as $\pi$ is uniformly continuous and $\mathsf{T}_{\mathcal{R}}$ is an $\alpha$-uniform generator of $\phi^r_t$. $\square$

## Appendix A. Modified Brin-Katok entropy Structure

Let $(X,T)$ be a topological system and let $\mu\in\mathcal{M}(X,T)$. In [11] T.Downarowicz defines $h^{BK}(\mu,\epsilon)$ for an ergodic measure $\mu$ as done in Subsection 3.2.2, but then he extends the function harmonically on the whole space $\mathcal{M}(X,T)$. Let $P$ be a finite measurable partition of $X$. By Shanon-MacMillan-Breiman theorem the sequence $-\frac{1}{n}\log\mu(P^n(x))$ is converging for $\mu$-almost every $x$. Moreover the limit $h(\mu,P,x)$ satisfies $h(\mu,P,x)=h(\mu,P,Tx)$ almost everywhere and $\int h(\mu,P,x)\,d\mu(x)=h(\mu,P)$.

**Theorem A.1.** *The Brin-Katok entropy structure for a topological system $(X,T)$ as defined in the proof of Lemma 3.6 is an entropy structure.*

For any finite Borel partition $P$ of $X$ we have $h(\mu,P)\geq h(\mu,\mathsf{diam}(P))$ for all $\mu\in\mathcal{M}(X,T)$. When $P$ is a clopen partition we let $Leb(P)$ be the Lebesgue constant of the open cover $P$. Then we have also $h(\mu,P)\leq h(\mu,Leb(P))$. Consequently if $(X,T)$ is a zero-dimensional system then the entropy structure $(h(\cdot,P_k))_k$ for a sequence $(P_k)_k$ of clopen partitions with $\mathsf{diam}(P_k)\xrightarrow{k}0$ is uniformly equivalent to the Brin-Katok entropy structure.

We deal now independently with a general topological system $(X,T)$. Let $(Y,S)$ be the product of $(X,T)$ with an irrational circle rotation $(\mathbb{S}^1,\mathsf{R})$. As already mentioned the system $(Y,S)$ has the small boundary property. Let $(P_k)_k$ be a nonincreasing sequence of partitions of $Y$ with small boundary and $\mathsf{diam}(P_k)\xrightarrow{k}0$. Let $\lambda$ be the Lebesgue measure on the circle. The sequence $(h(\cdot\times\lambda,P_k))_k$ defines an entropy structure of $(X,T)$ (by definition). By taking the distance $d_Y$ on $Y$ defined for all $y=(x,t)$, $y'=(x',t')\in Y=X\times\mathbb{S}^1$ by $d_Y(y,y')=\max\left(d_X(x,x'),d_{\mathbb{S}^1}(t,t')\right)$ we have for all $n\in\mathbb{N}$, for all $\epsilon>0$ and for all $y=(x,t)\in Y$

$$B_S(y,n,\epsilon)=B_T(x,n,\epsilon)\times B(t,\epsilon).$$

In particular we get $h^S(\mu\times\lambda,\epsilon,y)=h^T(\mu,\epsilon,x)$ and then by integrating with respect to $\mu\times\lambda$

$$h^S(\mu\times\lambda,\epsilon)=h^T(\mu,\epsilon).$$

To conclude the proof of the theorem it is enough to show the sequences $(h(\cdot\times\lambda,P_k))_k$ and $(h(\cdot\times\lambda,\epsilon_k))_k$ are equivalent for some (any) sequence $(\epsilon_k)_k$ of positive numbers with $\lim_k\epsilon_k=0$. As mentioned above we always have $h(\nu,P)\geq h(\nu,\mathsf{diam}(P))$ for any finite Borel partition $P$ of $Y$ and for any $\nu\in\mathcal{M}(Y,S)$. The theorem follows therefore from the following proposition:

**Proposition A.1.** *Let $(Y,S)$ be a topological system and let $P$ be a partition of $Y$ with small boundary. Then for all $\gamma>0$ there is $\delta>0$ such that*

$$\forall \nu\in\mathcal{M}(Y,S),\quad h(\nu,P)\leq h(\nu,\delta)+3\gamma.$$

*Proof.* We will show that for all $\gamma>0$ there is $\delta>0$ such that for any $\nu\in\mathcal{M}(Y,S)$ and for $\nu$-almost every $x$ :

$$h(\nu,P,x)\leq h(\nu,\delta,x)+3\gamma. \tag{8}$$

Let $\gamma>0$. In [7, 26] the authors only consider a single measure $\nu$ with a partition satisfying $\nu(\partial P)=0$. This last condition together the ergodic theorem allows to control the number of atoms of $P^n$ intersecting a $\nu$-typical dynamical ball of length $n$. To get uniform estimates in $\nu$ for an essential partition $P$, we use the combinatorial lemma of [8]. We may then apply verbatim the proof of Mañé [26] to get the desired inequality (8). As a sake of completeness we give now the details.

Fix $\gamma'\in]0,\gamma/2[$ so small that $\limsup_n \frac{1}{n}\log\binom{\lceil n\gamma'/\log\sharp P\rceil}{n}<\gamma/2$. By Lemma 6 in [8] there exists $\delta>0$ such that

$$\limsup_n \sup_{x\in X}\frac{1}{n}\log\sharp\{A^n\in P^n,\ B(x,n,\delta)\cap A^n\neq\emptyset\}<\gamma'. \tag{9}$$

In fact it follows from the proof in [8] that

$$\limsup_n \sup_{x\in X}\sup_{A^n}\frac{1}{n}\sharp\left\{k\in[0,n-1],\ A_k\neq P(S^kx)\right\}<\frac{\gamma'}{\log\sharp P}, \tag{10}$$

where the supremum holds over $A^n=\bigcap_{k=0}^{n-1}S^{-k}A_k\in P^n$ with $B(x,n,\delta)\cap A^n\neq\emptyset$.

For $n,k\in\mathbb{N}$ we let

$$E_k^n:=\{x\in X,\ \nu(P^n(x))\leq e^{-nk\gamma}\}\text{ and}$$

$$F_k^n:=\{x\in E_k^n,\ \exists A^n\in P^n\text{ with }\nu(A^n)\geq e^{-n(k-2)\gamma}\text{ and }B(x,n,\delta)\cap A^n\neq\emptyset\}.$$

Then, for $n$ large enough and for any fixed $A^n\in P^n$, there are at most $\binom{\lceil n\gamma'/\log\sharp P\rceil}{n}e^{\gamma' n}$ atoms $P^n(x)$ with $B(x,n,\delta)\cap A^n\neq\emptyset$ by (10). Therefore we have for $n$ large enough :

$$\begin{aligned}\nu(F_n^k) &\leq \sum_{\substack{A^n\in P^n,\\ \nu(A^n)\geq e^{-n(k-2)\gamma}}}\ \sum_{\substack{x\in E_k^n\\ B(x,n,\delta)\cap A^n\neq\emptyset}}\nu(P^n(x)),\\ &\leq e^{n(k-2)\gamma}\times\binom{\lceil n\gamma'/\log\sharp P\rceil}{n}e^{n\gamma'}\times e^{-nk\gamma}\leq e^{-\gamma n}.\end{aligned}$$

Therefore by Borel-Cantelli Lemma, every $x$ in a subset $E$ of full $\nu$-measure belongs to finitely many $E_n^k$, $n,k\in\mathbb{N}$. We may also assume that $-\frac{1}{n}\log\nu(P^n(x))$ is converging (to $h(\nu,P,x)$) for $x\in E$. Let $x\in E$ and $k\in\mathbb{N}$ with $k\gamma<h(\nu,P,x)\leq(k+1)\gamma$. For $n$ large enough, $x$ belongs to $E_k^n\setminus F_k^n$. Thus the dynamical ball $B(x,n,\delta)$ only intersects atoms $A^n\in P^n$ with $\nu(A^n)\leq e^{-n(k-2)\gamma}$, therefore $\nu(B(x,n,\delta))\leq e^{-n(k-3)\gamma}$ by (9). We get finally

$$\begin{aligned}h(\nu,\delta,x) &\geq (k-3)\gamma,\\ &\geq h(\nu,P,x)-3\gamma.\end{aligned}$$

□

## Appendix B. Proof of the inequality $h^\pi\geq h+u_1^T$

From Theorem 55 in [10] we have $h^\pi\geq h+\mathfrak{u}_1^T$ for a symbolic extension $\pi$ with an embedding of $(X,T)$, but in [10] the proof involves a delicate intermediate construction, *the enhanced system.* Here we give a direct proof of $h^\pi\geq h+u_1^T\geq h+\mathfrak{u}_1^T$.

**Lemma B.1.** *Assume $(X,T)$ is a zero-dimensional system admitting a symbolic extension $\pi : (Y,S) \to (X,T)$ with an embedding $\psi : (X,T) \to (Y,S)$, then we have*

$$h^\pi \geq h + u_1^T.$$

*In particular $u_1^T = 0$ when $(X,T)$ is a subshift.*

*Proof.* We let $\mathcal{Q} = (Q_k)_k$ be a nonincreasing sequence of clopen partitions with $\mathsf{diam}(Q_k) \xrightarrow{k} 0$. The sequence of affine upper semicontinuous functions $h_k = h(\cdot, Q_k)$, $k \in \mathbb{N}$ then defines an entropy structure of $(X,T)$. Recall $D$ denotes a convex distance on $\mathcal{M}(X,T)$ inducing the weak-$*$ topology. We let $Per_n(X,T) := \{x \in X,\ T^n x = x\}$ and $Per(X,T) = \bigcup_{n>0} Per_n(X,T)$. By a standard combinatorial argument there is for any $k$ a positive number $\epsilon_k \in ]0, 1/k[$ so small that for any $x \in Per_n(X,T)$ the number of $A \in Q_k^n$, such that there exists $y \in Per_n(X,T) \cap A$ with $D(\mu_y, \mu_x) < \epsilon_k$, is less than $e^{n/k}$. For a periodic point $x$ with minimal period $n$ we recall $p_k(\mu_x) = \frac{1}{n} \log \sharp\{\mu_y,\ D(\mu_y, \mu_x) < \epsilon_k \text{ and } y \in Per_n(X,T)\}$. The sequence $(p_k)_k$ is converging pointwisely to zero because there are only finitely many periodic points in $(X,T)$ with a given period as in the subshift $(Y,S)$. Therefore there exists a nondecreasing sequence of positive integers $(n_k)_k$ going to infinity such that for all $k$ we have $p_k(\mu_x) = 0$ for all periodic points $x$ with minimal period less than $n_k$. For $\mu \in \mathcal{M}(X,T)$ we let $p_k(\mu) = \int p_k(\mu_x)\, d\mu(x)$, $k \in \mathbb{N}$. By definition we have $u_1 = u_1^T = \lim_k \tilde{p_k}$. By Lemma 54 in [10] for all $\mu \in \mathcal{M}(X,T)$ there exists a sequence of $T$-invariant probability measures $(\mu_k)_k$ converging to $\mu$ with $p_k(\mu_k) \xrightarrow{k} u_1(\mu)$. For any periodic point $x$ with minimal period equal to $n$ we let $\gamma_k^x$ be the probability measure associated to $\sum_y \delta_{\psi(y)}$, where the sum holds over $y \in Per_n(X,T)$ with $D(\mu_y, \mu_x) < \epsilon_k$. Finally we let $\nu_k = \int \gamma_k^x\, d\mu_k(x) \in \mathcal{M}(Y,S)$. Observe that $\pi\gamma_k^x \xrightarrow{k} \mu_x$ and therefore $\pi\nu_k \xrightarrow{k} \mu$ by convexity of $D$. Let $P$ be the zero-coordinate partition of $Y$. By superharmonicity of $\nu \mapsto H_\nu(R|R')$ for any given clopen partitions $R$, $R'$ of $Y$ we have

$$\frac{1}{n_k} H_{\nu_k}(P^{n_k}|\pi^{-1}Q_k^{n_k}) \geq \frac{1}{n_k} \int_{Per(X,T)\setminus Per_{n_k-1}(X,T)} H_{\gamma_k^x}(P^{n_k}|\pi^{-1}Q_k^{n_k}) d\mu_k(x).$$

Then for any periodic point $x$ with minimal period $n_x \geq n_k$ we have :

$$\begin{aligned}
\frac{1}{n_k} H_{\gamma_k^x}(P^{n_k}|\pi^{-1}Q_k^{n_k}) &\geq \frac{1}{n_x} H_{\gamma_k^x}(P^{n_x}|\pi^{-1}Q_k^{n_x}), \text{ because } \left(\frac{1}{n} H_\xi(P^n|\pi^{-1}Q_k)\right)_n \searrow \text{ by Fact 2.2.5 in [12]},\\
&\geq \frac{1}{n_x}\left(H_{\gamma_k^x}(P^{n_x}) - H_{\gamma_k^x}(\pi^{-1}Q_k^{n_x})\right),\\
&\geq \frac{1}{n_x} \log \sharp\{y \in Per_{n_x}(X,T) \text{ with } D(\mu_y, \mu_x) < \epsilon_k\}\\
&\quad - \frac{1}{n_x} \log \sharp\{A \in Q_k^{n_x},\ \exists y \in Per_{n_x}(X,T) \cap A \text{ with } D(\mu_y, \mu_x) < \epsilon_k\},\\
&\geq p_k(\mu_x) - 1/k.
\end{aligned}$$

Therefore we get

$$p_k(\mu_k) = \int p_k(\mu_x) d\mu_k(x) \leq \frac{1}{n_k} H_{\nu_k}(P^{n_k}|\pi^{-1}Q_k^{n_k}) + 1/k.$$

The left member goes to $u_1(\mu)$ when $k$ goes to infinity. Let us now show the limsup in $k$ of the right member is not larger than $h^\pi(\mu) - h(\mu)$. We have for all $k'' \leq k' \leq k$

$$\begin{aligned}
\frac{1}{n_k} H_{\nu_k}(P^{n_k}|\pi^{-1}Q_k^{n_k}) &\leq \frac{1}{n_k} H_{\nu_k}(P^{n_k}|\pi^{-1}Q_{k''}^{n_k}), \text{ since } Q_k \text{ is finer than } Q_{k''},\\
&\leq \frac{1}{n_{k'}} H_{\nu_k}(P^{n_{k'}}|\pi^{-1}Q_{k''}^{n_{k'}}) \text{ because } \left(\frac{1}{n} H_\xi(P^n|\pi^{-1}Q_k^n)\right)_n \searrow \text{ as recalled above.}
\end{aligned}$$

The involved partitions being clopen, we have for any weak limit $\nu$ of $(\nu_k)_k$, by letting $k$ go to infinity :

$$\limsup_k \frac{1}{n_k} H_{\nu_k}(P^{n_k}|\pi^{-1}Q_k^{n_k}) \leq \frac{1}{n_{k'}} H_\nu(P^{n_{k'}}|\pi^{-1}Q_{k''}^{n_{k'}}).$$

As it holds for all $k'$ and $\pi\nu = \mu$ we get with $h_{k''} := h(\cdot, Q_{k''})$ :

$$\limsup_k \frac{1}{n_k} H_{\nu_k}(P^{n_k}|\pi^{-1}Q_k^{n_k}) \le h(\nu) - h_{k''}(\pi\nu) \le (h^\pi - h_{k''})(\mu).$$

We conclude the proof by letting $k'' \to +\infty$. □

## Appendix C. Uniform generators with small boundary for asymptotically expansive systems

**Proposition C.1.** *Any asymptotically expansive topological system with the small boundary property admits an essential uniform generator.*

*Proof.* Replacing the system by a zero-dimension strongly isomorphic extension, we can assume by Proposition 3.1 the initial system to be zero-dimensional. Let $(X, T)$ be such a zero-dimensional asymptotically expansive system. By the Main Theorem in [9] there exists, for a finite alphabet $\mathcal{A}$, a sequence of continuous equivariant maps $\psi_k : (X, T) \to (\mathcal{A}^Z, \sigma)$ converging pointwisely to an embedding $\psi$, such that the induced maps on $\mathcal{M}(X, T)$ are converging uniformly. Moreover it is shown that $\psi^{-1}$ extends continuously to a symbolic extension $\pi : (\overline{\psi(X)}, \sigma) \to (X, T)$. The maps $\psi_k$ encode the orbit of $x$ at some scales $\epsilon_k$ with $\epsilon_k \xrightarrow{k} 0$. Except for the so-called *free positions* corresponding to some letter $*$ in $\mathcal{A}$, which represents the coordinates we can freely use to encode the smaller scales, the other letters are fixed once for all :

$$\forall a \in \mathcal{A} \setminus \{*\},\ \psi^{-1}([a]) = \bigcup_k \psi_k^{-1}([a]).$$

Moreover the upper asymptotic density of $*$ in any $\psi_k(x)$ goes to zero uniformly in $x \in X$ when $k$ goes to infinity. Consequently we have $\psi\mu([*]) = 0$ for any $\mu \in \mathcal{M}(X, T)$.

The partition $P = \{\psi^{-1}(a),\ a \in \mathcal{A}\}$ defines a uniform generator (see Proposition 3.1). Let us now show $P$ is an essential partition. The maps $\psi_k$ being continuous we have for any $\mu \in \mathcal{M}(X, T)$

$$\begin{aligned}
\sum_{a\in\mathcal{A}} \mu\left(\mathsf{Int}(\psi^{-1}[a])\right) &\ge \sum_{a\in\mathcal{A}\setminus\{*\}} \mu\left(\mathsf{Int}(\psi^{-1}[a])\right), \\
&\ge \lim_k \sum_{a\in\mathcal{A}\setminus\{*\}} \mu\left(\mathsf{Int}(\psi_k^{-1}[a])\right), \\
&\ge \sum_{a\in\mathcal{A}\setminus\{*\}} \lim_k \mu\left(\psi_k^{-1}[a]\right), \\
&\ge \sum_{a\in\mathcal{A}\setminus\{*\}} \lim_k \psi_k\mu([a]), \\
&\ge \sum_{a\in\mathcal{A}\setminus\{*\}} \psi\mu([a]) = 1.
\end{aligned}$$

Therefore $P$ has a small boundary.

□

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