PERIODIC EXPANSIVENESS OF SMOOTH SURFACE DIFFEOMORPHISMS AND APPLICATIONS

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ABSTRACT. We prove that periodic asymptotic expansiveness introduced in [14] implies the equidistribution of periodic points to measures of maximal entropy. Then following Yomdin's approach [50] we show by using semi-algebraic tools that C^∞ interval maps and C^∞ surface diffeomorphisms satisfy this expansiveness property respectively for repelling and saddle hyperbolic points with Lyapunov exponents uniformly away from zero.

1. INTRODUCTION

In this paper we establish new relations between the entropy and the growth of periodic points of smooth dynamical systems. In general the topological entropy may not be equal to the exponential growth in n of n-periodic points. Of course it may be less, e.g. when the system has uncountably many periodic points, but it may also be larger : there are even minimal (therefore aperiodic) smooth systems with positive entropy [33]. However these two quantities coincide in many cases, in particular under some expansive and specification properties. Then one may also wonder if periodic points are equidistributed with respect to measures of maximal entropy.

For expansive homeomorphisms with the standard specification property R. Bowen proved in [7] the equality between the entropy and the growth of periodic points, together with the equidistribution of periodic points with respect to the unique invariant measure of maximal entropy. In particular these properties hold true for topologically transitive subshifts of finite type and Axiom A systems [8]. Also by a celebrated result of A. Katok [36] any $C^{1+\alpha}$ surface diffeomorphism admits hyperbolic horseshoes with entropy arbitrarily close to the topological entropy so that the exponential growth in n of saddle (hyperbolic) n-periodic points is larger than or equal to the topological entropy.

Here we will prove the following result:

Main Theorem. *Let $f : M \to M$ be a C^∞ diffeomorphism of a compact surface M (resp. a C^∞ interval map) with positive topological entropy $\mathsf{h}_{\mathsf{top}}(f) > 0$.*

Then for any $\delta \in]0, \mathsf{h}_{\mathsf{top}}(f)[$ the set Per_n^δ of saddle (resp. repelling) n-periodic points with Lyapunov exponents δ-away from zero grows exponentially in n as the topological entropy. Moreover these periodic points are equidistributed with respect to measures of maximal entropy, i.e.:

- $\limsup_{n \to +\infty} \frac{1}{n} \log \sharp \mathsf{Per}_n^\delta = \mathsf{h}_{\mathsf{top}}(f)$,
- *for all increasing sequences of positive integers $(n_k)_k$ satisfying*

$$\lim_{k \to +\infty} \frac{1}{n_k} \log \sharp \mathsf{Per}_{n_k}^\delta = \mathsf{h}_{\mathsf{top}}(f),$$

any weak-star limit μ of the sequence $\left(\frac{1}{\sharp \mathsf{Per}_{n_k}^\delta} \sum_{x \in \mathsf{Per}_{n_k}^\delta} \delta_x \right)_k$ is an f-invariant measure of maximal entropy, i.e.:

$$h(\mu) = \mathsf{h}_{\mathsf{top}}(f).$$

We list now some comments and earlier related works enlightening the above statement:

i. Periodic points with small Lyapunov exponents. For $1 \le r \le \infty$ V. Kaloshin [34] has proved that in Newhouse domains (i.e. C^r open sets with a dense subset of diffeomorphisms having an homoclinic tangency) generic C^r smooth surfaces diffeomorphisms have arbitrarily fast growth of saddle periodic points (see [1] for analytic examples). Kaloshin stated his result for finite r, but his proof also works for $r = \infty$. For these C^∞ surface diffeomorphisms we can therefore not replace the sets Per_n^δ with the sets Per_n of all n-periodic points in our Main Theorem. In fact, it follows from the Main Theorem that any family of saddle periodic points of a surface diffeomorphism with exponential growth larger than the topological entropy contains periodic points with an arbitrarily small Lyapunov exponent. Finally we also recall that by Kupka-Smale theorem [37, 44] all periodic points of a C^r generic diffeomorphism are hyperbolic (in particular the sets Per_n are finite for all positive integers n).

ii. Systems with Lyapunov exponents uniformly away from zero. When all invariant measures are hyperbolic with Lyapunov exponents uniformly away from zero then one may consider in the Main Theorem the set Per_n of n-periodic points (because in this case we have $\mathsf{Per}_n^\delta = \mathsf{Per}_n$ for some positive δ). Such nonuniformly hyperbolic smooth surface diffeomorphisms have been built and studied in [21, 45]. More recently P. Berger proved that Henon like diffeomorphisms satisfy this property and our result recover then

the equidistribution property shown in [4].

iii. Comparison with a weaker form due to Chung and Hirayama. A version of the above theorem was proved for $C^{1+\alpha}$ surface diffeomorphisms f in [23] (see also [29]), where the authors considered for some fixed $c, \delta > 0$ and for any positive integer n the subset $\mathsf{Per}^n_\delta(c)$ of $x \in \mathsf{Per}^n_\delta$ with $\|T_{f^i x} f^k\|, \|(T_{f^i x} f^k)^{-1}\| \geq ce^{k\delta}$ for all $k \in \mathbb{N}$ and for all $0 \leq i < n$. The union $\bigcup_n \mathsf{Per}^n_\delta(c)$ is the set of periodic points of its closure, say $K_{\delta,c} := \overline{\bigcup_n \mathsf{Per}^n_\delta(c)}$. Moreover $K_{\delta,c}$ is a uniformly hyperbolic set. According to the aforementioned pioneer work of R. Bowen the exponential growth in n of $\mathsf{Per}^\delta_n(c)$ coincides with the topological entropy restricted to $K_{\delta,c}$. Then by Katok's Theorem one concludes that the exponential growth in n of $\mathsf{Per}^\delta_n(c)$ goes to $\mathsf{h}_{\mathsf{top}}(f)$ when c and δ go to zero. Under C^∞ smoothness assumption our Main Theorem shows the stronger fact that the exponential growth in n of $\bigcup_{c>0} \mathsf{Per}^\delta_n(c)$ is equal to the topological entropy. By using Bowen's result Chung and Hirayama also proved that periodic points in $\mathsf{Per}^\delta_n(c)$ are equidistributed with respect to an invariant measure μ_c which is converging to a measure of maximal entropy when c goes to zero.

iv. The case of topological Markov shifts. A similar picture occurs in the non compact setting of connected topological Markov shifts with a countable set of states. Indeed the (Gurevic's) entropy of such a Markov shift, which may be defined as the supremum of the measure theoretic entropies of invariant ergodic probability measures, coincides with the supremum of the topological entropy of finite subgraphs. As for a $C^{1+\alpha}$ surface diffeomorphism the system thus contains a family of subshifts of finite type with entropy arbitrarily close to the entropy of the system. Although periodic points are equidistributed with respect to measures of maximal entropy for these finite subgraphs, this is false in general for the topological Markov shift. By a result of Vere-Jones [46] it holds true for the so called *strongly positive recurrent* topological Markov shifts (see Proposition 2.3 in [10] for some (equivalent) definitions of strongly positive recurrence).

v. Positive recurrence in Sarig's Markov representation. The above results must be compared with the recent work of O. Sarig about the coding of $C^{1+\alpha}$ surface diffeomorphisms. In [47] he built for any $\delta > 0$ a finite to one topological Markov shift extension of any surface diffeomorphism on a set of full measure for any hyperbolic ergodic measure with Lyapunov exponents δ-away from zero. Then the finite subgraphs of this Markov shift correspond to the hyperbolic sets. However it is unknown if the connected components of this Markov shift are strongly positive recurrent even for C^∞ surface diffeomorphisms.

vi. Existence and finiteness of ergodic measures of maximal entropy. Building on Yomdin's theory, S. Newhouse proved that any C^∞ map on a compact manifold admits an ergodic measure of maximal entropy [43] (see also [18]). J. Buzzi proved also in [18] that there were only finitely many such measures for C^∞ interval maps with positive topological entropy. It was very recently shown by using Sarig's Markov representation that it also holds true for surface diffeomorphisms [20]. In our Main Theorem we do not know how to identify the limits μ of periodic measures in Per^δ (in general the measures μ are not ergodic). For topologically transitive such systems, uniqueness of the measure of maximal entropy has been established in [18] [20] so that periodic points in Per^δ are actually equidistributed with respect to this measure.

vii. Lower bound on the growth of periodic points. By applying Gurevic's theory to the Markov representations of Sarig [47] and Buzzi [18] for respectively smooth surface diffeomorphisms and interval maps, these dynamical systems when they admit a measure of maximal entropy (in particular in the C^∞ case) satisfy the following estimate[1]:

$$\exists p \in \mathbb{N} \setminus \{0\}, \quad \liminf_{n \to +\infty,\ p|n} e^{-n\mathsf{h}_{\mathsf{top}}(f)} \sharp \mathsf{Per}_n > 0.$$

We will prove in the Appendices that we can replace in the above inequality the set Per_n by Per_n^δ for any positive δ less than the topological entropy. This follows from the construction of Sarig in the first case whereas for interval maps we have to slightly modify the Markov representation of Buzzi. In particular we get the following statement.

Corollary 1.1. *With the assumptions and notations of the Main Theorem, there exists a positive integer p such that :*

- $\lim_{n \to +\infty,\ p|n} \frac{1}{n} \log \sharp \mathsf{Per}_n^\delta = \mathsf{h}_{\mathsf{top}}(f),$
- *any weak-star limit of* $\left(\frac{1}{\sharp \mathsf{Per}_n^\delta} \sum_{x \in \mathsf{Per}_n^\delta} \delta_x \right)_{n,\ p|n}$ *is a measure of maximal entropy.*

viii. Open question. Does the Main Theorem also hold true for C^r surface diffeomorphims with finite $r > 1$ provided $\mathsf{h}_{\mathsf{top}}(f) > \Lambda^+(f)/r$ where $\Lambda^+(f)$ denotes the supremum over all invariant measures of the sum of the positive Lyapunov exponents?

The proof of the Main Theorem follows from the following general strategy: any subset of periodic points whose growth is larger than or equal to the topological entropy, but with zero local exponential growth, is equidistributed with respect to maximal measures. Moreover its exponential growth rate is in fact equal to the topological entropy. By local exponential growth

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[1]In the present paper $\mathbb{N} := \{0, 1, ...,\}$ will denote the set of nonnegative integers.

we mean the exponential growth in n of the n-periodic points lying in a arbitrarily small n-dynamical ball. From Katok's Theorem we then only have to check that $(\mathsf{Per}_n^\delta)_n$ has zero local exponential growth rate for a C^∞-surface diffeomorphism (or a C^∞ interval map). In [14] [15] such systems are said to be asymptotically periodic-expansive. This condition was used there to build so-called uniform generators. In our setting we get:

Corollary 1.2. *Let $f : M \to M$ be a C^∞ surface diffeomorphism (or interval map).*

Then for any $\delta > 0$ there exists a uniform δ-generator P, that is a finite measurable partition P of the complement of $\bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathsf{Per}_n \setminus \mathsf{Per}_n^\delta$ such that the maximal diameter of the elements of $\bigvee_{k=-n}^{n} f^k P$ goes to zero when n goes to infinity.

Moreover we may assume that with $\mathsf{per}^\delta(f) := \sup_{n \in \mathbb{N} \setminus \{0\}} \frac{\log \sharp \mathsf{Per}_n^\delta(f)}{n}$ the partition P satisfies

$$\sharp P \leq e^{\max\left(\mathsf{h}_{\mathrm{top}}(f), \mathsf{per}^\delta(f)\right)} + 1.$$

When the surface diffeomorphism is nonuniformly hyperbolic with Lyapunov exponents uniformly from zero (as discussed in *ii.*), then for some positive δ the uniform δ-generator is covering the whole surface.

2. LOCAL PERIODIC GROWTH, THE ABSTRACT FRAMEWORK

Let (X, T) be a topological dynamical system, i.e. X is a compact metrizable space and T is a continuous map from X into itself. We denote by $\mathsf{Per}(T)$ (or Per) the set of periodic points of (X, T). When $\mathcal{P}$ is an invariant subset of $\mathsf{Per}(T)$ and n is a positive integer we let $\mathcal{P}_n$ be the subset of n-periodic points in $\mathcal{P}$, that is

$$\mathcal{P}_n := \{x \in \mathcal{P}, \ T^n x = x\}.$$

For any invariant subset $\mathcal{P} \subset \mathsf{Per}$ we let

$$g_{\mathcal{P}} = \limsup_n \frac{1}{n} \log \sharp \mathcal{P}_n.$$

We also introduce the following local quantity. For any $\epsilon > 0$ we denote by $B_T(x, n, \epsilon)$ (or just $B(x, n, \epsilon)$) the n-dynamical ball at x of size ϵ :

$$B_T(x, n, \epsilon) = \bigcap_{0 \leq k < n} T^{-k} B(T^k x, \epsilon).$$

Then we let

$$g_{\mathcal{P}}^*(\epsilon) := \limsup_n \frac{1}{n} \sup_{x \in X} \log \sharp \left(\mathcal{P}_n \cap B(x, n, \epsilon) \right)$$

and

$$g_{\mathcal{P}}^* = \lim_{\epsilon \to 0} g_{\mathcal{P}}^*(\epsilon).$$

The local periodic growth $g^*_{\mathcal{P}}$ is defined in a similar way to the tail entropy h^* introduced by M. Misiurewicz in [41] and somehow represents for the periodic growth what the tail entropy does for the topological entropy. In general h^* and g^*_{Per} may differ for the same reasons as for the topological entropy and the (global) periodic growth. For systems with the standard specification property[2] we always have $g_{\mathsf{Per}} \geq \mathsf{h}_{\mathsf{top}}$ and $g^*_{\mathsf{Per}} \geq h^*$ [25].

We may also define $g^*_{\mathcal{P}}$ by using the infinite dynamical ball

$$B(x, \infty, \epsilon) := \bigcap_{\alpha < k < +\infty} T^{-k} B(T^k x, \epsilon),$$

with $\alpha = -\infty$ it T is invertible and $\alpha = -1$ if not. Observe that when p belongs to $\mathcal{P}_n \cap B(x, n, \epsilon)$ then we have

$$\mathcal{P}_n \cap B(x, n, \epsilon) \subset B(p, \infty, 2\epsilon).$$

Consequently we have:

Lemma 2.1.

$$g^*_{\mathcal{P}} = \lim_{\epsilon \to 0} \limsup_n \frac{1}{n} \sup_{x \in X} \log \sharp \left(\mathcal{P}_n \cap B(x, \infty, \epsilon) \right).$$

Alternatively we may also take the supremum over $x \in \mathcal{P}_n$ on the right hand side.

Recall a topological system (X, T) is *expansive* when there is $\epsilon > 0$ so that for all $x \in X$ the infinite dynamical ball $B(x, \infty, \epsilon)$ is reduced to the singleton $\{x\}$. The system will be said to be *asymptotically $\mathcal{P}$-expansive* when $g^*_{\mathcal{P}} = 0$. Aperiodic systems as well as expansive ones are obviously asymptotically $\mathcal{P}$-expansive for any $\mathcal{P} \subset \mathsf{Per}$.

Clearly if (X, T) is asymptotically $\mathcal{P}$-expansive with finite topological entropy then the growth in n of n-periodic points of $\mathcal{P}$ is at most exponential and we have more precisely:

Lemma 2.2.

$$g_{\mathcal{P}} \leq \mathsf{h}_{\mathsf{top}}(T) + g^*_{\mathcal{P}}.$$

We give now a lower bound for the entropy of any equidistributed measure for $\mathcal{P}$. For any n we let $\nu^{\mathcal{P}}_n$ be the atomic measure given by $\nu^{\mathcal{P}}_n := \frac{1}{\sharp \mathcal{P}_n} \sum_{x \in \mathcal{P}_n} \delta_x$, where δ_x is the Dirac measure at x. Clearly $\nu^{\mathcal{P}}_n$ is an invariant probability measure.

Lemma 2.3. *Let (X, T) and $\mathcal{P}$ be as above. Assume $\sharp \mathcal{P}_n < \infty$ for all n. Then for any weak-star limit μ of $(\nu^{\mathcal{P}}_{n_k})_k$, where $(n_k)_k$ is an increasing sequence of integers with $g_{\mathcal{P}} = \lim_k \frac{1}{n_k} \log \sharp \mathcal{P}_{n_k}$, we have*

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[2]For a continuous interval map this property is satisfied if and only if the map is topologically mixing [5].

$$h(\mu) \geq g_{\mathcal{P}} - g_{\mathcal{P}}^*.$$

Before the proof we recall some basic definitions and facts about entropy in ergodic theory [27]. For a Borel probability measure μ on X [3] and a finite Borel partition P of X we let $H_\mu(P)$ be the Shannon entropy of μ with respect to P:

$$H_\mu(P) := -\sum_{A \in P} \mu(A) \log \mu(A).$$

By concavity of $x \mapsto -x \log x$ on $[0, 1]$ we always have

(1) $$H_\mu(P) \leq \log \sharp P.$$

Observe also that the map $\nu \mapsto H_\nu(P)$ defined on the (compact) set of Borel probability measures endowed with the weak-star topology is continuous at μ whenever $\mu(\partial A) = 0$ for every $A \in P$. When Q is another finite Borel partition we may define the conditional quantity $H_\mu(P|Q)$ as follows:

$$H_\mu(P|Q) := -\sum_{B \in Q} \mu(B) H_{\mu^B}(P),$$

where μ^B is the conditional measure on B defined by $\mu^B(C) = \frac{\mu(B \cap C)}{\mu(B)}$ for any Borel set C. The following inequality is well-known:

(2) $$H_\mu(P) \leq H_\mu(Q) + H_\mu(P|Q).$$

If μ is moreover T-invariant the sequence $\left(\frac{H_\mu(P^n)}{n}\right)_n$, with $P^n = \bigvee_{k=0}^{n-1} T^{-k} P$, is nonincreasing and its limit is usually denoted by $h_\mu(P)$. Then the measure theoretic (or Kolmogorov-Sinai) entropy $h(\mu)$ of μ is defined as the supremum of $h_\mu(P)$ over all finite Borel partitions P. When $(P_k)_k$ is a sequence of finite partitions whose diameter is going to zero then one can show (see [27]) that:

$$h(\mu) = \lim_k h_\mu(P_k).$$

Let us go back now to the proof of Lemma 2.3.

Proof. Let μ be the limit of a subsequence $(\nu_{n_k}^{\mathcal{P}})_k$ and let $\epsilon > 0$. Let P be a partition of diameter less than ϵ with $\mu(\partial A) = 0$ for all $A \in P$. For any large n we let Q_n be a finite Borel partition of X finer than P^n such that any element of Q_n contains at most one point of $\mathcal{P}_n$.

For any invariant measure ν, the sequence $\left(\frac{H_\nu(P^n)}{n}\right)_n$ is nonincreasing. Therefore we have for any $0 < l \leq n_k$ (in what follows $\nu_{n_k} := \nu_{n_k}^{\mathcal{P}}$):

$$\frac{1}{l} H_{\nu_{n_k}}(P^l) \geq \frac{1}{n_k} H_{\nu_{n_k}}(P^{n_k}).$$

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[3] Of course all these notions may be defined in a more general context, but we will always consider topological dynamical systems on compact metrizable spaces in the present paper.

From (2) and (1) we get then:

$$\frac{1}{l}H_{\nu_{n_k}}(P^l) \geq \frac{1}{n_k}\left(H_{\nu_{n_k}}(Q_{n_k}) - H_{\nu_{n_k}}(Q_{n_k}|P^{n_k})\right),$$

$$\geq \frac{1}{n_k}\left(H_{\nu_{n_k}}(Q_{n_k}) - \sum_{A \in P^{n_k}} \nu_{n_k}(A)H_{\nu_{n_k}^A}(Q_{n_k})\right),$$

$$\frac{1}{l}H_{\nu_{n_k}}(P^l) \geq \frac{1}{n_k}\left(\log \sharp \mathcal{P}_{n_k} - \sup_{A \in P^{n_k}} \log \sharp (\mathcal{P}_{n_k} \cap A)\right).$$

As the diameter of any $A \in P$ is less than ϵ we have

$$\frac{1}{l}H_{\nu_{n_k}}(P^l) \geq \frac{1}{n_k}\left(\log \sharp \mathcal{P}_{n_k} - \sup_{x \in X} \log \sharp (\mathcal{P}_{n_k} \cap B(x, n_k, \epsilon))\right).$$

By continuity the left term is going to $\frac{1}{l}H_\mu(P^l)$ when k goes to infinity, whereas for the right term we have

$$\liminf_k \frac{1}{n_k}\left(\log \sharp \mathcal{P}_{n_k} - \sup_{x \in X} \log \sharp (\mathcal{P}_{n_k} \cap B(x, n_k, \epsilon))\right) \geq \lim_k \frac{1}{n_k} \log \sharp \mathcal{P}_{n_k} -$$

$$\limsup_k \frac{1}{n_k} \sup_{x \in X} \log \sharp (\mathcal{P}_{n_k} \cap B(x, n_k, \epsilon)),$$

$$\geq g_\mathcal{P} - g_\mathcal{P}^*(\epsilon).$$

Thus we get for all positive integer l:

$$\frac{1}{l}H_\mu(P^l) \geq g_\mathcal{P} - g_\mathcal{P}^*(\epsilon),$$

and by taking the limit in l:

$$h_\mu(P) \geq g_\mathcal{P} - g_\mathcal{P}^*(\epsilon).$$

The above inequality holds for any finite partition P whose elements have diameter less than ϵ and boundaries with zero μ-measure. Since these partitions generate the whole Borel sigma-algebra up to sets of zero μ-measure we get therefore:

$$h(\mu) \geq g_\mathcal{P} - g_\mathcal{P}^*(\epsilon).$$

Finally we conclude the proof by taking the limit when ϵ goes to zero.   □

By the variational principle [30, 31] the topological entropy is equal to the supremum of the measure theoretic entropy of all invariant measures. A measure whose entropy attains this supremum (and therefore is equal to the topological entropy) is said to be maximal or of maximal entropy. Such measures do not always exist (even in intermediate smoothness, i.e. for C^r smooth systems with $r < +\infty$ [42]). The following corollary gives a new criterion for the existence of maximal measures and the equidistribution of periodic points with respect to these measures beyond Bowen's result:

Corollary 2.1. *Assume T is asymptotically $\mathcal{P}$-expansive and $g_\mathcal{P} \geq \mathsf{h}_{\mathsf{top}}(T)$. Then we have*

- $g_\mathcal{P} = \mathsf{h}_{\mathsf{top}}(T)$,

- *any weak-star limit of $(\nu_{n_k}^{\mathcal{P}})_k$ with $g_{\mathcal{P}} = \lim_k \frac{1}{n_k} \sharp \mathcal{P}_{n_k}$ is a measure of maximal entropy.*

For a topological system (X, T) with the small boundary property a measure theoretic analogue of $g_{\mathcal{P}}^*$ was defined in [15]. A system is said to have the small boundary property[4] when it admits a basis of neighborhoods with null boundary, i.e. with null μ-measure for any invariant measure μ. Let $(\epsilon_k)_k$ be a decreasing sequence of real numbers going to zero. Then for any invariant measure μ we let

$$(3) \qquad g_{\mathcal{P}}^*(\mu) \;\; = \;\; \lim_k \limsup_{\nu_n \xrightarrow{n \to \infty} \mu} \mathfrak{P}_k(\nu_n)$$

with

$$\mathfrak{P}_k(\nu_n) = \frac{1}{n} \int \log \sharp \left(\mathcal{P}_n \cap B(x, n, \epsilon_k) \right) d\nu_n(x),$$

where ν_n are periodic measures associated to a periodic point in $\mathcal{P}_n$ with minimal period equal to n. When there is no such sequence $(\nu_n)_n$ converging to μ we let $g_{\mathcal{P}}^*(\mu) = 0$. Then we have the following variational principle (see Appendix B):

$$(4) \qquad g_{\mathcal{P}}^* = \sup_\mu g_{\mathcal{P}}^*(\mu).$$

Remark 2.1. *In the statement of Lemma 2.3 one can replace $g_{\mathcal{P}}^*$ by $g_{\mathcal{P}}^*(\mu)$. The easy proof is left as an exercise for the reader.*

3. Local periodic growth for C^r, $r > 1$, interval maps and surface diffeomorphisms

We investigate here the local periodic growth for C^r smooth systems $f : M \to M$ for a real[5] number $r \geq 1$ on a compact manifold M. It is known that for C^r generic diffeomorphims in dimension larger than one the growth of hyperbolic periodic points is superexponential in Newhouse domains [34] and thus these diffeomorphisms have infinite local exponential growth $g_{\mathcal{P}}$ with respect to $\mathcal{P} = \mathsf{Per}$. Similar explicit examples have been also built on the interval [35]. When $f : M \to M$ is a surface diffeomorphism (resp. a C^1 interval map) we let Per^δ for $\delta > 0$ be the set of saddle (resp. repelling) periodic points of f with Lyapunov exponents δ-away from zero. For any f-invariant measure μ we let $\chi^+(\mu, f)$ be the integral with respect to μ of the positive part of the largest Lyapunov exponent, i.e. $\chi^+(\mu, f) := \int \lim_n \frac{1}{n} \log^+ \|T_x f^n\| d\mu(x)$. We also denote by $R(f) = \sup_\mu \chi^+(\mu, f)$ its supremum over all invariant measures, which can be written as $R(f) := \lim_n \frac{1}{n} \log^+ \sup_{x \in X} \|T_x f^n\|$ (see [11]).

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[4]It was proved in [39] that on a finite dimensional manifold any dynamical system with countable periodic points have this property.

[5]The map f is C^r if it is $\lfloor r \rfloor$ times differentiable and its $\lfloor r \rfloor$-differential map is $r - \lfloor r \rfloor$-Holder.

Theorem 3.1. *Let f be a C^r interval (or circle) map with $r > 1$. Then for any $\delta > 0$ and any invariant measure μ we have*

$$g^*_{\mathsf{Per}^\delta}(\mu) \leq \frac{\chi^+(\mu, f)}{r-1},$$

in particular

$$g^*_{\mathsf{Per}^\delta} \leq \frac{R(f)}{r-1}.$$

For surface diffeomorphisms we will show the following upper bound:

Theorem 3.2. *Let $f : M \to M$ be a C^r surface diffeomorphism with $r > 1$. Then for any $\delta > 0$ and any invariant measure μ we have*

$$g^*_{\mathsf{Per}^\delta}(\mu) \leq \frac{2(3r-1)}{r(r-1)} \max\left(\chi^+(\mu, f), \chi^+(\mu, f^{-1})\right),$$

in particular

$$g^*_{\mathsf{Per}^\delta} \leq \frac{2(3r-1)}{r(r-1)} \max\left(R(f), R(f^{-1})\right).$$

The examples built in [17] show that the upper bound in Theorem 3.1 for C^r interval maps with finite $r > 1$ is sharp. But we do not think this is the case for surface diffeomorphisms in Theorem 3.2. Let us now give some consequences of the above theorems. First recall we have by Katok's theorem [36] and its noninvertible version [22, 49]:

Theorem 3.3. *Let f be a C^r surface diffeomorphism (or interval map) with $r > 1$ and let μ be an invariant measure with positive entropy. Then for any $0 < \delta < h(\mu)$:*

$$g_{\mathsf{Per}^\delta} \geq h(\mu),$$

in particular for any $\delta < h_{\mathsf{top}}(f)$,

$$g_{\mathsf{Per}^\delta} \geq h_{\mathsf{top}}(f).$$

Therefore any C^∞ surface diffeomorphism or interval map f is asymptotically Per^δ-expansive and satisfies $g_{\mathsf{Per}^\delta} \geq h_{\mathsf{top}}(f)$ for any $0 < \delta < h_{\mathsf{top}}(f)$. Stronger lower bounds follow from the Markov representations built in [47],[18] and from Gurevic's theory:

Theorem 3.4. [47][18]*(See Appendix A) Let f be a C^r surface diffeomorpshim (or interval map) with $r > 1$ admitting a measure of maximal entropy. Then for any $0 < \delta < h_{top}$ there is a positive integer p such that*

$$\liminf_{n \to +\infty, \ p|n} e^{-n h_{\mathsf{top}}(f)} \sharp \mathsf{Per}^\delta_n > 0.$$

The main statements of the Introduction follow then directly from Corollary 2.1.

Question 3.1. *Is any C^∞ diffeomorphism $f : M \to M$ with $\mathsf{dim}(M) \geq 3$ asymptotically Per^δ-expansive for all $0 < \delta < h_{\mathsf{top}}(f)$?*

A symbolic extension is a topological extension by a subshift over a finite alphabet (a symbolic extension does not need to be Markovian and the extension does not have to be finite-to-one). Existence of symbolic extensions for smooth systems has been widely studied. Such extensions are known for C^∞ smooth systems but also for C^r (with $r > 1$) interval and surface maps (it is still open in higher dimension for finite $r > 1$). In [15] the authors refine the abstract theory of symbolic extensions developed in [9] to build uniform generators (see Theorem 1.2 in the Introduction). This is done not only by controlling the entropy at different scales as in [9] but also but also by using the local periodic growth, captured by the map $\mu \mapsto g_{\mathcal{P}}^*(\mu)$. Uniform generators are directly related with symbolic extensions with a Borel embedding (see Theorem 1.2 in [15]). In this context we get with Theorems 3.1 and 3.2:

Corollary 3.1. *(See Appendix C) Let (M, f) be a C^r surface diffeomorphism, resp. a C^r interval map, with $r > 1$.*

Then for any $\delta > 0$, there is a symbolic extension $\pi : (Y, S) \to (M, f)$ and a Borel embedding $\psi : B \to Y$ with B a Borel set with full measure for every ergodic measure except periodic measures in $\bigcup_n \left(\mathrm{Per}_n \setminus \mathrm{Per}_n^\delta\right)$ such that $\psi \circ f = \sigma \circ \psi$ and $\pi \circ \psi = \mathsf{Identity}_B$ and

$$\sup_{\nu,\ S^*\nu=\nu\ and\ \pi^*\nu=\mu} h(\nu) = E(\mu),$$

where for any for any f-invariant measure μ we let:

$$E(\mu) := h(\mu) + \frac{2(3r-1)}{r(r-1)} \left(\chi^+(\mu, f) + \chi^+(\mu, f^{-1})\right),$$

$$resp.\ E(\mu) = h(\mu) + \frac{\chi^+(\mu, f)}{r - 1}.$$

Moreover the cardinality of the alphabet of Y may be chosen to be less than or equal to $e^{\max\left(\sup_\mu E(\mu), \mathrm{per}^\delta\right)} + 1$ with $\mathrm{per}^\delta := \sup_{n \in \mathbb{N} \setminus \{0\}} \frac{\log \sharp \mathrm{Per}_n^\delta}{n}$.

Corollary 1.2 stated in the introduction follows then from Theorem 1 in [15] by taking $r = \infty$. Indeed, in this case the alphabet of Y may be chosen less than or equal to $e^{\max(\mathsf{h}_{\mathsf{top}}(f), \mathrm{per}^\delta)} + 1$ and a uniform δ-generator is given by $P := \psi^{-1}Q$ with Q being the zero-coordinate partition of Y (for a proof we refer to Theorem 1.2 in [15] which relates uniform generators and symbolic extensions with a Borel embedding).

The upper bounds on the local periodic growth obtained above in Theorems 3.1 and 3.2 are reminiscent of the following ones obtained by J. Buzzi in [18] for the tail entropy of a C^r map f on a compact manifold of dimension d:

$$h^*(f) \leq \frac{dR(f)}{r}.$$

In fact, the proof of both estimates uses semi-algebraic tools introduced by Y.Yomdin in [50].

4. The case of interval maps

The proof of Theorem 3.1 will follow quite directly from the following reparametrization lemma of dynamical balls proved in [12]. For a $\mathcal{C}^r$ (with $r > 1$) smooth map f, we let $\|f\|_r := \max_{s=1,\dots,[r],r} \|f^{(s)}\|_\infty$ where $\|f^{(s)}\|_\infty$ is the usual supremum norm of the s-derivative of f for $s \leq [r]$ (where $[r]$ denotes the integer part of r) and $\|f^r\|_\infty$ is the $r - [r]$ Holder norm of $f^{([r])}$. We also let $H : [1, +\infty[\to \mathbb{R}$ be the function defined for all $t \geq 1$ by $H(t) = -\frac{1}{t}\log(\frac{1}{t}) - (1 - \frac{1}{t})\log(1 - \frac{1}{t})$.

Lemma 4.1. [12] *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a $\mathcal{C}^r$ circle map with $r > 1$ and let $\delta > 0$. Then for any positive integer p there is $\epsilon > 0$, depending only on $\|f^p\|_r$, such that for all $x \in \mathbb{S}^1$ and for all positive integers n there exists a finite collection of intervals $\mathcal{J}_n(x)$ with the following properties :*

(i) $\forall J \in \mathcal{J}_n(x)$, $\forall y, z \in J$, $|(f^n)'(y) - (f^n)'(z)| \leq \frac{1}{3}\sup_{t \in J}|(f^n)'(t)|$,

(ii) $\{y \in \mathbb{S}^1, \ |(f^n)'(y)| \geq e^{\delta n}\} \cap B(x, n, \epsilon) \subset \bigcup_{J \in \mathcal{J}_n(x)} J$,

(iii) $\log \sharp \mathcal{J}_n(x) \leq [n/p]\left(\frac{1}{r-1} + H([\lambda_n^+(x, f^p) - p\delta] + 3)\right)(\lambda_n^+(x, f^p) - p\delta) + [n/p]A + B$,

where A is a constant depending only on r, whereas B is a constant depending only on f and p (not on n). Also we have used the notation

$$\lambda_n^+(x, f^p) := \frac{1}{[n/p]}\sum_{j=0}^{[n/p]-1} \log^+|(f^p)'(f^{jp}x)|.$$

Proof of Theorem 3.1. For a fixed $\delta > 0$ we will in fact prove the following stronger upper bound for any f-invariant measure μ:

$$g^*_{\mathsf{Per}^\delta}(\mu) \leq \frac{\chi^+(\mu) - \delta}{r-1}.$$

Let μ be an f-invariant measure and let $0 < \gamma \ll 1$. We consider an integer $p = p(\mu, \gamma, r)$ so large that $\frac{1}{p}\int \log^+|(f^p)'|d\mu \simeq \chi^+(f, \mu)$, $\frac{A}{p} \ll 1$ (with $A = A(r)$ as in Lemma 7.1) and $H(p\gamma) \ll 1$. We apply Lemma 7.1 for f, p and δ and we let $\epsilon > 0$ be as in the statement. It follows from the first item there is at most one point of Per_n^δ in any $J \in \mathcal{J}_n(x)$ as f^n is expanding on any such interval J intersecting Per_n^δ (once one takes n with $e^{n\delta} > \frac{4}{3}$). Then for any periodic measure ν_n supported on the orbit of a periodic point $x \in \mathsf{Per}_n^\delta$ with minimal period n we have whenever ϵ_k is less than ϵ:

$$\mathfrak{P}_k(\nu_n) = \frac{1}{n}\int \log \sharp\left(\mathcal{P}_n \cap B(x, n, \epsilon_k)\right) d\nu_n(x) \leq \frac{1}{n}\int \log \sharp \mathcal{J}_n(x) d\nu_n(x).$$

If $\lambda_n^+(x, f^p) - p\delta \geq p\gamma$ we have $H([\lambda_n^+(x, f^p) - p\delta] + 3))(\lambda_n^+(x, f^p) - p\delta) \ll \lambda_n^+(x, f^p) - p\delta$ and if not, $H([\lambda_n^+(x, f^p) - p\delta] + 3)(\lambda_n^+(x, f^p) - p\delta) \leq p\gamma \log 2$.

Thus for $n >> \max(B, p)$ with B as in Lemma 7.1 we get

$$\frac{1}{n} \int \log \sharp \mathcal{J}_n(x) d\nu_n(x) \; \lesssim \; \frac{1}{p(r-1)} \int \left(\lambda_n^+(x, f^p) - p\delta\right) d\nu_n(x),$$

$$\lesssim \; \frac{1}{p(r-1)} \frac{1}{[n/p]} \sum_{j=0}^{[n/p]-1} \int \left(\log^+ |(f^p)'(f^{jp}x)| - p\delta\right) d\nu_n(x).$$

As all terms in this last sum are equal we have

$$\frac{1}{n} \int \log \sharp \mathcal{J}_n(x) d\nu_n(x) \lesssim \frac{1}{p(r-1)} \int \left(\log^+ |(f^p)'(x)| - p\delta\right) d\nu_n(x).$$

When ν_n is going to μ then this last term is going by continuity of the integrand to

$$\frac{1}{p(r-1)} \left(\int \log^+ |(f^p)'| d\mu - p\delta\right) \simeq \frac{\chi^+(f, \mu) - \delta}{r - 1}.$$

We conclude that $g^*_{\mathsf{Per}^\delta}(\mu) = \lim_k \limsup_{\nu_n \to \mu} \mathfrak{P}_k(\nu_n) \leq \frac{\chi^+(f,\mu)-\delta}{r-1}.$ $\qquad \square$

5. n-HYPERBOLIC HEXAGONS

In the next sections we consider a C^r surface diffeomorphism with $r > 1$. The proof of the Main Theorem for such a diffeomorphism follows the same strategy as the above one dimensional case. In Lemma 7.1 we reparametrized dynamical balls by intervals with bounded distortion so that they contain each at most one repelling periodic point. In dimension two we introduce now local hyperbolic hexagons which play somehow the same role as these intervals.

We first define a notion of "finite time local hyperbolic sets" involving cones and we recall some related terminology and notations.

5.1. **Cones.** Let $(E, \| \cdot \|)$ be a two-dimensional Euclidean vector space and v, w be two (nonzero) noncolinear vectors in E. The associated cone $C(v, w)$ of E is the subset of the vector space E given by $C(v, w) = \{\alpha v + \beta w, \; \alpha\beta \geq 0\}$.

The *aperture* of the cone $C(v, w)$, which will be denoted by $\mathsf{ap}(C(v, w))$, is the unsigned angle $\angle(v, w) \in]0, \pi[$ between v and w and the center of $C(v, w)$ is the line $\mathbb{R}(v + w)$. We also refer to the two unit vectors generating this line as oriented centers of the cone. A subset C of E is said to be a cone when there are $v, w \in E$ with $C = C(v, w)$. For $\alpha > 0$ and a cone C with aperture less than $\frac{\pi}{\alpha}$ we let $C(\alpha)$ be the cone with the same center as C but with $\mathsf{ap}(C(\alpha)) = \alpha \mathsf{ap}(C)$. Finally two cones C, C' are said transverse when $C \cap C' = \{0\}$ and α-transverse for $\alpha > 0$ when for any $(u, u') \in C \times C'$ the unsigned angle between u and u' is in $]\alpha, \pi - \alpha[$. A regular C^1 curve $\lambda : [0, 1] \to E$ is said tangent to a cone C when $0 \neq \lambda'(t) \in C$ for all $t \in [0, 1]$.

For any $0 < \alpha < \pi$ we fix $\mathcal{C}_\alpha$ the family of cones of $\mathbb{R}^2$ (endowed with the usual Euclidean norm) with aperture equal to α centered at the lines given

by the $[4/\alpha] + 1$-roots of the unity in $\{z \in \mathbb{C}, \ Re(z) \geq 0\} \simeq \mathbb{R} \times \mathbb{R}^+ \subset \mathbb{R}^2$. Clearly the cones C in $\mathcal{C}_\alpha$ but also the cones $C(1/2)$ for $C \in \mathcal{C}_\alpha$ are covering $\mathbb{R}^2$ and the cardinality of $\mathcal{C}_\alpha$ is less than $4/\alpha + 1$. Finally we let $\mathfrak{C}_\alpha$ be the collection of pairs of cones - later called bicones - $(C, C') \in \mathcal{C}_\alpha \times \mathcal{C}_\alpha$ given by α-transverses cones.

5.2. n-**hyperbolic set.**

Let n be a fixed nonnegative integer. We consider a sequence $(E_k, \|\cdot\|_k)_{k=0,\dots,n}$ of two-dimensional Euclidean spaces with $(E_0, \|\cdot\|_0) = (E_n, \|\cdot\|_n)$ and we denote by B_k the unit ball of $(E_k, \|\cdot\|_k)$ and by 0_k the zero of E_k for $k = 0, \dots, n$. We fix some isometry between $(E_0, \|\cdot\|_0)$ and $(\mathbb{R}^2, \|\cdot\|)$ with the usual Euclidean norm $\|\cdot\|$ and we again denote by $\mathfrak{C}_\alpha$ the corresponding (via the isometry) family of cones in E_0.

Let $\mathcal{F} := (f_k : B_k \to E_{k+1})_{0 \leq k < n}$ be a sequence of C^1 maps such that $f_k(0_k) = 0_{k+1}$ and f_k is a diffeomorphism onto its image for all k. For $0 < l \leq n$ we denote by $f^l = f_{l-1} \circ \dots \circ f_0$ the lth-composition defined on the l-dynamical ball $B(\mathcal{F}, l) := \{x \in B_0, \ f^j x \in B_j \text{ for } 0 \leq j < l-1\}$. By convention we let f^0 to be the identity map on B_0.

Definition 5.1. *For $n \in \mathbb{N}$, $C > 0$, $\delta > 0$ and $\alpha > 0$, an open subset U_n of $B(\mathcal{F}, n) \subset E_0$ is said to be (C, δ, α, n)-hyperbolic (or shortly n-hyperbolic) when there are two bicones (C_u, C_s) in $\mathfrak{C}_\alpha$ and two (transverse) C^∞ smooth unit vector fields $e_u : U_n \to C_u$ and $e_s : U_n \to C_s$ such that for any $y \in U_n$ we have :*

- *$T_y f^n (e_u(y)) \in C_u$ with $\|T_y f^n (e_u(y))\|_n \geq C e^{n\delta}$,*
- *$T_y f^n (e_s(y)) \in C_s$ with $\|T_y f^n (e_s(y))\|_n \leq C^{-1} e^{-n\delta}$.*

The vector fields e_u and e_s are called respectively the n-expanding and n-contracting fields of the n-hyperbolic set U_n.

If we let v_u be an oriented center of the cone C_u, then by changing e_u for $-e_u$ we may assume $\angle (v_u, e_u(y)) \leq \frac{\pi}{2}$ for all $y \in U_n$. As the cones C_u and C_s are transverse and have the same aperture we can write $v_u = \lambda e_s + \beta e_u$ with $|\lambda| \leq |\beta|$. Therefore, for C, α and δ fixed, provided $C e^{n\delta}$ is large enough, the vector $T_x f^n(v_u)$ belongs to $C_u(3/2)$ and $\|T_x f^n(v_u)\| \geq \frac{C}{3} e^{n\delta}$. Thus we may always assume $e_u = v_u$ for large n if we relax the first item of Definition 5.1 by replacing C_u by $C_u(3/2)$ and C by $C/3$.

Remark 5.1. *The n-expanding and n-contracting fields are not canonical. When working later on with the dynamical system given by a surface diffeomorphims (and not a sequence $\mathcal{F}$ as above), they will not a priori correspond with the Oseledets unstable and stable directions, which do not vary smoothly and are not globally defined. Indeed the n-expanding and n-contracting fields only depend on the n-first iterations of the dynamical system. However for a saddle hyperbolic n-periodic point y lying in a n-hyperbolic set U_n the vectors $e_s(y)$ and $T_y f^n e_u(y)$ associated to U_n are respectively close to the usual (Oseledets) stable and unstable spaces (whenever $C e^{n\delta}$ is large enough).*

5.3. **Hyperbolic hexagons.** We define in this subsection hyperbolic hexagons as a generalization of the usual notion of rectangles that we first recall now. Let U be an open set of E_0. For two transverse one dimensional foliations $\mathcal{F}_u$ and $\mathcal{F}_s$ on U a rectangle is the image of a bifoliation chart, i.e of a topological embedding $\phi : [0,1]^2 \to U$ which straightens simultaneously the stable and unstable foliations: for all x and y in $[0,1]$ the sets $\phi(\{x\} \times [0,1])$ and $\phi([0,1] \times \{y\})$ are pieces of a leaf of $\mathcal{F}_u$ and $\mathcal{F}_s$ respectively.

If $\mathcal{F}_u$ and $\mathcal{F}_s$ are generated by two transverse C^∞ smooth non vanishing vector fields e_u and e_s on U we introduce the following generalization of rectangles :

Definition 5.2. *With the previous notations an open subset H of U is said to be an hexagon when any two points in H may be joined by a C^1 regular curve in H which is either tangent everywhere to $C(e_u, e_s)$ or tangent everywhere to $C(e_u, -e_s)$.*

For $\epsilon \in \{-1, 1\}$ we let $\mathcal{F}_u^\epsilon$ and $\mathcal{F}_s^\epsilon$ be the oriented one dimensional foliations associated to ϵe_u and ϵe_s. Then an hexagon is an open set satisfying the following (oriented) accessibility-like property [6]:

Lemma 5.1. *An open subset H of U is an hexagon if and only if for any $x, y \in H$ there is $\epsilon_u, \epsilon_s \in \{-1, 1\}$ and $x = z_1, ..., z_p = y \in U$ such that for all $1 \leq i \leq p$ the points z_{i-1} and z_i are in the same leaf of $\mathcal{F}_\alpha$ and the path in this leaf from z_{i-1} to z_i is oriented as $\mathcal{F}_\alpha^{\epsilon_\alpha}$ for $\alpha = u$ or s.*

In the above statement we point out that p may depend on x and y, but ϵ_u, ϵ_s are independent of i.

When an hexagon H is a subset of a rectangle given by a bifoliation C^∞ smooth chart $\phi : [0,1]^2 \to U$ (it will be always the case in the following) then $\phi^{-1}(H)$ is an hexagon in $]0,1[^2$ for the usual foliations in vertical and horizontal lines. In this case we have:

Lemma 5.2. *Let H be an open subset of $]0,1[^2$ then the following conditions are equivalent :*

- *H is an hexagon for the horizontal and vertical foliations,*
- *there are $0 < a < b < 1$ and $c, d \in [a, b]$ as well as functions $\zeta < \eta :]a, b[\to]0, 1[$ such that:*
  - *η (resp. ζ) is lower semi-continuous (resp. upper semi-continuous),*
  - *η (resp. ζ) is nondecreasing on $]a, c[$ (resp. $]d, b[$) and nonincreasing on $]c, b[$ (resp. $]a, d[$),*
    with[7] $H = \left\{ (x, y) \in]0, 1[^2, \ \zeta(x) < y < \eta(x) \right\}.$
- *any pair of points in H may be joined by a monotone staircase function (with finitely many steps).*

---

[6]See [2] for the usual notion of accessibility in smooth dynamical systems.
[7]Such sets look like usual hexagons. This explains the terminology.

The easy proofs of the two above lemmas are left to the reader. We will now consider hexagons with respect to the contracting and expanding fields associated to a n-hyperbolic structure.

Definition 5.3. *Let U_n be a n-hyperbolic set of $\mathcal{F}$ as in the Subsection 5.2. Let e_u and e_s be the n-expanding and n-contracting associated vector fields.*

A subset R_n of U_n is called a n-hyperbolic hexagon when R_n is an hexagon with respect to e_u and e_s. When U_n is an (C, δ, α, n)-hyperbolic set with respect to the constants we will say that R_n is a (C, δ, α, n)-hyperbolic hexagon.

A set $\tilde{R}_n$ is called a generalized n-hyperbolic hexagon when there are C, δ, α such that $\tilde{R}_n$ is the Hausdorff limit of the closures of a sequence of (C, δ, α, n)-hyperbolic hexagons (then we also say that $\tilde{R}_n$ is a generalized (C, δ, α, n)-hyperbolic hexagon).

We say $x \in B(\mathcal{F}, n)$ is a n-periodic point for $\mathcal{F} := (f_k : B_k \to E_{k+1})_{0 \le k < n}$ when $f^n x = x$. The following key lemma will allow us to bound the number of periodic points by counting the number of hyperbolic hexagons covering a dynamical ball.

Lemma 5.3. *For all C, δ, α, there is an integer $N = N(C, \delta, \alpha)$ such that any generalized (C, δ, α, n)-hyperbolic hexagon carries with $n > N$ at most one n-periodic point for $\mathcal{F}$.*

Proof. Let us first consider a (C, δ, α, n)-hyperbolic hexagon R_n with respect to $(C_u, C_s) \in \mathfrak{C}_\alpha$. Assume by contradiction that R_n contains two n-periodic points p and q. We can assume (by exchanging e_u with $-e_u$ or/and e_s with $-e_s$) that there is a C^1 regular curve $\lambda : [0, 1] \to M$ with $\lambda(0) = p$ and $\lambda(1) = q$ such that $\lambda'(t) \in C(e_s, e_u)$ for all $t \in [0, 1]$. Thus one can write $\lambda'(t)$ as $\lambda'_u(t) + \lambda'_s(t)$ with $\lambda'_u(t) \in \mathbb{R}^+ e_u$ and $\lambda'_s(t) \in \mathbb{R}^+ e_s$. Also for $\gamma = u, s$ we let $x_\gamma := \int_{[0,1]} \lambda'_\gamma(t)dt \in C_\gamma$ and $w_\gamma := \int_{[0,1]} T_{\lambda(t)} f^n \left(\lambda'_\gamma(t) \right) dt \in C_\gamma$. For some positive constant $C(\alpha)$ depending only on α we have (we write $\| \cdot \|$ for $\| \cdot \|_0 = \| \cdot \|_n$ to simplify the notations):

$$
\begin{aligned}
\|w_u\| &\ge C(\alpha) \int_{[0,1]} \|T_{\lambda(t)} f^n \left(\lambda'_u(t) \right) \| dt, \\
&\ge CC(\alpha) e^{n\delta} \int_{[0,1]} \|\lambda'_u(t)\| dt, \\
&\ge CC(\alpha) e^{n\delta} \|x_u\|,
\end{aligned}
$$

and we get similarly

$$
\|w_s\| \le C^{-1} C^{-1}(\alpha) e^{-n\delta} \|x_s\|.
$$

Thus for n large enough $w_u - x_u \in C_u(2)$ and $w_s - x_s \in C_s(2)$. This contradicts the fact that $p - q = x_u + x_s = w_u + w_s$ and the transversality of the cones $C_u(2), C_s(2)$. Finally if $\tilde{p}$ and $\tilde{q}$ are n-periodic points of a generalized n-hyperbolic hexagon $\tilde{R}_n$, which is the Hausdorff limit of $\left(\overline{R_n^k} \right)_k$ where $(R_n^k)_k$ is a sequence of (C, δ, α, n)-hyperbolic hexagons. Then $p - q =$

$x_u + x_s \simeq w_u + w_s$ for any p, q in R_n^k (with some large k) respectively close to $\tilde{p}$ and $\tilde{q}$. It leads to the same contradiction. This concludes the proof of the lemma. $\qquad\square$

Remark 5.2. *In the above proof we used in a essential way that the n-expanding and n-contradicting fields, e_u and e_s, and their image under Tf^n lie in the same bicone (C_u, C_s). It is not enough to assume their image lies in another bicone of $\mathfrak{C}_\alpha$.*

6. SEMI-ALGEBRAIC TOOLS

We now recall some results of semi-algebraic geometry which will be used in the next section to build a collection of n-hyperbolic hexagons covering a given n-dynamical ball. We refer to [3] for an introduction to semi-algebraic geometry.

6.1. **Nash maps and their degree.** A semi-algebraic set of $\mathbb{R}^d$ is a set which may be written as a finite union of polynomial inequalities. Semi-algebraic sets are finite union of real analytic manifolds (also called Nash manifolds). A map $f : A \subset \mathbb{R}^d \to \mathbb{R}^e$ is called semi-algebraic when its graph $\Gamma_f := \{(x, f(x)), \ x \in A\} \subset \mathbb{R}^{d+e}$ is semi-algebraic. A real analytic semi-algebraic map on a Nash manifold is called a Nash map. Here we only consider Nash maps defined on cubes $]0, 1[^d$ for $d \in \mathbb{N}$ (by convention we let $]0, 1[^0 = \{0\}$).

The algebraic complexity of a Nash map may be quantified as follows. The complexity $\mathsf{comp}(f)$ of a Nash map $f :]0, 1[^d \to \mathbb{R}^e$ is the minimal integer n for which we have

$$\Gamma_f = \bigcup_{i=1,\ldots,n} \bigcap_{j=1,\ldots,n} \{P_{i,k} \ ?_{i,j} \ 0\}$$

for $?_{i,j} \in \{>, =\}$ and for polynomials $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_{d+e}]$ with total degree[8] less than or equal to n.

In the present paper we will work with another notion. The degree $\mathsf{deg}(f)$ of a real Nash map $f :]0, 1[^d \to \mathbb{R}$ is the minimal total degree of the vanishing polynomials of f, i.e. polynomials $P \in \mathbb{R}[X_1, \ldots, X_d, X_{d+1}] \setminus \{0\}$ with $P(x, f(x)) = 0$ for any $x \in]0, 1[^d$. The map f being real analytic we may assume the polynomial P to be irreducible. When f is a polynomial map defined by a polynomial $F \in \mathbb{R}[X_1, \ldots, X_d]$ the degree $\mathsf{deg}(f)$ of f is then equal to the (usual) total degree $\mathsf{deg}_t(F)$ of F. For a Nash map $\phi = (\phi_1, \ldots, \phi_e) :]0, 1[^d \to \mathbb{R}^e$ we let the degree of ϕ be the maximal degree of its components $\phi_1, \ldots, \phi_e$.

In general the graph of a real Nash map may not be written as the zero locus of a vanishing polynomials. However , the degree and the complexity

---

[8]The total degree of a monomial $\prod_l X_l^{\beta_l}$ is the sum $\sum_l \beta_l$ and the total degree of a polynomial is defined to be the largest total degree of its monomials.

of a Nash map are equivalent in the following sense (Proposition 4 in [24]) : there is a function $a = a_{d,e} : \mathbb{N} \to \mathbb{N}$ such that for any Nash map $f :]0,1[^d \to \mathbb{R}^e$ we have

(5) $\deg(f) \leq a(\mathsf{comp}(f))$ and $\mathsf{comp}(f) \leq a(\deg(f))$.

6.2. **Yomdin-Gromov Lemma.** A fundamental tool in the Reparametrization lemma of dynamical balls by hyperbolic hexagons is the following powerful lemma due to Yomdin and Gromov [50][32]. We recall the functional version of this lemma in dimension 3 (we later apply this lemma in local charts of the unit tangent bundle of a compact surface).

Definition 6.1. *A map $\phi = (\phi_1, ..., \phi_d) :]0,1[^d \to \mathbb{R}^d$ is said triangular when the ith-component ϕ_i of ϕ only depends on the ith-first coordinates, i.e. for all $(x_1, ..., x_d) \in]0,1[^d$ we have*

$$\phi(x_1, ..., x_d) = (\phi_1(x_1), \phi_2(x_1, x_2), ..., \phi_d(x_1, ..., x_d)).$$

Observe that triangular maps are stable under composition (when the composition is well defined).

When r is an integer the C^r norm $\|\psi\|_r$ of a C^r map $\psi : U \to \mathbb{R}^e$ for an open set $U \subset \mathbb{R}^d$ is defined as follows:

$$\|\psi\|_r := \max_{k=0,1...,r} \|\psi^{(k)}\|_\infty$$

with $\|\psi^{(k)}\|_\infty = \max_{\alpha \in \mathbb{N}^d, \, |\alpha|=k} \sup_{x \in U} \|\partial^\alpha \psi(x)\|$ for $k \in \mathbb{N}$.

Theorem 6.1. [32] *(see also [13][40]) Let r be a positive integer and let $f :]0,1[^3 \to \mathbb{R}^4$ be a Nash map. Then there is a family $\mathcal{F}$ of Nash maps $\phi = (\phi_1, \phi_2, \phi_3) :]0,1[^{d_\phi} \to]0,1[^3$ with $0 \leq d_\phi \leq 3$ such that:*
- *each $\phi \in \mathcal{F}$ with $d_\phi = 3$ is triangular and is a diffeomorphism onto its image,*
- $f^{-1}(]-1,1[^4) \subset \bigcup_{\phi \in \mathcal{F}} \phi(]0,1[^{d_\phi})$,
- $\|\phi\|_r, \|f \circ \phi\|_r \leq 1$ *for any $\phi \in \mathcal{F}$,*
- $\sharp\mathcal{F}$ *and $\max_{\phi \in \mathcal{F}} \deg(\phi)$ is bounded from above by a constant which depends only on $\deg(f)$ and r.*

The key point in the above statement lies in the last property : the number of reparametrizations used to control the derivatives of f depends only on the degree of f, not on its C^r norm $\| \cdot \|_r$. This lemma was introduced by Yomdin and Gromov [50] [32] to bound the local dynamical complexity of C^r maps through Taylor-Lagrange polynomial interpolation. More recently applications in diophantine geometry were discovered by Pila and Wilkie (see the survey [51]).

Remark 6.1. *The reparametrization maps ϕ are uniformly continuous (in fact $1-Lipschitz$) and therefore extend continuously on the closed cube $[0,1]^{d_\phi}$. Thus we may assume $d_\phi = 3$ for all ϕ in the above statement if one replaces the second item by*

$$f^{-1}(]-1,1[^4) \subset \bigcup_{\phi \in \mathcal{F}} \phi([0,1]^3).$$

Remark 6.2. *In [32] (also in [5]) the authors have worked with the complexity and not the degree, but this is irrelevant as these notions are equivalent by (5).*

6.3. **Degree of a composition.** For the purpose of the next section we need to control dynamically the degree of semi-algebraic maps. Using elimination the following rule of composition for the degree was proved in [11].

Lemma 6.1. *Let $\phi_0 :]0,1[^e \rightarrow]0,1[$ and $\phi_1, ..., \phi_e :]0,1[^d \rightarrow]0,1[$ be Nash maps then*

$$\deg(\phi_0(\phi_1, \phi_2, ..., \phi_e)) \leq \prod_{i=0,1,2,...,e} \deg(\phi_i).$$

The lemma is obviously satisfied for polynomial maps $(\phi_i)_i$. In fact we have in this case $\deg(\phi_0(\phi_1, \phi_2, ..., \phi_e)) \leq \deg(\phi_0) \max_{i=1,2,...,e} \deg(\phi_i)$.

We recall some basic facts of elimination theory used in the proof of Lemma 6.1. Let A be a factorial ring and $P, Q \in A[X]$. The resultant $\mathsf{Res}_X(P, Q) \in A$, which is the determinant of the Sylvester matrix of P and Q, satisfies the following properties (see [38]):

- $\mathsf{Res}_X(P, Q) = 0$ if and only if P and Q have a non constant common factor in $A[X]$,
- $\mathsf{Res}_X(P, Q)$ belongs to the ideal $\langle P, Q \rangle$ generated by P and Q.

When $A = \mathbb{R}[X_1, ..., X_n]$ the resultant $\mathsf{Res}_X(P, Q)$ is a polynomial in $X_1, ..., X_n$ and the total degree of $\mathsf{Res}_X(P, Q) \in \mathbb{R}[X_1, ..., X_n]$ is bounded from above by the product of the total degrees of P and Q, seen as polynomials in the $n+1$ variables $X_1, ..., X_n, X$ (Theorem 10.9 in [48]). In this case we note also that any common root $(x_1, ..., x_n, x) \in \mathbb{R}^{n+1}$ of P and Q satisfies,

$$\mathsf{Res}_X(P, Q)(x_1, ..., x_n) = 0.$$

We go back now to the proof of Lemma 6.1.

Proof. For $i = 0, 1, 2, ..., e$ we let P_i be a vanishing irreducible polynomial of ψ_i with minimal total degree. We eliminate the variable Y_1 in

$$\begin{cases} P_0(Y_1, ..., Y_e, Y) &= 0, \\ P_1(X_1, ..., X_d, Y_1) &= 0. \end{cases}$$

Observe that for all $(x_1, , ..., x_d, y_2, ..., y_e) \in [0,1]^{d-1+e}$ we have

$$\begin{cases} P_0\left((\phi_1(x_1, ..., x_d), y_2, ..., y_e, \phi_0\left(\phi_1(x_1, ..., x_d), y_2, ..., y_e\right)\right) &= 0, \\ \qquad\qquad\qquad\qquad\qquad P_1\left(x_1, ..., x_d, \phi_1(x_1, ..., x_d)\right) &= 0, \end{cases}$$

and therefore the polynomials P_0 and P_1 as elements of $\mathbb{R}[X_1, ..., X_d, Y_1, ..., Y_e, Y]$ have the common root

$$(x_1, ..., x_d, \phi_1(x_1, ..., x_d), y_2, ..., y_e, \phi_0(\phi_1(x_1, ..., x_d), y_2, ..., y_e)).$$

In particular the resultant in Y_1 of P_0 and P_1, say $Q_1 := \mathsf{Res}_{Y_1}(P_0, P_1)$ in $\mathbb{R}[X_1, ..., X_d, Y_2, ..., Y_e, Y]$, vanishes at $(x_1, ..., x_d, y_2, ..., y_e, \phi_0(\phi_1(x_1, ..., x_d), y_2, ..., y_e))$ for all $(x_1, ..., x_d, y_2, ..., e_d) \in]0, 1[^{d-1+e}$. Moreover we have $\mathsf{deg}_t(Q_1) \leq \mathsf{deg}_t(P_0)\mathsf{deg}_t(P_1)$. This proves the lemma for $e = 1$ because in this case Q_1 is a vanishing polynomial of $\phi_0 \circ \phi_1$.

For larger integers e we eliminate by induction on $1 \leq k < e$ the variable Y_{k+1} in

$$\begin{cases} Q_k(X_1, ..., X_d, Y_{k+1}, ..., Y_e, Y) &= 0, \\ P_k(X_1, ..., X_d, Y_{k+1}) &= 0. \end{cases}$$

with $Q_k = \mathsf{Res}_{Y_k}(P_k, Q_{k-1}) \in \mathbb{R}[X_1, ..., X_d, Y_{k+1}, ..., Y_e, Y]$. As $(P_i)_i$ are irreducible, the polynomials Q_k are not zero. Also $Q_e = \mathsf{Res}_{Y_e}(P_e, Q_{e-1})$ is a vanishing polynomial of $\psi_0(\psi_1, \psi_2, ..., \psi_d)$. Finally we have $\mathsf{deg}_t(Q_e) \leq \prod_{i=0,1,..,e} \mathsf{deg}_t(P_i) = \prod_{i=0,1,...,e} \mathsf{deg}(\psi_i)$. $\qquad\square$

Then we get by an immediate induction that the algebraic degree grows at most exponentially in n after n-compositions. This estimate on the dynamical degree will be crucial to control the number of hexagons covering a given dynamical ball (Lemma 7.2).

Corollary 6.1. *Let* $\psi_1, \psi_2, ..., \psi_n :]0, 1[^d \rightarrow]0, 1[^d$ *be Nash maps then*

$$\mathsf{deg}(\psi_n \circ ... \circ \psi_2 \circ \psi_1) \leq \mathsf{deg}(\psi_n \circ ... \circ \psi_2)\mathsf{deg}(\psi_1)^d \leq \mathsf{deg}(\psi_n) \prod_{i=1,...,n-1} \mathsf{deg}(\psi_i)^d.$$

This result first appears in [11] to compute the dimensional entropies of a product.

6.4. **Covering semialgebraic hyperbolic sets by hexagons.** We cover now an n-hyperbolic semi-algebraic set by a collection of n-hyperbolic hexagons with cardinality bounded by a function depending only on the algebraic complexity of this set. Let us specify the precise form of the semi-algebraic sets that we will consider.

Fix a pair (C_u, C_s) of cones in E_0 belonging to $\mathfrak{C}_\alpha$. Then we may choose an isometry between $(E_0, \| \cdot \|_0)$ and $(\mathbb{R}^2, \| \cdot \|)$ so that the center of C_u is mapped to $\{0\} \times \mathbb{R}$. Using the notations of Subsection 5.2 we consider a n-hyperbolic set U_n with respect to $\mathcal{F}$ and (C_u, C_s) of the following form : there are Nash maps $\phi :]0, 1[^2 \rightarrow B$ and $\psi :]0, 1[^3 \rightarrow \mathbb{T}$ (with B and $\mathbb{T}$ being respectively the Euclidean unit ball and unit sphere of $\mathbb{R}^2$) such that:

- ϕ is a diffeomorphism onto its image and with $\underline{U_n}$ denoting the isometric image of U_n in $\mathbb{R}^2$ we have

$$\underline{U_n} = \phi(]0, 1[^2),$$

- the n-expanding vector field $\underline{e_u}$ on $\underline{U_n}$ is given by

$$\underline{e_u}(\phi(t,s)) = \underline{v_u} = (0,1) \text{ for all } (t,s) \in]0,1[^2,$$

- the n-contracting vector field $\underline{e_s}$ on $\underline{U_n}$ is given by

$$\underline{e_s}(\phi(t,s)) = \psi(t,s,0) \text{ for all } (t,s) \in]0,1[^2.$$

We recall the n-expanding and n-contradicting vector fields given by the n-hyperbolic structure on U_n (cf. Definition 5.1) do not correspond to the Oseledets distribution.

Lemma 6.2. *Let (U_n, ϕ, ψ) be as above. Then there is a collection $\mathcal{H}$ of generalized n-hyperbolic hexagons such that*

$$U_n \subset \bigcup_{H_n \in \mathcal{H}} H_n \text{ and}$$

$$\sharp \mathcal{H} \leq P\left(\deg(\phi), \deg(\psi)\right),$$

for a universal polynomial P.

Proof. We have $U_n = \phi(]0,1[^2) = \bigcup_{0 < a < 1/2} \phi(I_a)$ with $I_a =]a, 1-a[^2$. Thus by compactness it is enough to cover for any $0 < a < 1/2$ the set $\phi(I_a)$ by generalized n-hyperbolic hexagons as in the lemma with a universal polynomial P which does not depend on a.

We fix once and for all $0 < a < 1/2$. As $\phi :]0,1[^2 \to \mathbb{R}^2$ is a diffeomorphism onto its image we may extend smoothly on the whole space $\mathbb{R}^2$ the restriction of $\underline{e_s}$ on $\phi(I_a)$ in such a way the extended C^∞ field, denoted by e_s^a, does not vanish and belongs also to the cone C_s. The vector fields $\underline{e_s^a}$ and $\underline{e_u} = (0,1)$ being smooth non vanishing vector fields on $\mathbb{R}^2$ lying in the transverse cones C_s and C_u there is a diffeomorphism $\Theta : [0,1]^2 \to \mathbb{R}^2 \supset \phi(]a, 1-a[^2)$ which straightens simultaneously the foliations associated to e_s^a and $\underline{e_u}$, i.e. $\Theta(\{t\} \times [0,1])$ and $\Theta([0,1] \times \{s\})$ are respectively tangent to $\underline{e_u}$ and $\underline{e_s^a}$ for all $(t,s) \in [0,1]^2$.

We let ∂I_a (resp. $\mathsf{ex}(I_a)$) be the boundary of I_a given by the four edges of the corresponding square (resp. the extreme set of I_a given by the four corners). A point x in $\partial \phi(I_a) = \phi(\partial I_a)$ will be said *singular* when :

- either $x \in \phi(\mathsf{ex}(I_a))$,
- or $\underline{e_s}$ is tangent to $\partial \phi(I_a)$ at x, but $\underline{e_s}$ is transverse [9] to $\partial \phi(I_a)$ on $U \setminus \{x\}$ for a neighborhood U of x,
- or $\underline{e_u}$ is tangent to $\partial \phi(I_a)$ at x, but $\underline{e_u}$ is transverse to $\partial \phi(I_a)$ on $U \setminus \{x\}$ for a neighborhood U of x.

Claim 1. *Let $\mathcal{S}$ be the set of singular points. Then we have*

$$\sharp \mathcal{S} \leq P\left(\deg(\phi), \deg(\psi)\right),$$

for a universal polynomial P (which does not depend on a)

---

[9]As ϕ and e_s are real analytic it is equivalent to say the image by ϕ of the edge of I_a containing $\phi^{-1}(x)$ is not an integral curve of the vector field e_s.

We consider the subset $\mathcal{S}_x$ of $[0,1]$ given by the x-coordinates of the singular points through the chart Θ, i.e.

$$\mathcal{S}_x = \pi(\Theta^{-1}\mathcal{S}),$$

where $\pi : [0,1]^2 \to [0,1]$ is the projection on the first coordinate. Let us reorder this set as follows

$$\mathcal{S}_x := \{x_1 < x_2 < ... < x_K\}.$$

Claim 2. *For any $1 \leq l < K$ there are monotone maps $\eta_1^l < ... < \eta_{N_l}^l :$ $]x_l, x_{l+1}[\to [0,1]$ with $N_l \leq \sharp\mathcal{S}$ such that*

$$(]x_l, x_{l+1}[\times[0,1]) \cap \Theta^{-1} \circ \phi(I_a) \quad := \bigcup_{1 \leq i < N_l}\{(x,y), \quad x \in]x_l, x_{l+1}[,$$
$$\eta_1^{i+1}(x) < y < \eta_1^{i+1}(x)\}.$$

As already observed in Lemma 5.2 the sets $E_l := \{(x,y), \ x \in]x_l, x_{l+1}[, \ \eta_1^{i+1}(x) < y < \eta_1^{i+1}(x)\}$ are hexagons for the horizontal and vertical foliations, and thus the sets $\Theta(E_l)$ are n-hyperbolic hexagons. Therefore Lemma 6.2 will be proved once we have shown Claim 1 and Claim 2.

Proof of Claim 1. For a singular point $x \notin \phi(\mathsf{ex}(I_a))$ either $\underline{e_u}$ or $\underline{e_s}$ is tangent to $\partial\phi(I_a)$ at $\Theta(x)$. Thus it is enough to prove that the wedge products $(0,1) \wedge \partial_t\tilde{\phi}(t,i)$ and $\psi(t,i,0) \wedge \partial_t\tilde{\phi}(t,i)$ (resp. $(0,1) \wedge \partial_s\tilde{\phi}(i,s)$ and $\psi(i,s,0) \wedge \partial_s\tilde{\phi}(i,s)$) for $i \in \{a, 1-a\}$ vanishes either for all $t \in]0,1[$ (resp. for all s) or for N values of t (resp. s) with $N \leq P(\deg(\phi), \deg(\psi))$ where P is a universal polynomial (independent of a). But these maps being Nash on $]0,1[$ either they vanish everywhere or the number of their zeroes is less than or equal to their degree. By using again elimination theory one easily checks that for any Nash maps $f, g :]0,1[\to \mathbb{R}$ the degrees of fg, $f+g$ and f' depends polynomially on the degree of f and g, i.e. there are universal polynomials $Q \in \mathbb{R}[X,Y]$ and $R \in \mathbb{R}[X]$ with $\deg(fg), \deg(f+g) \leq Q(\deg(f), \deg(g))$ and $\deg(f') \leq R(\deg(f))$. Moreover for a Nash maps $h :]0,1[^2 \to \mathbb{R}$ and for any $c \in]0,1[$ the degree of $h_c := h(c,.)$ is less than or equal to the degree of h. As the above wedge products are obtained by such operations on ψ and the coordinates of ϕ, this concludes the proof of the lemma. $\qquad\square$

Proof of Claim 2. It is easily checked from the definition of singular points that the set $(]x_l, x_{l+1}[\times[0,1]) \cap \phi(I_a)$ may be written as the union of "slices" of the form $\{(x,y), \ x \in]x_l, x_{l+1}[, \ \eta(x) < y < \zeta(x)\}$ with η and ζ being monotone (cf. Figure 1). Finally let us see why the number of such slices is less than or equal to $\sharp\mathcal{S}$. The boundary of $\Theta^{-1} \circ \phi(I_a)$ is a Jordan curve $\gamma : [0,1] \to [0,1]^2$ given by four C^∞ smooth Jordan arcs (corresponding to the image of the edges of I_a). Moreover the graph of any η_i^l (with η_i^l as in the statement of Claim 2) is a piece of γ, i.e. there is $0 \leq a_{i,l} < b_{i,l} \leq 1$ such that the graph of η_i^l coincides with $\gamma(]a_{i,l}, b_{i,l}[)$. Finally notice that $\mathcal{S} \subset \bigcup_{i,j}\{\gamma(a_{i,l}), \gamma(b_{i,l})\}$. Let $i \neq j$. We may assume $a_{j,l} > b_{i,l}$ and $\pi(\gamma(b_{i,l})) = x_l$ (the other cases being similar). Then either $\Theta^{-1} \circ \phi(\mathsf{ex}(I_a)) \cap \gamma([b_{i,l}, a_{j,l}]) \neq \emptyset$
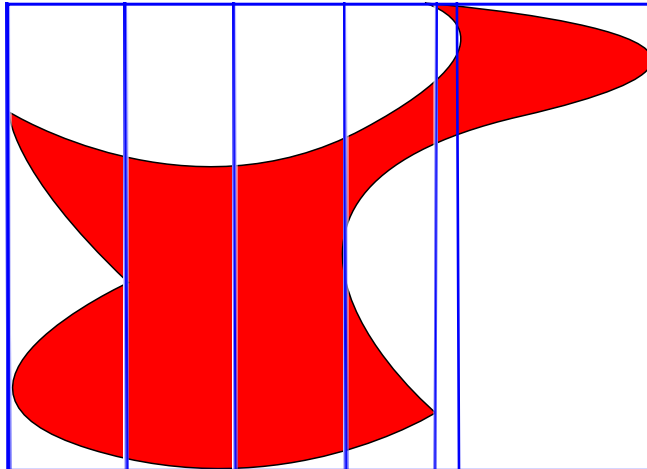
Figure 1: *The n-hyperbolic set $\phi(I_a)$ (in red) through a straightening chart of the n-contracting and n-expanding foliations. The blue vertical lines represent the expanding leaves in the domain of the chart at the singular points of $\partial\phi(I_a)$.*

or $\gamma(a_{j,l})$ and $\gamma(b_{i,l})$ belongs to the same C^∞ smooth (open) arc. But as $(x_{l+1} =)\pi(\gamma(a_{i,l})) > \pi(\gamma(b_{i,l})) = x_l \leq \pi(\gamma(a_{j,l}))$, there is in this case $t \in [b_{i,l}, a_{j,l}]$ such that the vertical line is tangent to γ at $\gamma(t)$, i.e. $\underline{e_u}$ is tangent to $\partial\phi(I_a)$ at $\Theta \circ \gamma(t)$. In both cases there is $t \in [b_{i,l}, a_{j,l}]$ such that $\Theta \circ \gamma(t)$ belongs to $\mathcal{S}$. In particular the number of function η_i^l above $]x_l, x_{l+1}[$ is less than or equal to $\sharp\mathcal{S}$.

$\square$

$\square$

7. Reparametrization Lemma

We prove now a Reparametrization Lemma of dynamical balls for surface diffeomorphisms as Lemma 7.1 for interval maps. We follow a general scheme in Yomdin's theory by first approximating locally our given C^r diffeomorphism by polynomials and by then applying semi-algebraic tools. The main novelty in the present paper consists in approximating not only the C^r map but also the action of its derivative on the unit tangent bundle. A similar approach was developed in [12] to build symbolic extensions of C^r smooth surface systems with $r > 1$.

7.1. **Local dynamic.** Let $f : M \to M$ be a C^r diffeomorphism on a Riemanian surface $(M, \|\cdot\|)$ with $r > 1$. We fix a point $x \in M$, a scale ϵ less than the radius of injectivity of $(M, \|\cdot\|)$ and two positive integers $p << n$. Let $\exp$ be the exponential map on the tangent bundle TM. For $0 \leq k \leq [n/p]$ (resp. $k = [n/p] + 1$) we let $(E_k, \|\cdot\|_k)$ be the Euclidean space $(T_{f^{kp}x}M, \|\cdot\|_{f^{kp}x})$ (resp. $(T_{f^n x}M, \|\cdot\|_{f^n x})$) and B_k be its unit ball. We define the ϵ-rescaled local dynamics of f^p at x till the time n as the

following sequence of functions $\mathcal{F}_x^n(\epsilon, p) = (F_k : B_k \subset E_k \to E_{k+1})_{k=0,\dots,[n/p]}$
: for any $0 \leq k < [n/p]$, we let

$$F_k = \left(\exp_{f^{(k+1)p}x}(\epsilon.) \right)^{-1} \circ f^p \circ \exp_{f^{kp}x}(\epsilon.)$$

and

$$F_{[n/p]} = \left(\exp_{f^n x}(\epsilon.) \right)^{-1} \circ f^{n-[n/p]p} \circ \exp_{f^{[n/p]p}x}(\epsilon.).$$

We also let $F^l = F_{l-1} \circ \dots \circ F_0$ for $0 < l \leq [n/p] + 1$ and $F^0 = \mathsf{Identity}$ so that we have $F^{[n/p]+1} = \left(\exp_{f^n x}(\epsilon.) \right)^{-1} \circ f^n \circ \exp_x(\epsilon.)$. When p divides n the image $\exp_x \left(B \left(\mathcal{F}_x^n(\epsilon, p), [n/p] \right) \right)$ by $\exp_x(\epsilon.)$ of the unit $[n/p]$-dynamical ball for $\mathcal{F}_x^n(\epsilon, p)$ is exactly the $[n/p]$-dynamical ball $B_{f^p}(x, [n/p], \epsilon)$ at x for f^p. In the following the point x will always be assumed to be a periodic point with period n, so that we have $(E_{[n/p]+1}, \| \cdot \|_{[n/p]+1}) = (E_0, \| \cdot \|_0)$.

7.2. Covering dynamical balls with hexagons.

We first generalize the notion of hyperbolic hexagon in the context of a smooth surface dynamical system as follows. A subset of M is called a local n-hyperbolic hexagon (resp. a local generalized n-hyperbolic hexagon) for f at $x \in \mathsf{Per}_n$ when it is for some k the image by $f^{-k} \circ \exp_{f^k x}$ of a n-hyperbolic hexagon (resp. generalized n-hyperbolic hexagon) of the sequence $\mathcal{F}_{f^k x}^n(\epsilon, p)$. Obviously Lemma 5.3 again holds true with this generalized definition : there is at most one n-periodic point in a generalized local n-hyperbolic hexagon for large enough n.

In the lemma below we consider a Riemannian (C^r smooth) surface $(M, \| \cdot \|)$. We denote again by $\| \cdot \|$ the bundle norm induced by the Riemannian structure of M on bundle maps $F : TM \to TM$ over $f : M \to M$ and we let m be its conorm, i.e. for any $x \in M$ we let

$$\| F(x, .) \| = \sup_{v \in T_x M \setminus \{0\}} \frac{\| F(x, v) \|_{f(x)}}{\| v \|_x}$$

and

$$m(F(x, .)) = \inf_{v \in T_x M \setminus \{0\}} \frac{\| F(x, v) \|_{f(x)}}{\| v \|_x}.$$

We fix once and for all a finite C^r atlas $\mathcal{A}$ of M. Then the C^r norm $\| f \|_r$ of any smooth map $f : M \to M$ with respect to $\mathcal{A}$ (the choice of $\mathcal{A}$ will be implicit later on) is the maximum of the C^r norms $\| \cdot \|_r$ (as defined in Subsection 6.2) of f over the collection of relevant charts.

Lemma 7.1. *Let $f : M \to M$ be a C^r surface diffeomorphism with $r > 1$ and let $\delta > 0$.*

Then for any integer $s > r$ and for any positive integer p with $p\delta \gg s$, there exists $\epsilon > 0$ depending only on $\| f^p \|_r$, such that for all positive integers n and for all $x \in \mathsf{Per}_n^\delta$, there exists a finite collection $\mathcal{J}_n(x)$ of subsets of M with the following properties:

(i) any $J \in \mathcal{J}_n(x)$ is a local generalized n-hyperbolic hexagon,

(ii) $\mathsf{Per}_n^\delta \cap B(x, n, \epsilon) \subset \bigcup_{J \in \mathcal{J}_n(x)} J$,

(iii) $\log \sharp \mathcal{J}_n(x) \leq [n/p]\frac{2}{r}\lambda_n^+(x, f^p) + 2[n/p]\left(\frac{1}{r-1} + \frac{1}{s}\right)\lambda_n(x, f^p) + [n/p]A + B$,

where A is a constant depending only on r, s, whereas B is a constant depending only on f, δ and p (not on x and n). Also we have used the notations

$$\lambda_n(x, f^p) := \frac{1}{[n/p]} \sum_{j=0}^{[n/p]-1} \log\left(\frac{\|T_{f^{pj}x}f^p\|}{m(T_{f^{pj}x}f^p)}\right) \quad and$$

$$\lambda_n^+(x, f^p) := \frac{1}{[n/p]} \sum_{j=0}^{[n/p]-1} \log^+ \|T_{f^{pj}x}f^p\|.$$

The upper bound in (iii) looks independent from the parameter δ but this is not the case as p is chosen large compared to s/δ. In fact by the already mentioned works of Kaloshin one can not replace the subset Per^δ of periodic points by the set Per of all periodic points in the above statement.

We prove now Theorem 3.2 assuming Lemma 7.1. Let μ be a f-invariant probability measure. We denote by $\chi(\mu, f) := \int \lim_n \frac{1}{n} \log \|T_x f^n\| d\mu(x)$ the largest (maybe nonpositive) Lyapunov exponent of μ. Fix $\delta > 0$ and $s \in \mathbb{N}$ with $s \gg r$. We let p be a positive integer such that

$$p\delta \gg s,$$
$$\frac{A}{p} \ll 1,$$
$$\frac{1}{p}\int \log^+ \|T_x f^p\| d\mu(x) \simeq \chi^+(\mu, f) \text{ and}$$
$$\frac{1}{p}\int \log\left(\frac{\|T_x f^p\|}{m(T_x f^p)}\right) d\mu(x) \simeq \chi(\mu, f) + \chi(\mu, f^{-1}).$$

As there is at most one periodic point in Per_n^δ in a local generalized n-hyperbolic hexagon, we get for any periodic measure ν_n supported on the orbit of a periodic point $x \in \mathsf{Per}_n^\delta$ with minimal period n and for k with $\epsilon_k \leq \epsilon$:

$$\mathfrak{P}_k(\nu_n) \leq \frac{1}{n}\int \log \sharp \mathcal{J}_n(x) d\nu_n(x).$$

Then for n large enough compared to B and for s large enough compared to r we have

$$\frac{1}{n}\int \log \sharp \mathcal{J}_n(x) d\nu_n(x) \lesssim \int \left(\frac{2}{pr}\lambda_n^+(x, f^p) + \frac{2}{p(r-1)}\lambda_n(x, f^p)\right) d\nu_n(x),$$

$$\lesssim \frac{2}{pr}\int \log^+ \|T_x f^p\| d\nu_n(x) + \frac{2}{p(r-1)}\int \log\left(\frac{\|T_x f^p\|}{m(T_x f^p)}\right) d\nu_n(x).$$

When ν_n is going to μ then by continuity of the integrand this last term is going to

$$\frac{2}{pr} \int \log^+ \|T_x f^p\| d\mu(x) + \frac{2}{p(r-1)} \int \log\left(\frac{\|T_x f^p\|}{m(T_x f^p)}\right) d\mu(x) \simeq$$
$$\frac{2}{r}\chi^+(\mu, f) + \frac{2}{(r-1)}\left(\chi(\mu, f) + \chi(\mu, f^{-1})\right),$$

and thus we conclude that

$$g^*_{\mathsf{Per}^\delta}(\mu) = \lim_k \limsup_{\nu_n \to \mu} \mathfrak{P}_k(\nu_n) \leq \frac{2(3r-1)}{r(r-1)} \max\left(\chi^+(\mu, f), \chi^+(\mu, f^{-1})\right).$$

This completes the proof of Theorem 3.2.

7.3. **Reparametrization lemma for the tangent map.** Let $\mathcal{F} = (F_k)_{0 \leq k \leq m}$ be a sequence of C^r maps with $r > 1$ from the unit Euclidean ball $B \subset \mathbb{R}^2$ to $\mathbb{R}^2$. We consider the action of the differential of $\mathcal{F}$ on the (trivial) unit fiber bundle $\mathfrak{E} := \{(x, u, v), x \in B, (u, v) \in S_x \times S_x\}$ where $S_x = T_x^1 \mathbb{R}^2 = \mathbb{T}$ is the unit circle. For any $0 \leq k \leq m$ and for any $u \in T_x^1 M$ we let $u_k = \frac{T_x F^k(u)}{\|T_x F^k(u)\|} \in T^1 M$. For two sequences of integers $\mathcal{A} = (a_0, ..., a_k, ...a_m) \in \mathbb{Z}^{m+1}$ and $\mathcal{B} = (b_0, ..., b_k, ...b_m) \in \mathbb{Z}^{m+1}$ we define the associated dynamical balls

$$B_{T\mathcal{F}}(\mathcal{A}, \mathcal{B}) := \bigcap_{0 \leq k \leq m} \Big\{ (x, u, v) \in B \times \mathbb{T} \times \mathbb{T} \mid F^k x \in B(0, 1),$$
$$\|T_{F^k x} F_k(u_k)\| \in [e^{a_k}, e^{a_k+1}],$$
$$\|T_{F^k x} F_k(v_k)\| \in [e^{-b_k-1}, e^{-b_k}] \Big\},$$

and

$$B^+_{T\mathcal{F}}(\mathcal{A}, \mathcal{B}) := \bigcap_{0 \leq k \leq m} \Big\{ (x, u, v) \in B \times \mathbb{T} \times \mathbb{T} \mid F^k x \in B(0, 2),$$
$$\|T_{F^k x} F_k(u_k)\| \in [e^{a_k-1}, e^{a_k+2}],$$
$$\|T_{F^k x} F_k(v_k)\| \in [e^{-b_k-2}, e^{-b_k+1}] \Big\}.$$

By applying Yomdin's reparametrization method to the tangent map we obtain the following lemma (we postpone its proof in the next section).

Lemma 7.2. *Let $\mathcal{F}$ be as above. Then for all integers $s > r$, for all $\alpha > 0$ and for all sequences $\mathcal{A}, \mathcal{B}$ of $m+1$ integers there is a family $\mathcal{I}_m = \mathcal{I}_m(\mathcal{A}, \mathcal{B})$ of Nash maps $\psi = (\psi_1, \psi_2, \psi_3) :]0, 1[^2 \times]0, 1[\times]0, 1[\to B \times \mathbb{T} \times \mathbb{T}$ continuously extendable on $[0, 1]^2 \times [0, 1] \times [0, 1]$ such that:*

(i) Any ψ is a diffeomorphism onto its image and $\psi(x, v, w) = (\psi_1(x), \psi_2(x, v), \psi_3(x, w))$ for all (x, v, w) (i.e. ψ_1; resp. ψ_2; resp. ψ_3 depends only on x; resp. x, v; resp. x, w).

(ii)

$$B_{T\mathcal{F}}(\mathcal{A}, \mathcal{B}) \subset \bigcup_{\psi \in \mathcal{I}_m} \psi([0, 1]^2 \times [0, 1] \times [0, 1]) \subset B^+_{T\mathcal{F}}(\mathcal{A}, \mathcal{B})$$

and for any $\psi \in \mathcal{I}_m$, for any $k = 0, ..., m + 1$, the sets

$$\bigcup_{(x,v)\in[0,1]^2\times[0,1]} T_{\psi_1(x)}F^k(\psi_2(x,v)) \quad and \quad \bigcup_{(x,w)\in[0,1]^2\times[0,1]} T_{\psi_1(x)}F^k(\psi_3(x,w))$$

are contained in cones of $\mathbb{R}^2$ with aperture less then $\alpha/2$,

(iii)

$$\sharp\mathcal{I}_m \leq BA^m \prod_{0\leq k\leq m} \max\left(1, \|F_k\|_r^{\frac{1}{r}}, \left(\frac{\|F_k\|_r}{e^{a_k}}\right)^{\frac{1}{r-1}}, \left(\frac{\|F_k\|_r}{e^{-b_k}}\right)^{\frac{1}{r-1}}\right)^2$$

$$\times \max\left(1, \left(\frac{\|F_k\|_r}{e^{a_k}}\right)^{\frac{1}{s}}\right) \times \max\left(1, \left(\frac{\|F_k\|_r}{e^{-b_k}}\right)^{\frac{1}{s}}\right),$$

(iv) for any $\psi \in \mathcal{I}_m$ the degree of ψ is less than A^{m+1},

where A is a constant depending only on r, s, whereas B depends only on α and on the C^r norms $\|\cdot\|_r$ of $\mathcal{F}$ (not on m).

Assuming the above Reparametrization Lemma we prove now Lemma 7.1. We fix once and for all $\delta > 0$ and $s > r$. We also let p and $n \geq p$ be integers as in Lemma 7.1 and we consider some $x \in \mathsf{Per}_n^\delta$.

For $y \in \mathsf{Per}_n^\delta$ we let v_y and w_y be respective representatives in T^1M of the unstable and stable directions at the hyperbolic periodic point y.

1. Reduction to periodic points with uniform angle. For $x \in M$, $\kappa \in \mathbb{Z}$ and a cone C in $T_{f^\kappa x}M$ we denote by $T\exp_{f^\kappa x}C$ the subset of $(y, v) \in T^1M$ with $y = \exp_{f^\kappa x}y'$ and $v = T_{y'}\exp_{f^\kappa x}(v')$ for some $y' \in T_{f^\kappa x}M$ and $v' \in C$.

Claim. *It is enough in item (ii) of Lemma 7.1 to cover by local hyperbolic hexagons the set*

$$E := \left\{z \in \mathsf{Per}_n^\delta \cap B(f^\kappa x, n, \epsilon) \mid (v_z, w_z) \in (T\exp_{f^\kappa x}C_u(1/2), T\exp_{f^\kappa x}C_s(1/2))\right\}$$

for some $0 \leq \kappa < n$ and for some bicones (C_u, C_s) of $T_{f^\kappa x}M \simeq \mathbb{R}^2$ in $\mathfrak{C}_\alpha$, the number α depending only on δ and f.

Indeed there is such an $\alpha = \alpha(\delta, f) > 0$ that for any (y, v, w) with $y \in B(x, n, \epsilon)$ and $v, w \in T_y^1M$ satisfying $\|T_yf^n(v)\| \geq e^{\delta n}$, $\|T_yf^n(w)\| \leq e^{-\delta n}$ the unsigned angle between $T_yf^\kappa(v)$ and $T_yf^\kappa(w)$ is in $]\alpha, \pi - \alpha[$ for some $0 \leq \kappa < n$ (one proves similarly that the angle between the stable and unstable directions is bounded from below by a constant depending only on δ and f on a set of μ-positive measure for any ergodic hyperbolic measure with Lyapunov exponents δ-away from 0). Also $z = f^\kappa y \in B(f^\kappa x, n, \epsilon)$ when x and y are n-periodic points and thus the desired cover is obtained by taking the pull back by f^κ of hexagons covering the above subset E of $\mathsf{Per}_n^\delta \cap B(f^\kappa x, n, \epsilon)$. Finally we note that in the Claim we may choose the

identification $T_{f^\kappa x}M \simeq \mathbb{R}^2$ in such a way C_u is centered at $\{0\} \times \mathbb{R}$.

2. Choice of the local dynamics. Let $m = [n/p]$. We consider the sequence $\mathcal{F} = (F_0, F_1, ..., F_m)$ obtained by letting $F_l \in \mathcal{F}_x^n(\epsilon, p)$ for $l \neq 0, m$ and F_0, F_m corresponding to the connected pieces of the orbit of x (with length less than $2p$) starting and finishing at $f^\kappa x$, i.e. if $\kappa \in [Kp, (K+1)p]$ with $K < m$ then we let

$$F_m = \left(\exp_{f^\kappa x}(\epsilon.)\right)^{-1} \circ f^{\kappa - Kp} \circ \exp_{f^{Kp}x}(\epsilon.) \text{ and}$$

$$F_0 = \left(\exp_{f^{(K+1)p}x}(\epsilon.)\right)^{-1} \circ f^{(K+1)p - \kappa} \circ \exp_{f^\kappa x}(\epsilon.)$$

Whenever $K = m$ we replace $(K+1)p$ by n in the definition of F_0. For simplicity we may assume $\kappa = 0$ and thus $\mathcal{F} = \mathcal{F}_x^n(\epsilon, p)$ in the following without loss of generality.

3. Fixing the growth of the derivative. For $y \in \mathsf{Per}_n^\delta \cap B(x, n, \epsilon)$ we denote by (y', v_y', w_y') the image of (y, v_y, w_y) through the local chart at x given by the inverse of the exponential map at x, that is $y = \exp_x y'$, $v_y = T_{y'}\exp_x(v_y')$ and $w_y = T_{y'}\exp_x(w_y')$ (we recall that we choose ϵ less than the radius of injectivity of M). Later on we consider the following set $\mathcal{E}$ of sequences $(\mathcal{A}, \mathcal{B})$:

$$\mathcal{E} := \{(\mathcal{A}, \mathcal{B}) \in \mathbb{Z}^{m+1} \times \mathbb{Z}^{m+1}, \exists y \in \mathsf{Per}_n^\delta \cap B(x, n, \epsilon) \text{ with } (y', v_y', w_y') \in B_{T\mathcal{F}}(\mathcal{A}, \mathcal{B})\}.$$

We recall that $H : [1, +\infty[\to \mathbb{R}$ is defined for all $t \geq 1$ by

$$H(t) = -\frac{1}{t}\log(\frac{1}{t}) - (1 - \frac{1}{t})\log(1 - \frac{1}{t}).$$

Claim. *There is a constant D depending only on f and p such that*

$$(6) \qquad \sharp\mathcal{E} \leq 100^{[n/p]}\left(e^{(\lambda_n(x, f^p)[n/p] + D)H(p\delta/2)}\right)^2.$$

Let us prove this claim. We consider the sequences $(l_k'(y))_{0 \leq k \leq m}$ and $(l_k''(y))_{0 \leq k \leq m}$ of positive integers defined for all $k = 0, ..., m$ and for all $y \in \mathsf{Per}_n^\delta \cap B(x, n, \epsilon)$ as follows (recall $(v_{y'})_k = \frac{T_y F^k(v_{y'})}{\|T_y F^k(v_{y'})\|} \in T^1 M$ and similarly $(w_{y'})_k = \frac{T_y F^k(w_{y'})}{\|T_y F^k(w_{y'})\|} \in T^1 M$):

$$l_k'(y) = \left[\log\left(\frac{\|T_{F^k y'}F_k\left((w_{y'})_k\right)\|}{m(T_{F^k y'}F_k)}\right)\right] + 1$$

and

$$l_k''(y) = \left[\log\left(\frac{\|T_{F^k y'}F_k\|}{\|T_{F^k y'}F_k\left((v_{y'})_k\right)\|}\right)\right] + 1.$$

Clearly we have

$$\max(l_k'(y), l_k''(y)) \leq \left[\log\left(\frac{\|T_{F^k y'}F_k\|}{m(T_{F^k y'}F_k)}\right)\right] + 1.$$

Then we may choose $\epsilon > 0$ so small compared to $1/\|f^p\|_{\min(2,r)}$ that for any $y \in B(x, n, \epsilon)$ we have

$$\log\left(\frac{\|T_{F^k y'} F_k\|}{m(T_{F^k y'} F_k)}\right) \leq \log\left(\frac{\|T_0 F_k\|}{m(T_0 F_k)}\right) + 1.$$

Since x is a saddle n-periodic point with Lyapunov exponents δ-away from zero we have

$$\sum_{k=0}^{m} \log\left(\frac{\|T_0 F_k\|}{m(T_0 F_k)}\right) \geq n\delta.$$

Observe also that

$$\left|\sum_{k=0}^{m} \log\left(\frac{\|T_0 F_k\|}{m(T_0 F_k)}\right) - m\lambda_n(x, f^p)\right| < 5 \sup_{z \in M, \ 0 \leq k \leq 2p} \log\left(\frac{\|T_z f^k\|}{m(T_z f^k)}\right).$$

We will use the following combinatorial fact [12]:

Fact. *For any real number $\sigma > 0$ the number of sequences of positive integers $(k_0, ..., k_m)$ with $\sum_{i=0}^{m} k_i \leq (m+1)\sigma$ is less than $e^{(m+1)\sigma H(\sigma)}$.*

When applying the above Fact to the sequences $(l'_k(y))_{0 \leq k \leq m}$ and $(l''_k(y))_{0 \leq k \leq m}$ with

$$(m+1)\sigma = \sum_{k=0}^{m} \left(\log\left(\frac{\|T_0 F_k\|}{m(T_0 F_k)}\right) + 1\right)$$

we obtain that the number of such sequences of positive integers satisfy for some constant D depending only on f and p:

$$\sharp\{(l'_k(y))_{0 \leq k \leq m}, (l''_k(y))_{0 \leq k \leq m})\} \quad \leq \quad e^{2(\lambda_n(x, f^p)m + D)H(p\delta/2)}.$$

Indeed with $D = 5 \sup_{z \in M, \ 0 \leq k \leq 2p} \log\left(\frac{\|T_z f^k\|}{m(T_z f^k)}\right)$ we have :

$$(m+1)\sigma \leq m\lambda_n(x, f^p) + D \text{ and}$$

$$\sigma \geq \frac{n\delta}{m+1} \geq \frac{p\delta}{2} >> 1.$$

This implies the inequality (6) of the claim as for $\epsilon > 0$ small enough we have also

$$\left|l'_k(y) - \left[\log T_{F^k y'} F_k \left((w_{y'})_k\right)\right] + \log m(T_0 F_k)\right| < 3,$$

$$\left|l''_k(y) - \left[\log \frac{1}{\|T_{F^k y'} F_k \left((v_{y'})_k\right)\|}\right] - \log \|T_0 F_k\|\right| < 3.$$

4. The n-hyperbolic structure. To obtain the combinatorial Inequality (6) we just needed that x belongs to Per_n^δ (in the definition of $\mathcal{E}$ we could have considered general triples (y', v', w') with $y \in B(x, n, \epsilon)$ and $v', w' \in T_y^1 M$).

We fix now sequences $\mathcal{A} = (a_0, ..., a_k, ...a_m)$ and $\mathcal{B} = (b_0, ..., b_k, ...b_m)$ with $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}$, in particular we have

$$\sum_k a_k \geq n\delta \text{ and } \sum_k b_k \geq n\delta.$$

Here we have used the fact that the sequences $\mathcal{A}$ and $\mathcal{B}$ are associated to a triple (y, v_y, w_y) with $y \in \mathsf{Per}_n^\delta$ and v_y, w_y being the usual unstable and stable spaces of the hyperbolic point y. The maps $(F_k)_k$ in $\mathcal{F}$ are defined on the tangent spaces along the orbit of x. By a change of fixed isometric charts, we may assume $F_k : B \subset \mathbb{R}^2 \to \mathbb{R}^2$ (we did not change the notations for simplicity). Then we apply Lemma 7.2 to $\mathcal{F}$ with $2s$ instead of s. Let $\psi = (\psi_1, \psi_2, \psi_3) \in \mathcal{I}_m$ such that there is $z \in \mathsf{Per}_n^\delta$ with $(v_z, w_z) \in (T\mathsf{exp}_x C_u(1/2), T\mathsf{exp}_x C_s(1/2))$ and (z', v_z', w_z') in the image of ψ. Let $(x_z, t_z, s_z) \in [0, 1]^2 \times [0, 1] \times [0, 1]$ be such that $(z', v_z', w_z') = \psi(x_z, t_z, s_z)$.

Claim. *The sets* $\psi_1(]0, 1[^2)$ *are* $(m + 1)$-*hyperbolic sets for* $\mathcal{F}$.

Indeed the vector fields $(\psi_1(x), \psi_2(x, t_z))_{x \in \psi_1(]0,1[^2)}$ and $(\psi_1(x), \psi_3(x, s_z))_{x \in \psi_1(]0,1[^2)}$ defines an hyperbolic structure on $\psi_1(]0, 1[^2)$ by letting $e_s(\psi_1(x)) = \psi_2(x, t_z)$ and $e_u(\psi_1(x)) = \psi_3(x, s_z)$, because for any $x \in]0, 1[^2$ we have $\psi(x) \in B_{T\mathcal{F}}^+(\mathcal{A}, \mathcal{B})$ by item (ii) of Lemma 7.2 and thus :

- $\log \|T_{\psi_1(x)} F^{m+1}(\psi_2(x, t_z))\| \geq \sum_{k=0}^m (a_k - 1) \gtrsim mp\delta$,
- $-\log \|T_{\psi_1(x)} F^{m+1}((x, \psi_3(x, s_z))\| \geq \sum_{k=0}^m (b_k - 1) \gtrsim mp\delta$.

Moreover since the angle between $T_{\psi_1(x)} F^{m+1}(\psi_2(x, v_z))$ and $T_{\psi_1(y)} F^{m+1}(\psi_2(y, v_z))$, respectively between $T_{\psi_1(x)} F^{m+1}(\psi_3(x, w_z))$ and $T_{\psi_1(y)} F^{m+1}(\psi_3(y, w_z))$, is less than $\alpha/2$ for any $y \in]0, 1[^2$. But as pointed out in Remark 5.2 it is not enough to control the aperture of the two cones given by the images of e_s and e_u under TF^{m+1}. We need also to check that these images lie in C_s and C_u respectively. As z is in Per_n^δ the vectors $\psi_2(x_z, t_z)$ and $\psi_3(x_z, s_z)$ are the usual stable and unstable (Oseledets) directions of the saddle hyperbolic periodic point z. Therefore they are invariant under $T_{\psi_1(x_z)} F^{m+1}$. Since we have $(v_z, w_z) \in (T\mathsf{exp}_x C_u(1/2), T\mathsf{exp}_x C_s(1/2))$ we get finally for any $x \in]0, 1[^2$:

- $\psi_2(x, v_z)$ and $T_{\psi_1(x)} F^{m+1}(\psi_2(x, v_z))$ lie in C_s,
- $\psi_3(x, v_z)$ and $T_{\psi_1(x)} F^{m+1}((x, \psi_3(x, w_z))$ lie in C_u.

As already mentioned (see the discussion below Definition 5.1) we may assume the n-expanding field e_u to be constant equal to the oriented center $(0, 1)$ of the cone C_u.

5. Cover by hexagons. For any Nash map $\psi = (\psi_1, \psi_2, \psi_3)$ as above, the set $\psi_1(]0, 1[^2)$ may be covered by a collection of generalized $(m + 1)$-hyperbolic hexagons for $\mathcal{F}$ with cardinality less than $P(\mathsf{deg}(\psi))$ for some universal polynomial P by Lemma 6.2. According to the upper bound on the degree of ψ given by item (iv) of Lemma 7.2 we have for some universal constant A:

Any $\psi_1(]0,1[^2)$ may be covered by a collection $\mathcal{C}(\psi)$ of generalized $(m+1)$-hyperbolic

(7) $\qquad\qquad$ *hexagons with $\sharp\mathcal{C}(\psi) \leq A^{m+1}$.*

6. Bounding the cardinality of the cover.

By letting $\epsilon > 0$ small enough when defining the local dynamics $\mathcal{F}_x^n(\epsilon,p)$ we may assume for all k that:

$$\|F_k\|_r \simeq \|T_0 F_k\| \text{ since we have } F_k(0) = 0$$

and then

$$\max\left(\frac{\|F_k\|_r}{e^{a_k}}, \frac{\|F_k\|_r}{e^{-b_k}}\right) \lesssim \frac{\|T_0 F_k\|}{m(T_0 F_k)},$$

so that by Lemma 7.2 (iii) we get (the constants A and B may change at each of the following steps):

$$\sharp\mathcal{I}_m(\mathcal{A},\mathcal{B}) \leq BA^m \sup_{(\mathcal{A},\mathcal{B})} \prod_{0 \leq k \leq m} \max\left(1, \|F_k\|_r^{\frac{1}{r}}, \left(\frac{\|F_k\|_r}{e^{a_k}}\right)^{\frac{1}{r-1}}, \left(\frac{\|F_k\|_r}{e^{-b_k}}\right)^{\frac{1}{r-1}}\right)^2$$

$$\times \max\left(1, \left(\frac{\|F_k\|_r}{e^{a_k}}\right)^{\frac{1}{2s}}\right) \times \max\left(1, \left(\frac{\|F_k\|_r}{e^{-b_k}}\right)^{\frac{1}{2s}}\right),$$

$$\leq BA^m \prod_{0 \leq k < m} \max\left(1, \|T_0 F_k\|^{\frac{1}{r}}, \left(\frac{\|T_0 F_k\|}{m(T_0 F_k)}\right)^{\frac{1}{r-1}}, \left(\frac{\|T_0 F_k\|}{m(T_0 F_k)}\right)^{\frac{1}{r-1}}\right)^2,$$

$$\times \max\left(1, \left(\frac{\|T_0 F_k\|}{m(T_0 F_k)}\right)^{\frac{1}{s}}\right).$$

Lastly observe $[n/p]\lambda_n^+(x, f^p) \simeq \sum_k \log^+ \|T_0 F_k\|$ and similarly $[n/p]\lambda_n(x, f^p) \simeq \sum_k \log\left(\frac{\|T_0 F_k\|}{m(T_0 F_k)}\right)$. Therefore we obtain

(8) $\qquad \sharp\mathcal{I}_m(\mathcal{A},\mathcal{B}) \leq BA^m e^{[n/p]\frac{2}{r}\lambda_n^+(x,f^p) + 2[n/p]\left(\frac{1}{r-1} + \frac{1}{2s}\right)\lambda_n(x,f^p)}.$

7. Conclusion.

If H is a generalized $(m+1)$-hyperbolic hexagon for $\mathcal{F}$ then $\exp_x H$ is a local generalized n-hyperbolic hexagon at x for the corresponding n-hyperbolic structure. Thus the collection $\mathcal{J}_n(x)$ of local generalized n-hyperbolic hexagons given by

$$\mathcal{J}_n(x) := \bigcup_{(\mathcal{A},\mathcal{B})\in\mathcal{E}} \left(\bigcup_{\psi\in\mathcal{I}_m(\mathcal{A},\mathcal{B})} \exp_x \mathcal{C}(\psi)\right)$$

satisfies the conclusion Lemma 7.1. Indeed one checks easily that

$$\mathsf{Per}_n^\delta \cap B(x,n,\epsilon) \subset \bigcup_{J\in\mathcal{J}_n(x)} J.$$

Moreover by combining Inequalities (6), (7) and (8) and by choosing p so large that $H(p\delta) < 1/s$ we get with A depending only on r, s and B on f, δ and p:

$$\log \sharp \mathcal{J}_n(x) \leq [n/p] \frac{2}{r} \lambda_n^+(x, f^p) + 2[n/p] \left(\frac{1}{r-1} + \frac{1}{s} \right) \lambda_n(x, f^p) + [n/p]A + B.$$

This completes the proof of Lemma 7.1.

Remark 7.1. *The cone C_u being fixed centered at the y-axis it would have been equivalent to reparametrize the dynamical ball*

$$\bigcap_{0 \leq k \leq m} \left\{ (x, u) \quad \in B \times \mathbb{T} \mid \quad F^k x \in B(0, 1), \|T_{F^k x} F_k(u_k)\| \in [e^{a_k}, e^{a_k+1}], \right.$$

$$\left. \|T_{F^k x} F_k(v_k)\| \in [e^{-b_k-1}, e^{-b_k}\| \right\},$$

with $v = (0, 1)$ in Lemma 7.2. Even if it involves four dimensional maps we prefer the general version given in Lemma 7.2.

7.4. **Proof of the reparametrization Lemma.** We prove now the reparametrization lemma (Lemma 7.2) for the tangent map stated in the previous subsection. For simplicity we assume $r \in \mathbb{N} \setminus \{0, 1\}$. The general case follows the same lines.

7.4.1. *An adapted space of smooth functions.* For any integer $s > r$ we let $C_{\mathfrak{E}}^{r,s}(]0, 1[^2 \times]0, 1[\times]0, 1[)$ (resp. $D_{\mathfrak{E}}^{r,s}(]0, 1[^2 \times]0, 1[\times]0, 1[))$ be the set of functions from $]0, 1[^2 \times]0, 1[\times]0, 1[$ to $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ (resp. $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$) of the form $\psi(x, v, w) = (\psi_1(x), \psi_2(x, v), \psi_3(x, w))$ for $(x, v, w) \in]0, 1[^2 \times]0, 1[\times]0, 1[$ where:

- the map ψ_1 admits bounded continuous partial derivatives $\partial^\gamma \psi_i$ with $\gamma \in \mathbb{N}^2$ satisfying $|\gamma| \leq r$,
- for $i \in \{2, 3\}$ the maps ψ_i admit bounded continuous partial derivatives $\partial^\gamma \psi_i$ with $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2 \times \mathbb{N}$ satisfying $|\gamma_1| \leq r - 1$ and $|\gamma_1| + \gamma_2 \leq s$.

In particular the map ψ_1 (resp. ψ_2, ψ_3) is C^r (resp. C^{r-1}). Moreover the maps ψ_2, ψ_3 and their derivatives w.r.t. x of order less than r are C^{s-r+1} w.r.t. v and w. In fact we will consider in the following only such maps ψ where ψ_2 and ψ_3 and their derivative w.r.t. x are real analytic w.r.t. v and w. In other terms we can take s as large as we want.

We endow these vector spaces with the following norm $\| \cdot \|_{r,s}$. For $1 \leq k \leq r$ and $0 \leq l \leq s$ we first let $p_{k,l}(\psi)$ be the maximum of the supremum norms of $\partial_x^\nu \psi_1$, $\partial_x^{\alpha_1} \partial_v^{\alpha_2} \psi_2$ and $\partial_x^{\beta_1} \partial_w^{\beta_2} \psi_3$ over all $\nu, \alpha_1, \beta_1 \in \mathbb{N}^2$, $\alpha_2, \beta_2 \in \mathbb{N}$ with $|\nu| = k$, $|\alpha_1| \leq k - 1$, $|\alpha_1| + \alpha_2 = l$, $|\beta_1| \leq k - 1$ and $|\beta_1| + \beta_2 = l$. When $k = 0$ we let $p_{0,l}(\psi)$ be the supremum norm of ψ_1 (for any l). Then we let $\|\psi\|_{r,s}$ be the norm defined as:

$$\|\psi\|_{r,s} := \max_{0 \leq k \leq r, 0 \leq l \leq s} p_{k,l}(\psi).$$

For a C^s map ψ we recall the C^s norm $\|\psi\|_s$ is given by the maximum over $\nu \in \mathbb{N}^4$ with $|\nu| \leq s$ of the supremum norms of $\partial^\nu \psi$. In particular when a map ψ in $C^{r,s}_{\mathfrak{E}}(]0,1[^2 \times]0,1[\times]0,1[)$ is C^s we have $\|\psi\|_{r,s} \leq \|\psi\|_s$.

7.4.2. *Proof of Lemma 7.2 by induction on the length of $\mathcal{F}$.* We fix $\mathcal{F}$, $\mathcal{A}$, $\mathcal{B}$, $\alpha > 0$ and $s > r$ as in the statement of the lemma. We consider the (nonautonomous) cocycle $\mathcal{G}$ over $\mathcal{F}$ given by the normalized action of $\mathcal{F}$ on the unit tangent bundle. More precisely we let $\mathcal{G} = (G_k)_{0 \leq k \leq m}$ be the following sequence of functions of $\mathfrak{E}$: for all $(x, v, w) \in \mathfrak{E}$,

$$G_k(x,v,w) = (F_k x, v_k, w_k) = \left(F_k(x), \frac{T_x F_k(v)}{\|T_x F_k(v)\|}, \frac{T_x F_k(w)}{\|T_x F_k(w)\|} \right).$$

Also we define the iterated sequence $(G^k)_{0 \leq k \leq m+1}$ inductively by

$$G^0 = \mathsf{Identity}$$

and for $0 \leq k \leq m$ and for all $(x,v,w) \in \mathfrak{E}$ with $x \in \bigcap_{l=0}^{k} F^{-l} B$,

$$G^{k+1}(x,v,w) = G_k \circ G^k(x,v,w).$$

By an induction on the length m of the sequence $\mathcal{F}$ we will prove the following claim.

Claim. *For any $m \in \mathbb{N} \cup \{-1\}$ and for any sequences $\mathcal{A}$ and $\mathcal{B}$ of $m+1$ integers, there is a family of Nash maps $\Psi_m = \{\psi^m\}$ with $\psi^m :]0,1[^2 \times]0,1[\times]0,1[\to B \times \mathbb{T} \times \mathbb{T}$ and a universal constant A' such that:*

(1) $\psi^m(x,v,w) = (\psi_1^m(x), \psi_2^m(x,v), \psi_3^m(x,w))$,

(2) for any $0 \leq k \leq m+1$, the map $G^k \circ \psi^m$ belongs to $C^{r,s}_{\mathfrak{E}}(]0,1[^2 \times]0,1[\times]0,1[)$ and

$$\|G^k \circ \psi^m\|_{r,s} \leq 1,$$

(3)

$$B_{T\mathcal{F}}(\mathcal{A}, \mathcal{B}) \subset \bigcup_{\psi^m \in \Psi_m} \psi^m([0,1]^2 \times [0,1] \times [0,1]) \subset B_{T\mathcal{F}}^+(\mathcal{A}, \mathcal{B}),$$

(4)

$$\begin{aligned}
\sharp \Psi_m \ \leq A' \ &\max\left(1, \|F_m\|^{\frac{1}{r}}_r, \left(\frac{\|F_m\|_r}{e^{a_{m+1}}} \right)^{\frac{1}{r-1}}, \left(\frac{\|F_m\|_r}{e^{-b_{m+1}}} \right)^{\frac{1}{r-1}} \right)^2 \\
&\times \max\left(1, \left(\frac{\|F_m\|_r}{e^{a_{m+1}}} \right)^{\frac{1}{s}} \right) \times \max\left(1, \left(\frac{\|F_m\|_r}{e^{-b_{m+1}}} \right)^{\frac{1}{s}} \right) \\
&\times \sharp \Psi_{m-1}.
\end{aligned}$$

(5) for any map $\psi^m \in \Psi_m$ there is $\psi^{m-1} \in \Psi_{m-1}$ and a Nash map $\phi_m :$ $]0,1[^2 \times]0,1[\times]0,1[\to]0,1[^2 \times]0,1[\times]0,1[$ with $\deg(\phi_m) \leq A'$ such that $\psi^m = \psi^{m-1} \circ \phi_m$. Moreover for any $\psi^{-1} \in \Psi_{-1}$ we have $\deg(\psi^{-1}) \leq A'$.

The above claim implies easily Lemma 7.2 as follows. We consider a fixed cover $\mathcal{U}_\alpha$ of $]0,1[^4=]0,1[^2\times]0,1[\times]0,1[$ by subcubes with diameter less than α/π. For any such subcube $S \in \mathcal{U}_\alpha$ we let $\eta_S :]0,1[^4 \to S$ be the affine (homothetic) reparametrization of S. Then the map $G^k \circ \psi^m$ being 1-Lipschitz for any $0 \le k \le m+1$ the image of $G^k \circ \psi^m \circ \eta_S$ is contained in a subcube with diameter less than α/π. But for any $u, v \in \mathbb{T}$ we have

$$
\begin{aligned}
\|u-v\| &= 2\left|\sin\left(\frac{\angle(u,v)}{2}\right)\right|, \\
&\ge 2\frac{|\angle(u,v)|}{\pi}.
\end{aligned}
$$

so that with $\psi = (\psi_1, \psi_2, \psi_3) = \psi^m \circ \eta_S$ the sets

$$
\bigcup_{(x,v)\in[0,1]^2\times[0,1]} T_{\psi_1(x)}F^k(\psi_2(x,v)) \text{ and } \bigcup_{(x,w)\in[0,1]^2\times[0,1]} T_{\psi_1(x)}F^k(\psi_3(x,w))
$$

are contained in cones of $\mathbb{R}^2$ with aperture less then $\alpha/2$. Together with item (3) of the Claim we get item (ii) of Lemma 7.2 by letting $\mathcal{I}_m := \{\psi^m \circ \eta_S, \ \psi^m \in \Psi_m \text{ and } S \in \mathcal{U}_\alpha\}$. Finally :

- for all nonnegative integers m we have

$$
\begin{aligned}
\sharp\mathcal{I}_m &\le \sharp\mathcal{U}_\alpha \times \sharp\Psi_m, \\
&\le \sharp\mathcal{U}_\alpha \times \sharp\Psi_{m-1} \times A' \max\left(1, \|F_m\|_r^{\frac{1}{r}}, \left(\frac{\|F_m\|_r}{e^{a_{m+1}}}\right)^{\frac{1}{r-1}}, \left(\frac{\|F_m\|_r}{e^{-b_{m+1}}}\right)^{\frac{1}{r-1}}\right)^2 \\
&\quad \times \max\left(1, \left(\frac{\|F_m\|_r}{e^{a_{m+1}}}\right)^{\frac{1}{s}}\right) \times \max\left(1, \left(\frac{\|F_m\|_r}{e^{-b_{m+1}}}\right)^{\frac{1}{s}}\right) \\
&\le \sharp\mathcal{U}_\alpha \times \sharp\Psi_{-1} \times A'^{m+1} \prod_{0\le k\le m} \max\left(1, \|F_k\|_r^{\frac{1}{r}}, \left(\frac{\|F_k\|_r}{e^{a_k}}\right)^{\frac{1}{r-1}}, \left(\frac{\|F_k\|_r}{e^{-b_k}}\right)^{\frac{1}{r-1}}\right)^2, \\
&\quad \times \max\left(1, \left(\frac{\|F_k\|_r}{e^{a_k}}\right)^{\frac{1}{s}}\right) \times \max\left(1, \left(\frac{\|F_k\|_r}{e^{-b_k}}\right)^{\frac{1}{s}}\right).
\end{aligned}
$$

so that we obtain the estimate (iii) on the number of reparametrization maps in Lemma 7.2 by letting $B = \sharp\mathcal{U}_\alpha \times \sharp\Psi_{-1}$ and $A \ge A'$,
- for all $\psi = \psi^m \circ \eta_S \in \mathcal{I}_m$ there are $\psi_{m-1} \in \Psi_{m-1}$ and a Nash map ϕ_m with $\deg(\phi_m) \le A'$ (where A' is a universal constant) and $\psi^m = \psi^{m-1} \circ \phi_m$ so that:

$$
\begin{aligned}
\deg(\psi) &= \deg(\psi^m \circ \eta_S), \\
&= \deg(\psi^m), \\
&= \deg(\psi^{m-1} \circ \phi_m),
\end{aligned}
$$

then by Corollary 6.1:

$$
\begin{aligned}
\deg(\psi) \;&\le\; \deg(\phi_m)^4 \times \deg(\psi^{m-1}), \\
&\le\; A'^4 \deg(\psi^{m-1}), \\
&\le\; A'^{4(m+1)} \times \max_{\psi^0 \in \Psi_{-1}} \deg(\psi^{-1}), \\
&\le\; A'^{4m+5}.
\end{aligned}
$$

so that we get the upper bound $\deg(\psi) \le A^m$ in Lemma 7.2 (iv) on the degree of the reparametrization maps by letting $A \ge A'^5$.

This concludes the proof of Lemma 7.2.

7.4.3. *Proof of the induction.* Now we go back to the proof of the claim which follows from Yomdin's approach : we first renormalized the reparametrization maps to kill the derivatives of order r and then we apply Yomdin-Gromov Lemma to the Taylor-Lagrange interpolating polynomials of degree $r-1$ to conclude.

Let $\mathcal{A} = (a_k)_{k \in \mathbb{N}}$ and $\mathcal{B} = (b_k)_{k \in \mathbb{N}}$ be two sequences of integers. By arguing inductively on m we build the family of Nash maps Ψ_m for the sequences $\mathcal{A}^m$ and $\mathcal{B}^m$ given respectively by the $m+1$-first terms of $\mathcal{A}$ and $\mathcal{B}$.

1. The base case of the induction. We let $\mathfrak{A} = (\zeta_1, ..., \zeta_p)$ be a Nash atlas of $\mathbb{T}$, i.e. the maps ζ_l, $1 \le l \le p$, are Nash diffeomorphisms from $]-1,1[$ to $\mathbb{T}$ with $\bigcup_{l=1,...,p} \zeta_l(]-1,1[) = \mathbb{T}$.

For the family $\Psi_{-1} = \{\psi^{-1}\}$ it is enough to let $\psi^{-1} = (\psi_1^{-1}, \psi_2^{-1}, \psi_3^{-1})$ be such that ψ_1^{-1} are homotheties covering the unit ball B whereas $\psi_2^{-1}(x,v)$ and $\psi_3^{-1}(x,w)$ do not depend on x and are equal to $\zeta_i(v)$ and $\zeta_j(w)$ for $1 \le i,j \le p$.

We assume now the collection Ψ_m already built for some integer $m \in \mathbb{N} \cup \{-1\}$.

2. Killing the highest derivatives by homothetic renormalizations. The tangent map TF_{m+1} acts naturally on $\mathfrak{E}$ by letting for any $(x,v,w) \in \mathfrak{E}$:

$$
TF_{m+1}(x,v,w) = (F_{m+1}x, T_x F_{m+1}v, T_x F_{m+1}w).
$$

As the map $\rho := G^{m+1} \circ \psi^m$ belongs to $C_{\mathfrak{E}}^{r,s}(]0,1[^2 \times]0,1[\times]0,1[)$ it has the form $\rho(x,v,w) = (\rho_1(x), \rho_2(x,v), \rho_3(x,w))$ for $(x,v,w) \in]0,1[^2 \times]0,1[\times]0,1[$. In this case the Faa di Bruno formula the derivatives of the composition $TF_{m+1} \circ G^{m+1} \circ \psi^m$ gives for some universal polynomial R depending only on r and s:

$$
\|TF_{m+1} \circ G^{m+1} \circ \psi^m\|_{r,s} \le \|F_{m+1}\|_r R\left(\|G^{m+1} \circ \psi^m\|_{r,s}\right).
$$

By induction hypothesis we have $\|G^{m+1} \circ \psi^m\|_{r,s} \le 1$. Thus for some constant $C = C(r,s)$ we get

$$
(9) \qquad \|TF_{m+1} \circ G^{m+1} \circ \psi^m\|_{r,s} \;\le\; C\|F_{m+1}\|_r.
$$

We cover the unit square $]0,1[^2$ into isometric open subsquares $\{\mathcal{S}\}$ with diameter less than

$$\min\left(\left(\frac{1}{C\|F_{m+1}\|_r}\right)^{\frac{1}{r}}, \left(\frac{e^{a_{m+1}}}{C\|F_{m+1}\|_r}\right)^{\frac{1}{r-1}}, \left(\frac{e^{-b_{m+1}}}{C\|F_{m+1}\|_r}\right)^{\frac{1}{r-1}}\right).$$

Similarly we cover the second and third factor $]0,1[$ by isometric open subintervals $\{\mathcal{R}\}$ and $\{\mathcal{T}\}$ with diameter respectively less than

$$\left(\frac{e^{a_{m+1}}}{C\|F_{m+1}\|_r}\right)^{\frac{1}{s}} \text{ and } \left(\frac{e^{-b_{m+1}}}{C\|F_{m+1}\|_r}\right)^{\frac{1}{s}}.$$

For any $\psi = (\psi_1, \psi_2, \psi_3) \in C^{r,s}_{\mathfrak{E}}(]0,1[^2\times]0,1[\times]0,1[)$ and $(u,v,w) \in (\mathbb{R}^+)^3$ we denote by $(u,v,w)\cdot\psi$ the function $(u\psi_1, v\psi_2, w\psi_3)$. Then for any $\mathcal{S}, \mathcal{R}, \mathcal{T}$ we let $\phi^{m+1} = (\phi_1^{m+1}, \phi_2^{m+1}, \phi_3^{m+1})$ be the coordinatewise affine (homothetic) reparametrization $\phi^{m+1} :]0,1[^2\times]0,1[\times]0,1[\to \mathcal{S} \times \mathcal{R} \times \mathcal{T}$ so that

$$
\begin{aligned}
I &:= p_{r,s}\left((1, e^{-a_{m+1}}, e^{b_{m+1}}) \cdot (TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1})\right), \\
&\leq p_{r,s}\left((\mathfrak{a}_m, \mathfrak{b}_m, \mathfrak{c}_m) \cdot (TF_{m+1} \circ G^{m+1} \circ \psi^m)\right),
\end{aligned}
$$

with $\mathfrak{a}_m, \mathfrak{b}_m, \mathfrak{c}_m$ defined as follows

- $\mathfrak{a}_m = \frac{1}{C\|F_{m+1}\|_r}$,
- $\mathfrak{b}_m = \max_{\alpha,\beta} \frac{(\min(1, \mathfrak{a}_m e^{a_{m+1}}))^{\beta/s+|\alpha|/(r-1)}}{e^{a_{m+1}}}$,

- $\mathfrak{c}_m = \max_{\alpha,\beta} \frac{\left(\min(1, \mathfrak{a}_m e^{-b_{m+1}})\right)^{\beta/s+|\alpha|/(r-1)}}{e^{-b_{m+1}}}$,

where the maxima holds over $(\alpha, \beta) \in \mathbb{N}^2 \times \mathbb{N}$ with $|\alpha| \leq r-1$ and $|\alpha|+\beta = s$. For such pairs (α, β) we have $\frac{\beta}{s} + \frac{|\alpha|}{r-1} \geq \frac{\beta+|\alpha|}{s} \geq 1$. Thus we get

$$\mathfrak{b}_m \leq \mathfrak{a}_m \text{ and } \mathfrak{c}_m \leq \mathfrak{a}_m,$$

and therefore by Inequality (9):

$$I \leq p_{r,s}\left(\frac{TF_{m+1} \circ G^{m+1} \circ \psi^m}{C\|F_{m+1}\|_r}\right) \leq 1.$$

3. Taylor-Lagrange approximation. In our setting the multivariate Taylor-Lagrange Inequality may be written as follows for $\psi = (\psi_1, \psi_2, \psi_3) \in C^{r,s}_{\mathfrak{E}}(]0,1[^2\times]0,1[\times]0,1[)$ at $(x_0, v_0, w_0) \in]0,1[^2\times]0,1[\times]0,1[$:

$$\left\|\psi(x,v,w) - \sum_{\nu\in\mathbb{N}^2,\ |\nu|=r-1} \frac{(x-x_0)^\nu}{\nu!}(\partial^\nu\psi_1(x_0), 0, 0)\right.$$

$$\left.- \sum_{(\alpha,\beta,\gamma)} \frac{(x-x_0)^\alpha(v-v_0)^\beta(w-w_0)^\gamma}{\alpha!\beta!\gamma!}\partial_x^\alpha\partial_v^\beta\partial_w^\gamma\psi(x_0, v_0, w_0)\right\|_{r,s} \leq C'p_{r,s}(\psi),$$

where the last sum holds over all $(\alpha, \beta, \gamma) \in \mathbb{N}^2 \times \mathbb{N} \times \mathbb{N}$ with $|\alpha| < r-1$, $|\alpha|+\beta < s$ and $|\alpha|+\gamma < s$ and where C' depends only on r and s (for $r \notin \mathbb{N}$

we should here consider the Taylor-Lagrange polynomial of degree $[r]$). One may change the constant C in the previous paragraph 2., so that we have

$$I = p_{r,s}\left((1, e^{-a_{m+1}}, e^{b_{m+1}}) \cdot (TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1})\right) \leq 1/C'.$$

Thus there is a polynomial $P = (P_1, P_2, P_3) : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ with $P(x, v, w) = (P_1(x), P_2(x, v), P_3(x, w))$ for all $(x, v, w) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ and with $\deg_t(P_1) < r$ and $\deg_t(P_2), \deg_t(P_3) < s$ such that

$$II := \|P - (1, e^{-a_{m+1}}, e^{b_{m+1}}) \cdot \left(TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1}\right)\|_{r,s} \leq 1.$$

4. Applying Yomdin-Gromov Lemma. By applying the Yomdin-Gromov algebraic lemma to P for the smoothness parameter s (see Remark 7.2 below), we get Nash maps $\theta :]0, 1[^2 \times]0, 1[\times]0, 1[\to]0, 1[^2 \times]0, 1[\times]0, 1[$ reparametrizing the set

$$P^{-1}\left(B(0, 2) \times B(0, 1 + e) \times B(0, 2)\right)$$
$$\bigcup$$
$$\left(TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1}\right)^{-1}\left(B(0, 1) \times B(0, e^{a_{m+1}+1}) \times B(0, e^{-b_{m+1}})\right)$$
$$\bigcup$$
$$\left(\psi^m \circ \phi^{m+1}\right)^{-1}\left(B_{T\mathcal{F}}(\mathcal{A}, \mathcal{B})\right).$$

Moreover the maps θ have the form $\theta(t, s, u) = (\theta_1(t), \theta_2(t, s), \theta_3(t, u))$ for all (t, s, u) where the degree of θ is bounded by some universal constant $A = A(s)$. Also we have $\theta \in D_{\mathfrak{E}}^{r,s}(]0, 1[^2 \times]0, 1[\times]0, 1[)$ and $P \circ \theta \in C_{\mathfrak{E}}^{r,s}(]0, 1[^2 \times]0, 1[\times]0, 1[)$ with

$$\|\theta\|_{r,s} \leq \|\theta\|_s \leq 1 \text{ and } \|P \circ \theta\|_{r,s} \leq \|P \circ \theta\|_s \leq 1.$$

By applying again Faa di Bruno formula we get for some universal polynomial R depending only on r and s:

$$\begin{aligned} III &:= \|(1, e^{-a_{m+1}}, e^{b_{m+1}}) \cdot \left(TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1} \circ \theta\right)\|_{r,s}, \\ &\leq \|\left(P - (1, e^{-a_{m+1}}, e^{b_{m+1}}) \cdot \left(TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1}\right)\right) \circ \theta\|_{r,s} + \|P \circ \theta\|_{r,s}, \\ &\leq R\left(II, \|\theta\|_{r,s}\right) + \|P \circ \theta\|_{r,s}. \end{aligned}$$

Therefore there is a universal constant $C = C(r, s)$ such that

$$(10) \quad III = \|(1, e^{-a_{m+1}}, e^{b_{m+1}}) \cdot \left(TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1} \circ \theta\right)\|_{r,s} \leq C.$$

By composing with D/C-homotheties with $D = 1 - 1/e$ (we already used this trick, e.g. to prove Lemma 7.2 from the Claim) we may assume that the ith component, for $i = 2$ and 3, of $TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1}$ are respectively $De^{a_{m+1}}$- and $De^{-b_{m+1}}$-Lipschitz. In particular as $D = 1 - 1/e < 1 < e - 1$

the image of $\psi^m \circ \phi^{m+1} \circ \theta$ is contained in $B_{T\mathcal{F}}^+(\mathcal{A}^{m+1}, \mathcal{B}^{m+1})$ and

$$\text{(11)} \qquad \| \left(TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1} \circ \theta \right)_i \|_0 \leq$$

$$2 \inf_{(t,s)} \| \left(TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1} \circ \theta \right)_i (t,s)\|.$$

Moreover these components satisfy also for $\alpha \in \mathbb{N}^2$ and $\beta \in \mathbb{N}$ with $|\alpha| \leq r-1$ and $0 \neq |\alpha| + \beta \leq s$:

$$\text{(12)} \qquad \sup_{(t,s)} \|\partial^{(\alpha,\beta)} \left(TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1} \circ \theta \right)_i (t,s)\| \leq$$

$$\inf_{(t,s)} \| \left(TF_{m+1} \circ G^{m+1} \circ \psi^m \circ \phi^{m+1} \circ \theta \right)_i (t,s)\|.$$

5. Control of the $\|\cdot\|_{r,s}$-norm of the normalized action. We may also control the derivatives of the normalized action according to the following fact:

Fact. *There is a homogeneous universal polynomial $R \in \mathbb{R}[X_0, ..., X_r]$, such that for any non vanishing C^r smooth map u we have for any α with $|\alpha| = r$:*

$$\sup_x \|\partial^\alpha (u/\|u\|)(x)\| \leq \frac{R(\|u\|_0, \|u\|_1, ..., \|u\|_r)}{\inf_x \|u(x)\|^{\deg_t(R)}}.$$

Together with Equation (11) and Equation (12) we indeed obtain for another constant $C = C(r,s)$:

$$\|G^{m+2} \circ \psi^m \circ \phi^{m+1} \circ \theta\|_{r,s} \leq C.$$

By composing again by $1/C$-homotheties we may assume $C = 1$. Then we let Ψ_{m+1} be the collection of maps $\Psi_{m+1} = \{\psi^m \circ \phi^{m+1} \circ \theta\}$. One checks easily that this family satisfies the properties of the Claim. This concludes the proof of the Reparametrization Lemma 7.2.

Remark 7.2. *In the above proof we apply Yomdin-Gromov Lemma to a 4-dimensional Nash map $f :]0,1[^2 \times]0,1[\times]0,1[\to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ of the form*

$$f(x,v,w) = (f_1(x), f_2(x,v), f_3(x,w)).$$

This follows from the version stated in Subsection 6.2. Indeed we may first apply this last version to the map $(x,v) \mapsto (f_1(x), f_2(x,v))$. Then if we let $\phi :]0,1[^2 \times]0,1[\to]0,1[^2 \times]0,1[$ with $\phi(x,v) = (\phi_1(x), \phi_2(x,v))$ be the triangular Nash reparametrization maps we apply again Theorem 6.1 to each map $(x,w) \mapsto (f_1 \circ \phi_1(x), f_3(\phi_1(x),w))$. Denoting by $\phi' = (\phi'_1, \phi'_2)$ the resulting reparametrization maps it is then enough to consider the family

$$\{(x,v,w) \mapsto (\phi_1 \circ \phi'_1(x), \phi_2(\phi'_1(x),v), \phi'_2(x,w))\}.$$

Remark 7.3. *From the proof we may in fact explicit the dependence in δ of $\sharp \mathsf{Per}_n^\delta$. Let us be more precise for a C^∞ surface diffeomorphisms f. For any $\gamma > 0$ there exists an integer $N = N(f, \delta, \gamma)$ such that for any n larger than N we have*

$$\sharp Per_n^\delta \leq \alpha(\delta, f)^{-3} e^{\gamma n}$$

where $\alpha(\delta, f)$ is the angle as in the proof of Lemma 7.1 assuming Lemma 7.2 on page 27.

Appendices

APPENDIX A. GROWTH OF $(Per_n^\delta)_n$

A.1. **The case of C^r surface diffeomorphisms with $r > 1$.** Let $f : M \to M$ be a C^r with $r > 1$ surface diffeomorphism. For any $\chi > 0$ Sarig build a finite to one Markov extension $\pi_\chi : (\Sigma(\mathcal{G}_\chi), \sigma) \to (M, f)$ such that $\pi_\chi(\Sigma(\mathcal{G}_\chi))$ has full measure for any hyperbolic ergodic invariant measure with Lyapunov exponents χ-away from zero. In fact for any $\alpha < 1$ we may choose the parameters in the construction of $\mathcal{G}_\chi$ so that we have:

Lemma A.1.

$$\pi_\chi\left(\mathsf{Per}(\Sigma(\mathcal{G}_\chi), \sigma)\right) \subset \mathsf{Per}_{\alpha\chi}(M, f).$$

Proof. Any n-periodic point in Sarig's graph is given by a closed finite (bi)chain of so called overlapping charts $\Psi := (\psi_1^+, \psi_1^-) \to (\psi_2^+, \psi_2^-)... \to (\psi_n^+, \psi_n^-) = (\psi_1^+, \psi_1^-)$.

By Proposition 3.4 in [47] the gradient of $f_i := \left(\psi_{i+1}^+\right)^{-1} \circ f \circ \psi_i^+$ takes the form

$$f_i(u, v) = (Au + h_1(u, v), Bv + h_2(u, v))$$

with $A < e^{-\chi}$, $B > e^\chi$ and $\|\nabla h_j\| < \epsilon$ for $j = 1, 2$ where ϵ may be chosen small compared to χ and $1 - \alpha$. In particular for ϵ small enough the differential map Tf_i preserves the cone $\{(u, v) \in \mathbb{R}^2, |v| \geq |u|\}$ and $\|Tf_i(u, v)\|' > e^{\alpha\chi}\|v\|'$ for all (u, v) in this cone with the product norm $\|(x, y)\|' = \max(|x|, |y|)$. In particular the largest Lyapunov exponent at $x = \pi_\chi(\Psi)$ is bigger then $\alpha\chi$. $\square$

It follows from Gurevic's theory (e.g. see the proof of Theorem 1.1 in [47]) that whenever f has a measure of maximal entropy then for some positive integer p we have $\liminf_{p|n} \sharp\mathsf{Per}_n^\delta e^{-n\mathsf{h}_{\mathsf{top}}(f)} > 0$ for any $0 < \delta < \mathsf{h}_{\mathsf{top}}(f)$. In particular these lower bounds holds for C^∞ surface diffeomorphisms.

A.2. **The case of C^r interval maps with $r > 1$.** We need here to introduce a slightly modified version of Buzzi-Hofbauer diagram. This diagram is built from the symbolic dynamic associated to a partition into monotone branches. We do not recall the whole construction and refer the reader to the original paper [18] for details. Therein the author used the "natural partition" given by the complementary set of the critical points. However we may also work with other partitions. Let $\mathsf{h}_{\mathsf{top}}(f) > \delta > 0$ and $\alpha > 0$. For C^r

interval maps f there exists a countable partition $P = P_\alpha$ of $[0,1] \setminus \{f' = 0\}$ into intervals I so that [19]

(13) $$\forall x, y \in I, \quad \frac{|f'(x)|}{|f'(y)|} \leq e^\alpha.$$

Then we recall that in the Markov representation associated to this partition we let an arrow from the vertex $A_{-q}...A_0 \in P^{q+1}$ to the vertex $B_{-p}...B_0 \in P^{p+1}$ whenever $p \leq q + 1$ and $B_{-p}...B_1 = A_{-p+1}...A_0$ and $B_0 \cap f^{q+1}\left(\bigcap_{k=0}^q f^{k-q} A_k\right) = f^p\left(\bigcap_{k=0}^p f^{k-p} B_k\right)$.

Buzzi proved that there is a Borel semiconjugation from the Markov shift to the interval map which induces an entropy-preserving bijection between measures of positive entropy (see also [16]). In particular if the interval map admits a maximal measure then so does the Markov shift. In this case applying the already mentioned works of Gurevic there is a vertex γ in the graph so that the number a_n of closed paths of length n at $\alpha = A_{-q}...A_0$ satisfies for some p

(14) $$\liminf_{p|n} a_n e^{-n h_{top}(f)} > 0.$$

By letting $I_\gamma := f^q\left(\bigcap_{k=0}^q f^{k-q} A_k\right)$ these paths corresponds to disjoints n-monotone branches $I \subset I_\gamma$ with $I_\gamma \subset f^n(I)$. The number a'_n of the intervals I with length less than $e^{(\alpha-1)n h_{top}(f)}|I_\gamma|$ also satisfies $\liminf_{p|n} a'_n e^{-n h_{top}(f)} > 0$. For these intervals I we have $\sup_{x \in I} |(f^n)'(x)| \geq e^{(1-\alpha)n h_{top}(f)}$ and therefore according to the distortion property (13) we get

$$\inf_{x \in I} |(f^n)'(x)| \geq e^{((1-\alpha)h_{top}(f)-\alpha)n}.$$

Then any n-periodic point in I has a Lyapunov exponent larger $e^{(1-\alpha)h_{top}(f)-\alpha}$ which is larger than δ if one takes α small enough. Thus we conclude that

$$\forall n \in \mathbb{N} \setminus \{0\}, \ \sharp Per_n^\delta \geq a'_n$$

and thus

$$\liminf_{p|n} \sharp Per_n^\delta e^{-n h_{top}(f)} > 0.$$

Appendix B. Variational principle for the local periodic growth

We prove here the variational principle (4) stated in Section 2:

(15) $$g_{\mathcal{P}}^* = \sup_{\mu} g_{\mathcal{P}}^*(\mu).$$

Proof. Clearly we have for all k and for all ergodic periodic measures ν_n with minimal period n:

$$\mathfrak{P}_k(\nu_n) \leq \frac{1}{n} \sup_{x \in X} \log \sharp \left(\mathcal{P}_n \cap B(x, n, \epsilon_k) \right),$$

and thus $g_{\mathcal{P}}^* \geq \sup_{\mu} g_{\mathcal{P}}^*(\mu)$. Let us show now the other inequality. Fix $\alpha > 0$. We will prove that $g_{\mathcal{P}}^* \leq \sup_{\mu} g_{\mathcal{P}}^*(\mu) + 2\alpha$. For any k there is a positive integer n_k and a point x_k in X satisfying:

$$
\begin{aligned}
g_{\mathcal{P}}^*(\epsilon_k) &= \limsup_n \frac{1}{n} \sup_{x \in X} \log \sharp \left(\mathcal{P}_n \cap B(x, n, \epsilon_k) \right), \\
&< \frac{1}{n_k} \log \sharp \left(\mathcal{P}_{n_k} \cap B(x_k, n_k, \epsilon_k) \right) + \alpha.
\end{aligned}
$$

Moreover we may assume the sequence $(n_k)_k$ is going to infinity. As already observed we have also for any $p_k \in \mathcal{P}_{n_k} \cap B(x_k, n_k, \epsilon_k)$ the following inclusion:

$$B(x_k, n_k, \epsilon_k) \subset B(p_k, n_k, 2\epsilon_k).$$

We choose such a periodic point p_k that its minimal period $P_k \leq n_k$ satisfy

$$\sharp \mathcal{P}_{P_k} \cap B(x_k, n_k, \epsilon_k) \geq \frac{1}{n_k} \sharp \mathcal{P}_{n_k} \cap B(x_k, n_k, \epsilon_k).$$

We also let $(a_l)_l$ be a nondecreasing sequence of positive integers going to infinity with $\epsilon_{a_l} \geq 2\epsilon_l$ for all large enough l. Then the associated periodic measure ν_{p_k} satisfies for large k:

$$
\begin{aligned}
\mathfrak{P}_{a_k}(\nu_{p_k}) &= \frac{1}{P_k} \int \log \sharp \left(\mathcal{P}_{P_k} \cap B(x, P_k, \epsilon_{a_k}) \right) d\nu_{p_k}(x), \\
&\geq \frac{1}{n_k} \int \log \sharp \left(\mathcal{P}_{P_k} \cap B(x, n_k, 2\epsilon_k) \right) d\nu_{p_k}(x), \\
&\geq \frac{1}{n_k} \log \sharp \left(\mathcal{P}_{n_k} \cap B(x_k, n_k, \epsilon_k) \right) - \frac{\log n_k}{n_k}, \\
&> g_{\mathcal{P}}^*(\epsilon_k) - 2\alpha.
\end{aligned}
$$

We may assume the period P_k of p_k goes to infinity with k ; otherwise for large k we would have $g_{\mathcal{P}}^*(\epsilon_k) \leq 2\alpha$ and thus $g_{\mathcal{P}}^* \leq 2\alpha$ (recall $\mathcal{P}_n$ is finite for all n). For any weak-star limit of $(\nu_{p_k})_k$, say $\mu = \lim_k \nu_{p_{\varphi(k)}}$, we have finally

:

$$
\begin{aligned}
g_{\mathcal{P}}^*(\mu) &\geq \lim_l \limsup_k \mathfrak{P}_l(\nu_{p_{\varphi(k)}}), \\
&\geq \limsup_k \mathfrak{P}_{a_{\varphi(k)}}(\nu_{p_{\varphi(k)}}), \\
&\geq \lim_k g_{\mathcal{P}}^*(\epsilon_{\varphi(k)}) - 2\alpha = g_{\mathcal{P}}^* - 2\alpha.
\end{aligned}
$$

This concludes the proof of the lemma. $\qquad\square$

j

Appendix C. Proof of Corollary 3.1

The statement below is a consequence of the main result of [15] (Theorem 5.5 and 6.5 therein). We refer to [27] and [9] for the notion of superenvelope of the entropy structure and its significance in the theory of symbolic extensions.

Theorem C.1. [15] *Let (X, T) be a topological dynamical system and $\mathcal{P}$ an invariant subset of $\mathsf{Per}(T)$ with $\sharp \mathcal{P}_n < +\infty$ for all n. We also assume that X admits a basis of neighborhoods [10] with null measure boundary for any T-invariant measure μ which is not a periodic measure in $\bigcup_n \left(\mathsf{Per}_n \setminus \mathsf{Per}_n^\delta \right)$.*

Let E be an affine bounded superenvelope of the entropy structure of (X, T) satisfying $E(\mu) \geq h(\mu) + g_{\mathcal{P}}^*(\mu)$ for all T-invariant probability measures μ (in particular $g_{\mathcal{P}}^* < +\infty$).

Then there is a symbolic extension $\pi : (Y, S) \to (X, T)$ and a Borel embedding $\psi : B \to Y$ with B a Borel set with full measure for every ergodic measure except periodic measures in $\bigcup_n \left(\mathsf{Per}_n \setminus \mathcal{P}_n \right)$ such that $\psi \circ f = \sigma \circ \psi$ and $\pi \circ \psi = \mathsf{Identity}_B$ and

$$\sup_{\nu, \ S^*\nu=\nu \ and \ \pi^*\nu=\mu} h(\nu) = E(\mu).$$

Moreover the cardinality of the alphabet of Y may be chosen to be less than or equal to $e^{\max\left(\sup_\mu E(\mu), \mathsf{per}^{\mathcal{P}}\right)} + 1$ with $\mathsf{per}^{\mathcal{P}} := \sup_{n \in \mathbb{N} \setminus \{0\}} \frac{\log \sharp \mathcal{P}_n}{n}$.

By [39] the existence of the satisfactory basis of neighborhoods is ensured whenever X is finite dimensional. Then it follows from the previous works on symbolic extensions for C^r systems with $r > 1$ given in [12, 28] that the functions

$$\mu \mapsto h(\mu) + \frac{2(3r-1)}{r(r-1)} \left(\chi^+(\mu, f) + \chi^+(\mu, f^{-1}) \right)$$

and

$$\mu \mapsto h(\mu) + \frac{\chi^+(\mu, f)}{r - 1}$$

are respectively superenvelopes of the entropy structure for a C^r surface diffeomorphism and a C^r interval map f with $r > 1$. Thus, together with Theorems 3.1 and 3.2 we get Corollary 3.1.

Finally let us explain how we deduce the above theorem from [15]. In fact Theorem C.1 corresponds to Theorem 5.5 [10] in [15] where the function $g_{\mathcal{P}}^*$ is replaced by another function u_1 defined as the limit in k of $\widetilde{\left((\mathfrak{Q}_k)^h\right)}$ where:

---

[10]In [15] the authors work with systems with the small boundary property. But as we only need to embedd periodic points in $\mathcal{P}$ the weaker assumption in Theorem C.1 is sufficient.

- $$\mathfrak{Q}_k(\nu_n) = \frac{1}{n} \int \log \sharp \left(\mathcal{Q}_n \cap B(x, n, \epsilon_k) \right) d\nu_n(x),$$

  where $\mathcal{Q}_n$ is the subset of $\mathcal{P}$ of periodic points with minimal period n and ν_n is an ergodic invariant measure supported on $\mathcal{Q}_n$,
- $(\mathfrak{Q}_k)^h$ is the harmonic extension of the function $\mathfrak{Q}_k$ (it was first defined only for ergodic periodic measures) by letting $\mathfrak{Q}_k(\nu) = 0$ for ergodic non periodic measures,
- $\left((\mathfrak{Q}_k)^h \right)^{\tilde{}}$ is the upper semi-continuous envelope of $(\mathfrak{Q}_k)^h$, i.e. the smallest upper semi-continuous function larger than $(\mathfrak{Q}_k)^h$.

Thus we only have to show that any affine superenvelope E of the entropy structure with $E - h \geq g_{\mathcal{P}}^* = \lim_k (\mathfrak{P}_k)^{\tilde{}}$ also satisfies $E - h \geq u_1 = \lim_k \left(\left((\mathfrak{Q}_k)^h \right)^{\tilde{}} \right)$. As the function $E - h$ is harmonic and $\mathfrak{P}_k \geq (\mathfrak{Q}_k)^h$ on ergodic measures this follows from Lemma 8.2.13 as in the proof of Lemma 8.2.14 in [27]. This concludes the proof of Theorem C.1 from Theorem 5.5 in [15].

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