ASYMPTOTIC h-EXPANSIVENESS RATE OF C^∞ MAPS

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Abstract. We study the rate of convergence to zero of the tail entropy of C^∞ maps. We give an upper bound of this rate in terms of the growth in k of the derivative of order k and give examples showing the optimality of the established rate of convergence. We also consider the case of multimodal maps of the interval. Finally we prove that homoclinic tangencies give rise to C^2 robustly non h-expansive dynamical systems.

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1. Introduction

Topological Entropy. A dynamical system (f, M) is defined by a continuous map $f : M \to M$ on a compact topological space M. The topological entropy $h(f)$ of (f, M) introduced by Adler, Konheim and McAndrew [1] estimates the dynamical complexity of the system by counting the exponential growth rate of distinguishable orbits at arbitrarily small scales. The topological entropy is a topological invariant, i.e. it is invariant under topological conjugacy. In this pioneer work [1] the authors use finer and finer open covers as the decreasing scale to define the topological entropy. Later Bowen [5] gave an equivalent definition for metric spaces M with distance d (in the present paper we will only consider C^∞ smooth manifolds M endowed with a Riemannian metric). Let us recall Bowen's definition.

For any subset $\Lambda \subset M$, $\varepsilon > 0$ and $n \in \mathbb{N}$, a subset $K \subset M$ is said (n, ε)-spanning Λ if for any $x \in \Lambda$ there exists $y \in K$ such that $d(f^i x, f^i y) < \varepsilon$ for $i = 0, 1, ..., n-1$. Let $r_n(f, \Lambda, \varepsilon)$ denote the smallest cardinality of any (n, ε)-spanning set of Λ. The ε-topological entropy of Λ is defined by

$$h_d(f, \Lambda, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(f, \Lambda, \varepsilon).$$

Letting $\varepsilon \to 0$, define the topological entropy of f on Λ by

$$h_d(f, \Lambda) = \lim_{\varepsilon\to 0} h_d(f, \Lambda, \varepsilon).$$

Denote $h_d(f, \varepsilon) = h_d(f, M, \varepsilon)$ and $h_d(f) = h_d(f, M)$. By an easy argument of compactness one then can prove $h_d(f)$ is equal to the topological entropy as defined in [1] by using open covers. In particular $h_d(f) = h(f)$ does not depend on the metric d. However this is not the case of $h_d(f, \varepsilon)$. If d_1 and d_2 are two equivalent metrics then there exists $C > 1$ such that $h_{d_2}(f, C\varepsilon) \leq h_{d_1}(f, \varepsilon) \leq h_{d_2}(f, C^{-1}\varepsilon)$ for all $\varepsilon > 0$. In the present paper we endow compact smooth manifolds with Riemannian metrics. As such metrics are equivalent, the ε-entropy of f is well defined up to some constant $C > 1$ as above. From now the distance d on M is fixed and we forget the index d in the above definitions.

Tail entropy and h-expansiveness. The tail entropy $h^*(f)$ of a topological system (f, M) first appeared in [29] (initially Misiurewicz called it topological conditional entropy). It is the entropy remaining at arbitrarily small scales. The tail entropy bounds the default of upper semi-continuity of the entropy of invariant Borel probability measures (see [42] for the entropy of invariant measures). This property established in [29] is certainly the main motivation to consider this quantity. As for the topological entropy Bowen gave a definition of the tail entropy for metric spaces replacing iterated open covers by dynamical balls in the definition of Misiurewicz. We present two equivalent definitions.

Given $x \in M$, $n \in \mathbb{N}$, denote the n-step dynamical ball $B_n(f, x, \varepsilon)$ consisting of all such points $y \in M$ that

$$d(f^i y,\ f^i x) < \varepsilon,\ \ i = 0, 1, \cdots, n-1.$$

Define the upper ε-tail entropy $\overline{h}^*(f, \varepsilon)$ as follows :

$$\overline{h}^*(f, \varepsilon) = \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in M} r_n(f, B_n(f, x, \varepsilon), \delta).$$

The lower ε-tail entropy $\underline{h}^*(f,\varepsilon)$ is the maximal entropy of infinite dynamical balls. More precisely let $B_\infty(f,x,\varepsilon)=\bigcap_{n\in\mathbb{N}}B_n(f,x,\varepsilon)$. Define the lower ε-tail entropy as follows :

$$\underline{h}^*(f,\varepsilon)=\sup_{x\in M}h(f,\,B_\infty(f,x,\varepsilon)).$$

One easily finds that $\underline{h}^*(f,\varepsilon)\leq\overline{h}^*(f,\varepsilon)$ for all $\varepsilon>0$. Moreover by an argument of compactness Bowen (Proposition 2.2 of [6]) has shown that for all $\varepsilon>0$ we have in fact the equality $\overline{h}^*(f,\varepsilon)=\underline{h}^*(f,\varepsilon)$ and we denote from now on this quantity by $h^*(f,\varepsilon)$. Also we can define the tail entropy $h^*(f)$ as follows :

$$h^*(f)=\lim_{\varepsilon\to 0}h^*(f,\varepsilon).$$

Like the topological entropy, the tail entropy is a topological invariant and thus $h^*(f)$ does not depend on the metric d, however $h^*(f,\varepsilon)$ may depend on d. But as already noted this is not important in our smooth setting up to rescale balls by a uniform constant.

The dynamical system (f,M) is called entropy expansive (h-expansive) when there exists $\varepsilon>0$ such that $h^*(f,\varepsilon)=0$ and asymptotically entropy expansive (asymptotically h-expansive) when $h^*(f)=0$. As noticed above the measure theoretical entropy is upper semi-continuous for asymptotically entropy expansive maps and therefore such maps always admit an invariant measure of maximal entropy.

The notion of ε-tail entropy is broadly used in the calculation of entropy, since by Theorem 2.4 of [6] it bounds the difference of ε-entropy and the whole entropy[1]:

$$|h(f)-h(f,\varepsilon)|\leq h^*(f,\varepsilon). \tag{1}$$

For any f-invariant Borel probability measure μ and for any finite Borel partition P with diameter less than ε we have also

$$|h(\mu)-h(\mu,P)|\leq h^*(f,\varepsilon). \tag{2}$$

We present now another notion introduced by Newhouse in [34] as the ε-local entropy. We first define a notion of local entropy for invariant measures. Let μ be an f-invariant probability measure and let $\varepsilon>0$ we put

$$h_{\mathrm{loc}}(\mu,\varepsilon):=\lim_{1\neq\sigma\to 1}\inf_{F,\ \mu(F)\geq\sigma}\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in F}r_n(f,F\cap B_n(f,x,\varepsilon),\delta).$$

In [12] it is shown like for the tail entropy that $h_{\mathrm{loc}}(\mu,\varepsilon)$ may be written by using infinite dynamical balls as follows

$$h_{\mathrm{loc}}(\mu,\varepsilon)=\lim_{1\neq\sigma\to 1}\inf_{F,\ \mu(F)\geq\sigma}\sup_{x\in F}h(f,F\cap B_\infty(f,x,\varepsilon)).$$

Finally we let $h_{\mathrm{loc}}(f,\varepsilon):=\sup_\mu h_{\mathrm{loc}}(\mu,\varepsilon)$ be the ε-local entropy of f. Clearly we have $h_{\mathrm{loc}}(f,\varepsilon)\leq h^*(f,\varepsilon)$. Moreover by the so called tail variational principle proved in [16], $\lim_{\varepsilon\to 0}h_{\mathrm{loc}}(f,\varepsilon)=h^*(f)$. However we do not know if $h_{\mathrm{loc}}(f,\varepsilon)=h^*(f,\varepsilon)$ (or even $\geq h^*(f,\varepsilon/10)$) for any $\varepsilon>0$. Newhouse proved (Theorem 1.2 of [34]) that $h_{\mathrm{loc}}(f,\varepsilon)$ also satisfies Inequality (1) and the following estimate for the entropy of measures finer than Inequality (2):

$$|h(\mu)-h(\mu,P)|\leq h_{\mathrm{loc}}(\mu,\varepsilon).$$

[1]However this inequality is in general quite rough and both members may have a different order of magnitude, even for asymptoically h-expansive systems. See Proposition 2.2.

The ε-local entropy is defined through invariant measures and we do not know if it can be expressed in a topological way. Conversely we ignore any satisfactory measure quantity $h^*(\mu,\varepsilon)$ such that a variational principle $h^*(f,\varepsilon)=\sup_\mu h^*(\mu,\varepsilon)$ holds and such that $(h^*(.,\varepsilon))_\varepsilon$ defines an entropy structure (See [16] for the theory of entropy structures).

Local volume growth. We introduce now the local volume growth which is closely related with the local entropy. We assume here that f is C^r with $r\geq 1$.

A C^r map σ from an open set $U\supset[0,1]^k$ of $\mathbb{R}^k$ to M, which is a diffeomorphism onto its image, is called a k-disk. For any k-disk σ and for any Borel subset E of $[0,1]^k$ we denote by $|\sigma|_E|$ the k-volume of σ on E, i.e. $|\sigma|_E|=\int_E\|\Lambda^k D_t\sigma\|_k d\lambda(t)$ where $d\lambda$ is the Lebesgue measure on $[0,1]^k$. Then for any $\varepsilon>0$ we define the ε-local k-volume growth $v_k^*(f,\varepsilon)$ of f as follows :

$$v_k^*(f,\varepsilon)=\sup_{\sigma,\ k-\text{disk}}\limsup_{n\to\infty}\frac{1}{n}\sup_{x\in M}\log\left|f^{n-1}\circ\sigma|_{\sigma^{-1}(B_n(f,x,\varepsilon))}\right|.$$

By using Pesin theory Newhouse [34] proved that for $C^{1+\alpha}$ dynamical systems,

$$h_{\text{loc}}(f,\varepsilon)\quad\leq\quad\max_k v_k^*(f,2\varepsilon)$$

where k takes over all numbers not more than the center unstable dimensions of invariant measures with positive entropy. For surface diffeomorphisms with nonzero topological entropy the only possible value of k is one by Ruelle inequality [37], i.e. in this case we have

$$h_{\text{loc}}(f,\varepsilon)\quad\leq\quad v_1^*(f,2\varepsilon). \tag{3}$$

Finally let us define the local k-volume growth $v_k^*(f)$ of f for any $1\leq k\leq\dim(M)$ as

$$v_k^*(f)=\lim_{\varepsilon\to 0}v_k^*(f,\varepsilon).$$

Yomdin's entropy theory of C^r smooth maps. In [43] Yomdin introduced semi-algebraic tools to study the local complexity of smooth maps and proved in this way Shub's entropy conjecture for C^∞ maps. This famous conjecture [39] states that the topological entropy $h(f)$ has always the logarithm of the spectral radius $\text{sp}(f)$ in homology as a lower bound for differentiable maps. It follows from the inequality $\log\text{sp}(f)\leq h(f)+\max_{1\leq k\leq\dim(M)}v_k^*(f)$ together with the following estimate on the local volume growth established in [43] for any C^r map f:

$$v_k^*(f)\quad\leq\quad\frac{kR(f)}{r}, \tag{4}$$

with $R(f)=\lim_n\frac{1}{n}\log^+\sup_{x\in M}\|D_xf^n\|$. Loosely speaking, the larger the differential order is, the more regular the dynamical complexity is. Using the estimate (4), Yomdin [43] and Newhouse [34] in the setting of C^∞ maps showed entropies in both topological and measure theoretic sense are upper semi-continuous.

Later Buzzi [14] further observed that Yomdin's work in fact implied directly (without refering to local volume growth) that

$$h^*(f)\leq\frac{\dim(M)R(f)}{r}. \tag{5}$$

Consequently, all C^∞ maps are asymptotically h-expansive.

Misiurewicz-like examples. In the early seventies Misiurewicz [30] produced C^r diffeomorphisms f_r without any measure of maximal entropy for any finite r, in particular $h^*(f_r) \neq 0$. In fact in this example one can compute

$$h^*(f_r) \geq \frac{R(f)}{r}$$

(it corresponds to the converse inequality of (5) up to the factor $\dim(M)$). The main idea consists in accumulating smaller and smaller horseshoes at a periodic point which admits a homoclinic tangency. Later Buzzi built in the same spirit a C^r interval map with $h^*(f) = \frac{R(f)}{r}$ and then by considering the product of such systems one can see that inequality (5) is sharp for noninvertible maps. See also [18], [13] for related recent works.

Rate of convergence of the tail entropy for C^∞ systems. As stated above C^∞ systems are asymptotically h-expansive, i.e. $\lim_{\varepsilon\to 0} h^*(f,\varepsilon) = 0$. This paper is devoted to the study of the rate of convergence in the previous limit. This was first investigated by Yomdin in [45] for analytic surface diffeomorphisms. He proved by using "analytic unit reparametrization of semi-algebraic sets" via Bernstein inequalities that

$$h_{\mathrm{loc}}(f,\varepsilon) \leq h^*(f,\varepsilon) \leq C(f)\frac{\log|\log\varepsilon|}{|\log\varepsilon|}$$

for any $\varepsilon > 0$ and for some constant $C(f)$ depending only on f. More recently Liao [26] proved that for any compact analytic manifold M, there exists a universal function $a : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\varepsilon\to 0} a(\varepsilon) = 0$ such that for any analytic map f on M, the ε-tail entropy satisfies

$$h^*(f,\varepsilon) \leq C(f)\, a(\varepsilon)$$

for some constant $C(f)$, independent of ε. Here we investigate the case of general C^∞ maps and give an explicit rate of convergence in terms of the growth in k of the supremum norms of the derivatives of order k by using "C^k unit reparametrizations of semi-algebraic set" as in the proof of the entropy conjecture by Yomdin [43]. In the same spirit of the previously mentioned sharp C^r examples we will then produce various examples, proving optimality of the established rate of convergence. We precise moreover as in [45] the modulus of upper semicontinuity of the topological entropy for some dynamical systems.

h-expansiveness and homoclinic tangencies. Hyperbolic systems are known to be expansive and therefore h-expansive. In fact they are robustly (h-)expansive for the C^1 topology, i.e. for any hyperbolic system f there exists $\varepsilon > 0$ and a C^1 neighborhood $\mathcal{U}$ of f such that $h^*(g,\varepsilon) = 0$ for any $g \in U$. Note that any C^1 robustly expansive diffeomorphism is Axiom A as shown by Mane [28]. For interval maps hyperbolicity is C^r open and dense for any r [25] but the celebrated Newhouse phenomenon claims this is no more the case for diffeomorphisms in higher dimensions [32]. In [27] Liao, Viana and Yang proved that any diffeomorphism C^1 far from homoclinic tangencies is C^1-robustly h-expansive. In Theorem G and Theorem H we prove that C^2 interval maps and diffeomorphisms in higher dimensions with a non degenerate homoclinic tangency are not C^2-robuslty h-expansive, which gives

somehow a reverse to the previous result of [27].

To resume non h-expansiveness of smooth map is produced by homoclinic tangencies while the rate of convergence of the ε-tail entropy is related with the growth of higher derivatives.

2. Statements of results

2.1. **Explicit rate for ultradifferentiable maps.**

2.1.1. *Ultradifferentiable maps.* An arbitrary sequence of positive real numbers $\mathcal{M} = (M_k)_{k\in\mathbb{N}}$ with $M_0 \geq 1$ will always be called a weight sequence. A quite usual condition on the weight is the logarithmic convexity. A weight sequence $(M_k)_k$ is called logarithmic convex if for all $k \in \mathbb{N}\setminus\{0\}$ we have

$$2\log M_k \leq \log M_{k+1} + \log M_{k-1}.$$

We will use two important properties of logarithmic convex weights :

- $\left((M_k/M_0)^{1/k}\right)_k$ is nondecreasing;
- $M_k M_l \leq M_0 M_{k+l}$ for all $k, l \in \mathbb{N}$.

A logarithmic convex weight $\mathcal{M} = (M_k)_k$ is called logarithmic superconvex if $\frac{\log(M_k/M_0)}{k}$ goes to infinity when k goes to infinity.

Note that logarithmic convex weights are quite general: if a weight $(M_k)_k$ satisfies $\liminf_k M_k^{1/k} = +\infty$ then there exists a logarithmic convex weight $(M'_k)_k$ with $M'_k \leq M_k$ and $M_l = M'_l$ for infinitely many $l \in \mathbb{N}$.

Let $O \subset \mathbb{R}^s$ be an open set and let $\mathcal{M} = (M_k)_k$ be a weight sequence, we define the spaces $U^{\mathcal{M}}(O,\mathbb{R}^t)$ and $V^{\mathcal{M}}(O,\mathbb{R}^t)$ of respectively U- and V-ultradifferentiable maps with respect to $\mathcal{M}$ as follows

$$U^{\mathcal{M}}(O,\mathbb{R}^t) = \Big\{f = (f_1,\cdots,f_t) \in C^\infty(O,\mathbb{R}^t) : \exists h > 0,\ s.t.\ \max_{i=1,\cdots,t}\sup_{r\in\mathbb{N}}\frac{\|D^r f_i\|_\infty}{h^r M_r} < \infty\Big\},$$

and

$$V^{\mathcal{M}}(O,\mathbb{R}^t) = \Big\{f = (f_1,\cdots,f_t) \in C^\infty(O,\mathbb{R}^t) : \max_{i=1,\cdots,t}\|D^r f_i\|_\infty \leq M_r\Big\}.$$

The previous setting of ultradifferentiable maps is well adapted to the characterization of quasi-analytic maps. An ultradifferentiable class $U/V^{\mathcal{M}}$ is said quasi-analytic if there is no nontrivial function in $U/V^{\mathcal{M}}$ with compact support. The famous Denjoy-Carleman theorem claims that an ultradifferentiable class $U/V^{\mathcal{M}}$ is quasi-analytic if and only if

$$\sum_{k\in\mathbb{N}}\frac{1}{\inf_{j\geq k} M_j^{1/j}} < +\infty.$$

In the present paper we study the entropy of smooth maps on a compact smooth manifold M of dimension m. Let g be a smooth Riemannian metric on M. We consider the exponential map exp_g associated to g and we denote by R_{inj} its radius of injectivity. The first derivative is the important map in the estimation of the entropy. Given a weight $\mathcal{M}$ we work in the following on the spaces $C^{\mathcal{M}}_{U/V}(M)$ (resp. $Diff^{\mathcal{M}}_{U/V}(M)$) of C^∞ maps (resp. diffeomorpshims) $f : M \to M$ whose first derivative is U/V-ultradifferentiable with respect to $\mathcal{M}$ through the local charts given by the exponential map, i.e. for any $x \in M$ and for any R, satisfying $f(B_M(x,R)) \subset B_M(f(x), R_{inj})$, we have

$$D(exp_{f(x)}^{-1} \circ f \circ exp_x) \in U/V^{\mathcal{M}}(B_{\mathbb{R}^m}(0,R), \mathbb{R}^{m^2}).$$

2.1.2. *Algebraic Lemma.* We present in this section the main semi-algebraic tool used in Yomdin's entropy theory.

For any integer r and for any C^r map we let $\|f\|_r$ be the supremum norm of the derivatives of order no more than r:

$$\|f\|_r := \max_{k=1,\cdots,r} \|D^k f\|.$$

The following algebraic lemma was stated by Gromov in [20].

Algebraic Lemma. Let $P \in (\mathbb{R}[X_1, \cdots, X_l])^m$ be a real m-vector polynomial in l variables of total degree r. Then there exists an integer $C_{r,l,m}$ depending only on r and m (but not on the coefficients of P) and continuous maps $\phi_1, \cdots, \phi_{C_{r,l,m}} : [0,1]^l \to [0,1]^l$, such that :

- $P^{-1}([0,1]^m) = \bigcup_{i=1,\cdots,C_{r,l,m}} \phi_i([0,1]^l)$;
- ϕ_i is analytic on $(0,1)^l$ for each i;
- $\|P \circ \phi_i\|_r \leq 1$ and $\|\phi_i\|_r \leq 1$ for each i.

A complete proof of this lemma may be found in [8] or [36].

We need to estimate the algebraic complexity $C_{r,l,m}$ in the previous lemma. In Section 3.3 we are going to show $C_{r,1,m}$ grows polynomially with r.

2.1.3. *Main result.* Let us first set some notations.

When $a = (a_k)_{k\in\mathbb{N}}$ is a non decreasing unbounded sequence of positive real numbers with $a_0 = 0$, we will consider the inverse function a^{-1} of a defined for all positive real numbers x by

$$a^{-1}(x) := \sup\{l \in \mathbb{N},\ a_l \leq x\} \in \mathbb{N}.$$

The inverse function a^{-1} of a is an unbounded non decreasing function. Observe also that if $a = (a_k)_{k\in\mathbb{N}}$ and $b = (b_k)_{k\in\mathbb{N}}$ are two sequences as above with $a_k \geq b_k$ for all $k \in \mathbb{N}$, then $a^{-1}(x) \leq b^{-1}(x)$ for all $x > 0$.

For a logarithmic super convex weight $\mathcal{M} = (M_k)_k$ we denote by $G_{\mathcal{M}}$ the inverse function of $a^{\mathcal{M}} = (a_k^{\mathcal{M}})_k$ with $a_0^{\mathcal{M}} = 0$ and $a_k^{\mathcal{M}} = \frac{\log^+(M_k/M_0)}{k}$ for $k \neq 0$.

For integers $0 \leq l \leq m$ and for a real number $D \geq 1$, we call a weight $\mathcal{M} = (M_k)_k$ (l,m,D)-admissible when $M_0 \geq e$ and for all integers $k > 0$:

$$\frac{\log(M_k/M_0)}{k} \geq \log k + \frac{2k\log(2^{2m+l} C_{k,l,m} k^{2l})}{Dl}.$$

A weight is said (l,m)-admissible (resp. admissible) if it is (l,m,D)-admissible for some D (resp. for some l,m,D). Admissible logarithmic convex weights are logarithmic superconvex.

For an U-ultradifferentiable map $f \in C_U^{\mathcal{M}}(M)$ with respect to a logarithmic convex admissible weight $\mathcal{M}$ the rates of convergence to zero of the ε-tail entropy,

$h^*(f,\varepsilon)$, and of the ε-local volume growths, $(v_l^*(f,\varepsilon))_{l\leq m}$, are related the growth in r of M_r as follows.

Theorem A. *Let M be a compact smooth Riemannian manifold of dimension m, $0 \leq l \leq m$ be an integer, D be a positive real number and $\mathcal{M} = (M_n)_n$ be a (l,m,D)-admissible weight.*

Then for all $f \in C_V^{\mathcal{M}}(M)$ and for all $0 < \varepsilon < \min(1/R_{inj}^2, 1)$, we have

$$v_l^*(f,\varepsilon) \leq \frac{(2D+1)l}{G_{\mathcal{M}}\left(|\log\varepsilon|/2\right)} \log M_0 \quad \left(\textit{resp. } h^*(f,\varepsilon) \leq \frac{(2D+1)m}{G_{\mathcal{M}}\left(|\log\varepsilon|/2\right)} \log M_0\right).$$

If f is in $C_U^{\mathcal{M}}(M)$ for some logarithmic superconvex weight $\mathcal{M} = (M_k)_k$ then f is in $C_V^{\tilde{\mathcal{M}}}(M)$ for the logarithmic convex weight $\tilde{M} = (\tilde{M}_k)_k$ with $(\tilde{M}_k)_k = (ab^k M_k)_k$ for some constants a and b depending on f. Then one easily sees that there exists a constant $C = C(a,b)$ such that for all $x > C$ we have

$$G_{\tilde{\mathcal{M}}}(x) \geq G_{\mathcal{M}}(x - C).$$

Therefore we get the following estimates for U-ultradifferentiable classes:

Corollary B. *Let M be a compact smooth Riemannian manifold of dimension m, $0 \leq l \leq m$ be an integer and $\mathcal{M} = (M_n)_n$ be a (l,m)-admissible weight (resp. (m,m)-admissible).*

Then for all $f \in C_U^{\mathcal{M}}(M)$, there exists a constant $C = C(f,M,\mathcal{M}) \geq 1$, such that for all $0 < \varepsilon < 1/C$, we have

$$v_l^*(f,\varepsilon) \leq \frac{C}{G_{\mathcal{M}}\left(|\log(C\varepsilon)|/2\right)} \quad \left(\textit{resp. } h^*(f,\varepsilon) \leq \frac{C}{G_{\mathcal{M}}\left(|\log(C\varepsilon)|/2\right)}\right).$$

Since $C_{r,1,m}$ grows polynomially with r as shown in Proposition 3.7, the weight $(k^{k^2})_k$ is $(1,2)$ admissible. Together with Inequality (3) we get as a consequence

Corollary C. *Let M be a compact Riemannian surface. Let f be a $C_U^{(k^{k^2})_k}$ diffeomorphism then for any $0 < \varepsilon < \zeta(M)$ (for some constant $\zeta(M)$ depending only on the Riemannian surface M)*

$$h_{\text{loc}}(f,\varepsilon) \leq \frac{C \log|\log\varepsilon|}{|\log\varepsilon|}$$

for some constant $C = C(f)$.

Analytic maps corresponds to the U-ultradifferentiable class with respect to the weight $\mathcal{M} = (k^k)_k$. In particular the above Corollary applies to analytic maps. We get in this way a new proof of Yomdin's result [45] for the ε-local entropy with a real approach, i.e. by using C^k reparametrizations instead of analytic unit reparametrizations of semi-algebraic sets. However Corollary C is more general as it states that analytic maps are not the largest U-ultradifferentiable class for which the rate in $\frac{\log|\log\varepsilon|}{|\log\varepsilon|}$ applies.

2.2. **Rate for multimodal maps of the interval.** We consider in this section multimodal maps, i.e. continuous piecewise (with a finite number of pieces) monotone maps of the unit interval. Such a map $f : [0,1] \to [0,1]$ is said l-multimodal if $[0,1]$ may be subdivided into l and not more intervals of monotonicity for f. We also let $L(f)$ be the least length of any subinterval of this minimal partition.

It was proved by Misiurewicz that multimodal maps are asymptotic h-expansive [29]. We can in fact give a precise estimate of the rate of entropy of the ε-tail entropy for smooth multimodal maps.

Theorem D. *Let f be a C^1 multimodal map. Then we have for any $0<\varepsilon<L(f)$*

$$h^*(f,\varepsilon)\leq \frac{\log^+\|f'\|_\infty}{|\log w(f',\varepsilon)|}$$

where $w(f',.)$ is the modulus of continuity of f', defined for any $\varepsilon>0$ as $w(f',\varepsilon):=\sup_{|x-y|<\varepsilon}|f'(x)-f'(y)|$.

For C^l l-multimodal maps f one can get an upper bound for any ε, independently from $L(f)$.

Theorem E. *Let f be a C^l l-multimodal map. Then we have for any $0<\varepsilon<1$*

$$h^*(f,\varepsilon)\leq \frac{\log^+\|f\|_l}{|\log\varepsilon|}.$$

By Markov inequality, for any integer $r>0$ there exists a constant $C(r)$ such that, for any polynomial $P:[0,1]\to[0,1]$ of degree r, one has $\|P\|_r\leq C(r)$. Therefore, Theorem E yields

Corollary F. *For any polynomial $P:[0,1]\to[0,1]$ of degree r, we have for any $0<\varepsilon<1$*

$$h^*(P,\varepsilon)\leq \frac{\log^+ C(r)}{|\log\varepsilon|}.$$

We will show in the next section that the above upperbound in $1/\log\varepsilon$ is sharp (See Propostion 2.2).

Remark 2.1. We can not expect to get an upper bound in $\frac{C}{|\log\varepsilon|}$ for any ε with C independent from the degree r in the above Corollary. See Remark 7.4.

2.3. Non h-expansive examples in dimension one.

2.3.1. *Homoclinic tangency.* Let $2\leq r\leq+\infty$. Let f be a C^r interval map and Λ be a hyperbolic repeller. We say that $f|_\Lambda$ has a non degenerate homoclinic tangency if there exists a critical point (local extremum) $c\in[0,1]$ such that $c\in W^u(\Lambda)$, $f^k(c)\in\Lambda$ for some $k>0$ and c is non degenerate for f^k, i.e. $(f^k)^{(l)}(c)\neq 0$ for some finite $l\leq r$. Here the unstable manifold $W^u(\Lambda)$ of Λ is defined as the set of points $x\in[0,1]$, such that for any neighborhood V of Λ, the point x belongs to $f^n(V)$ for some positive integer n. This notion is similar to the notion of homoclinic tangency for diffeomorphisms in higher dimensions. However this picture is not persistent under C^r perturbations contrarily to higher dimensions (Newhouse phenomenon).

Theorem G. *Let f be a C^2 interval map with a non degenerate homoclinic tangency. Then there exists $C=C(f)$ such that for any $0<\varepsilon<\frac{1}{2}$ we have*

$$h_{\mathrm{loc}}(f,\varepsilon)\geq\frac{C}{|\log\varepsilon|}.$$

In the next statement we see with the example of the quadratic map that the inequality (1), $\forall\varepsilon>0,\ h(f)-h(f,\varepsilon)\leq h_{loc}(f,\varepsilon)$, which holds for any continuous dynamical system (f,M), may be quite rough.

Proposition 2.2. *The quadratic map f_4 given by $f_4(x) = 4x(1-x)$ for all $x \in [0,1]$ has a homoclinic tangency at the repulsing fixed point* 0*. In particular we have*

$$h_{\text{loc}}(f_4, \varepsilon) \geq O\left(\frac{1}{|\log \varepsilon|}\right),$$

but for any $\alpha < 1$,

$$h(f_4) - h(f_4, \varepsilon) \leq o\left(\frac{\varepsilon^\alpha}{|\log \varepsilon|}\right).$$

By the already mentioned result of Kozlovscki, Shen and van Strien [25] hyperbolic and thus h-expansive maps form an open and dense set in the C^r topology for any finite r. But we do not know what is the Lebesgue typical rate for a one parameter family.

Question. What is the Lebesgue typical rate of $h^*(f_a, \varepsilon)$ (or $h_{\text{loc}}(f_a, \varepsilon)$) in the quadratic family $f_a(x) = ax(1-x)$?

2.4. **Non h-expansive C^2 robust examples in Newhouse domains for diffeomorphisms in higher dimensions.** We prove every map in Newhouse domains are not h-expansive and we give an explicit lower bound of the ε-tail entropy:

Theorem H. *Let M be a compact smooth surface. Assume $f \in \text{Diff}^r(M)$ $(r \geq 2)$ has a hyperbolic basic set whose stable and unstable manifolds are tangent at some point. Then for any C^r neighborhood $\mathcal{V}$ of f in $\text{Diff}^r(M)$, there exists a C^r open set $\mathcal{U} \subset \mathcal{V}$ and a constant $C > 0$ such that for any $f \in \mathcal{U}$,*

$$\limsup_{\varepsilon \to 0} h_{\text{loc}}(f, \varepsilon)|\log \varepsilon| > C. \tag{6}$$

Consequently, everyone in $\mathcal{U}$ is not h-expansive.

This theorem may be considered as a converse of the already mentioned result of Gang, Liao and Viana. Diffeomorphisms C^1 far from the set of diffeomorphisms exhibiting an homoclinic tangency are robustly h-expansive whereas in an C^2 open dense subset all systems are non h-expansive.

Remark 2.3. The previous lower bound on the tail entropy also holds on Newhouse intervals in any one-parameter family which unfolds a quadratic homoclinic tangency generically, for example in the conservative Henon family $(x, y) \mapsto (y, -x + a - y^2)$.

Since polynomial maps are dense in the space of C^r maps on any open bounded set of $\mathbb{R}^d$ (see also Proposition 2.2 and the above Remark for explicit examples), we have the following Corollary which contradicts Conjecture 6.2 of Yomdin [45] :

Corollary I. *There exist non h-expansive polynomial maps satisfying (6).*

Remark 2.4. Corollary I shows that analytic maps may have exponential dynamical complexity in any scale. However, in the setting of geometry, any l-dimensional analytic manifold $A \subset \mathbb{R}^m$ always has no exponential complexity in any scale, due to the property that for a constant $C(A)$, for any cube Q_t^m of the size t in $\mathbb{R}^m$ and for any affine $L : \mathbb{R}^m \to \mathbb{R}^m$,

$$\text{vol}(L(A) \cap Q_r^m) \leq C(A)t^l,$$

see Corollary 6.4 of [45].

Question. In the previous section we establish in dimension one that the rate of convergence of polynomials was in $1/|\log \varepsilon|$. Does it hold true in higher dimensions?

2.5. **Arbitrarily slow convergence for C^∞ maps and sharpness of Theorem A.** For general C^∞ maps the convergence to zero of the ε-tail entropy may be arbitrarily slow.

Theorem J. *Let M be a compact smooth manifold of dimension larger than one (resp. of dimension one). For any function $a : \mathbb{R}^+ \to \mathbb{R}^+$ with $a(t) \to 0$ as $t \to 0$, there exists a C^∞ diffeomorphism (resp. non invertible map) f_a on M and $\zeta(f_a) > 0$ such that*

$$h_{\mathrm{loc}}(f_a, \varepsilon) \geq a(\varepsilon) \quad \textit{for any} \quad \varepsilon \in (0, \zeta(f_a)).$$

In fact such a map f_a may be produced C^∞ arbitrarily close to a map g with an interval of homoclinic tangency. Moreover one can ensure this perturbation f_a to be volume preserving when this is the case of g. Gonchenko, Turaev and Shilnikov [19] have shown that volume preserving surface diffeomorphisms with such an interval of tangency are C^∞ dense in Newhouse domains. Therefore we get as a corollary:

Corollary K. *Let M be a compact smooth surface. For any function $a : \mathbb{R}^+ \to \mathbb{R}^+$ with $a(t) \to 0$ as $t \to 0$, there exists a C^∞ dense subset $\mathcal{F}_a$ of volume preserving diffeomorphisms in Newhouse domains, such that we have for all $f \in \mathcal{F}_a$ and for all ε small enough, $0 \leq \varepsilon < \zeta(f)$,*

$$h_{\mathrm{loc}}(f, \varepsilon) \geq a(\varepsilon).$$

We also prove the estimate obtained in Theorem A is sharp in the following sense.

Theorem L. *Let M be a compact smooth manifold of dimension larger than one (resp. of dimension one). For any nondecreasing and convex function $a : \mathbb{R}^+ \to \mathbb{R}^+$ with $a(t) \to 0$ as $t \to 0$ and $a(t) \geq t^{1/6}$ for all t, there exists a logarithmic convex weight $\mathcal{M} = (M_k)_k$ satisfying $\frac{\log M_0}{3H(|\log \varepsilon|)} \leq a(\varepsilon)$ for all $\varepsilon > 0$, where H is the inverse map of $k \mapsto \frac{\log(M_k/M_0)}{k}$, with the following property.*

For any ε small enough, $0 < \varepsilon < \zeta(M, a)$, there exists $f_\varepsilon \in Diff^{\mathcal{M}}_V(M)$ (resp. $f_\varepsilon \in C^{\mathcal{M}}_V(M)$) with

$$h^*(f_\varepsilon, \varepsilon) \geq a(\varepsilon) \geq \frac{\log M_0}{3H(|\log \varepsilon|)}.$$

2.6. **Modulus of upper semicontinuity of the topological entropy.** We state now, in the same spirit of [45], how our uniform estimates on the ε-local entropy may be used to explicit a modulus of continuity of the topological entropy for the C^0 topology.

Proposition 2.5. *Let $f \in C^0(M)$ and let G be a subset of $C^0(M)$ such that $h^G_{\mathrm{loc}}(\varepsilon) := \sup_{g \in G} h_{\mathrm{loc}}(g, \varepsilon)$ goes to zero when ε goes to zero and*

$$M_0(G) := \sup_{g \in G} \max(\|Dg\|_\infty, 2) < \infty.$$

Let p_ε be the least integer satisfying $\frac{1}{p_\varepsilon} \log r_{p_\varepsilon}(f, \varepsilon/4) - h(f, \varepsilon/4) \leq h^G_{\mathrm{loc}}(\varepsilon)$. Then for any ε and for any $g \in G$ with $d(g, f) := \sup_{x \in M} d(gx, fx) \leq \varepsilon$ we have

$$h(g) \leq h(f) + 2h^G_{\mathrm{loc}}(N(\varepsilon))$$

where $N(\varepsilon)$ denotes the inverse function of $\varepsilon \mapsto \frac{\varepsilon}{4} M_0(G)^{-p_\varepsilon}$, i.e. $N(\varepsilon)$ is the smallest positive real number such that $\frac{N(\varepsilon)}{4} M_0(G)^{-p_{N(\varepsilon)}} = \varepsilon$.

This result is limited as it seems to be difficult to estimate p_ε for a given $f \in C^0(M)$. In Section 5 we prove Proposition 2.5 and apply it to some (elementary) examples.

3. Rate of convergence for ultradifferentiable maps

In this section we devote to study the tail entropy and local volume growth for general C^∞ smooth maps beyond analytic maps. We are going to start by improving the classical semi-algebraic theory used by Yomdin [44], Gromov [20] and Buzzi [14].

3.1. **Buzzi estimates on the tail entropy via the algebraic lemma.** Following Yomdin's and Gromov's works to bound the local volume growth, Buzzi proved asymptotic h-expansiveness for C^∞ maps. As a first step, he proved the following upper bound of the tail entropy of some iterate of a C^r map f (see the proof of Theorem 2.2 in [14]). We let *exp* denote the exponential map of the Riemanian manifold M. To simplify the notation we write then $\|D^{k+1}\varepsilon^{-1}f^p\varepsilon\|_\infty$ for $\|D^{k+1}\varepsilon^{-1}exp^{-1}_{f^px}\circ f^p\circ exp_x(\varepsilon.)\|_\infty$ for all $0\leq k\leq r-1$.

Proposition 3.1. *Let $r>1$, $l\leq m\in\mathbb{N}$ and $p\in\mathbb{N}$. Let $f\in C^r(M)$ with the dimension of M equal to m and $\varepsilon>0$ such that we have $\|D^{k+1}\varepsilon^{-1}f^p\varepsilon\|_\infty\leq \max(\|Df^p\|,1)$ for all $1\leq k<r$. Let $C_{r,l,m}$ as in the Algebraic Lemma and let $\tilde{C}_{r,l,m}=2^{l+2m}C_{r,l,m}$. Then*

$$v_l^*(f^p,\varepsilon)\leq \frac{l}{r}\left(\log^+\|Df^p\|+2\log B_r\right)+\log\tilde{C}_{r,l,m}$$

and

$$h^*(f^p,\varepsilon)\leq \frac{m}{r}\left(\log^+\|Df^p\|+2\log B_r\right)+\log\tilde{C}_{r,m,m}$$

where B_r is the r^{th} Bell number.

For completeness we reproduce with few changes the proof of the above proposition by Buzzi. We first recall Faa di Bruno formula for the derivative of a composition.

Lemma 3.2.

$$(f\circ g)^{(k)}=\sum_{l=1}^k f^{(l)}\circ g\times B_k^l(g',g'',\cdots,g^{(k-l+1)})$$

with B_k^l the so-called Bell polynomials given by

$$\begin{aligned}&B_k^l(X_1,\cdots,X_{k-l+1})\\ =\;&\sum\frac{k!}{j_1!j_2!\cdots j_{k-l+1}!}\left(\frac{X_1}{1!}\right)^{j_1}\left(\frac{X_2}{2!}\right)^{j_2}\cdots\left(\frac{X_{k-l+1}}{(k-l+1)!}\right)^{j_{k-l+1}}\end{aligned}$$

where the sum holds over all $j_1,j_2,\cdots,j_{k-l+1}\in\mathbb{N}$ with $\sum j_i=l$ and $\sum ij_i=k$.

The r^{th} bell number $B_r:=\sum_{l=1}^r B_r^l(1,\cdots,1)$ counts the class of all partitions of $\{1,\cdots,n\}$. It also counts the class of all distributions of n labeled balls among n indistinguishable urns. Therefore B_r is less than the cardinality of the class of distributions of n labeled balls among n labeled urns, the latter class having r^r members.

Proposition 3.1 follows from the following lemma by considering for all n and for a fixed $x \in M$ the family of maps $(T_n)_{n\in\mathbb{N}}$ given by

$$T_n = \varepsilon^{-1} exp^{-1}_{f^{p(n+1)}x} \circ f^p \circ exp_{f^{pn}x}(\varepsilon\,\cdot).$$

We refer to the original work of Yomdin and Buzzi for this step but we reproduce the proof of the lemma (Proposition 3.3 of [14]).

Lemma 3.3. *Let $\sigma : [0,1]^l \to \mathbb{R}^m$ be a C^r l-disk with $\|\sigma\|_r < 1$ and $(T_n : (-2,2)^m \to \mathbb{R}^m)_n$ be a family of C^r maps with $\|D^{k+1}T_n\|_\infty \leq A_n$ for all $1 \leq k \leq r-1$ for a sequence $(A_n)_n$ satisfying $A_n \geq \max(\|DT_n\|_\infty, 1)$ for all n. Then for all n there exists a family $\mathcal{F}_n := (\psi_n : [0,1]^l \to [0,1]^l)$ of continuous maps, real analytic on $(0,1)^l$, such that with $T^n := T_n \circ \cdots \circ T_1$:*

- $\|T^n \circ \sigma \circ \psi_n\|_r \leq 1$;
- $\|D\left(T^k \circ \sigma \circ \psi_n\right)\|_\infty \leq 1$ *for all* $0 \leq k \leq n$;
- $\bigcap_{k=0,1,\cdots,n}(T^k \circ \sigma)^{-1}((-1,1)^m) \subset \bigcup_{\psi_n \in \mathcal{F}_n} \psi_n([0,1]^l)$;
- $\sharp\mathcal{F}_{n+1} \leq \tilde{C}_{r,l,m}\sharp\mathcal{F}_n \cdot \left(A_n B_r^2\right)^{\frac{l}{r}}$.

Proof. We argue by induction on n. Assume the lemma holds for n and let us prove it for $n+1$. For all $\psi_n \in \mathcal{F}_n$ we have by Faa di Bruno formula:

$$\begin{aligned} &\|D^r(T_{n+1} \circ T^n \circ \sigma \circ \psi_n)\|_\infty \\ \leq\ & \sum_{k=1}^r \|D^k T_{n+1}\| B_r^k \left(\|D\left(T^n \circ \sigma \circ \psi_n\right)\|_\infty, \cdots, \|D^{(r-k+1)}\left(T^n \circ \sigma \circ \psi_n\right)\|_\infty\right). \end{aligned}$$

and then by the induction hypothesis and the hypothesis on the higher derivatives of T_{n+1}:

$$\begin{aligned} \|D^r(T_{n+1} \circ T^n \circ \sigma \circ \psi_n)\|_\infty &\leq \sum_{k=1}^r \|D^k T_{n+1}\| B_r^k(1, \cdots, 1) \\ &\leq A_{n+1} B_r. \end{aligned}$$

Therefore, up to subdivide $[0,1]^l$ into $\left(A_{n+1}B_r^2\right)^{\frac{l}{r}}$ subcubes and to reparametrize them affinely from $[0,1]^l$, we can assume

$$\|D^r\left(T_{n+1} \circ T^n \circ \sigma \circ \psi_n\right)\|_\infty \leq 1/B_r. \tag{7}$$

Now if P is the r^{th} Lagrange polynomial at the center of $[0,1]^l$ of $T_{n+1} \circ T^n \circ \sigma \circ \psi_n$ there exists by Algebraic Lemma a family of maps $(\phi : [0,1]^l \to [0,1^l])$ satisfying

$$P^{-1}([-2,2]^m) = \bigcup_{i=1,\cdots,4^m C_{r,l,m}} \phi_i([0,1]^l).$$

In particular as we have by Taylor formula $\|T_{n+1} \circ T^n \circ \sigma \circ \psi_n - P\|_\infty \leq 1$, the maps $\psi_{n+1} := \psi_n \circ \phi$ satisfy

$$\bigcap_{k=0,1,\cdots,n+1} (T^k \circ \sigma)^{-1}((-1,1)^m) \subset \bigcup_{\psi_{n+1} \in \mathcal{F}_n} \psi_{n+1}([0,1]^l).$$

Moreover $\|P\circ\phi_i\|_r\leq 1$ and $\|\phi_i\|_r\leq 1$ for each i and therefore by using again Faa di Bruno formula together with (7) we get:

$$\begin{aligned}\|T_{n+1}\circ T^n\circ\sigma\circ\psi_{n+1}\|_r &\leq 1+\|(P-T_{n+1}\circ T^n\circ\sigma\circ\psi_n)\circ\phi\|_r\\ &\leq 1+\sum_{k=1}^r\|D^k(P-T_{n+1}\circ T^n\circ\sigma\circ\psi_n)\|B_r^k(1,\cdots,1)\\ &\leq 1+\sum_{k=1}^r\|D^r(T_{n+1}\circ T^n\circ\sigma\circ\psi_n)\|B_r^k(1,\cdots,1)\leq 2.\end{aligned}$$

Up to subdivide again $[0,1]^l$ into 2^l isometric subcubes and to reparametrize them affinely from $[0,1]^l$ we get $\|T_{n+1}\circ T^n\circ\sigma\circ\psi_{n+1}\|_r\leq 1$.

Finally, for all $0\leq k\leq n$ we have

$$\begin{aligned}\|D\left(T^k\circ\sigma\circ\psi_{n+1}\right)\|_\infty &= \|D\left(T^k\circ\sigma\circ\psi_n\circ\phi\right)\|_\infty\\ &\leq \|D\left(T^k\circ\sigma\circ\psi_n\right)\|_\infty\|D\phi\|_\infty\leq 1.\end{aligned}$$

This proves the statement for $n+1$ and concludes the proof by induction of Lemma 3.3.

□

Then following Yomdin, we use the following lemma which relies the ε-local volume growth and the ε-tail entropy of f with these of its iterates f^p to kill the constant term $\frac{2l}{r}\log B_r+\log\tilde{C}_{r,l,m}$ in Proposition 3.1. It follows from two facts. First the (ε,np)-Bowen ball for f is contained in the (ε,n)-Bowen ball for f^p with the same center. Secondly the growth of any l-disk under f^k with $0\leq k\leq p$ is uniformly bounded by $\max(1,\|Df\|)^{pl}$ and for any scale δ there exists a smaller scale δ' such that a (np,δ') spanning set for f^p is (δ,n) spanning for f.

Lemma 3.4. *Let $f\in C^1(M)$, and $g\in C^0(M)$, $\varepsilon>0$ and $p\neq 0$ be an integer. For any integer l less than or equal to the dimension of M, we have*

$$\begin{aligned}v_l^*(f,\varepsilon) &\leq v_l^*(f^p,\varepsilon)/p\,;\\ h_{\mathrm{loc}}(g,\varepsilon) &\leq h_{\mathrm{loc}}(g^p,\varepsilon)/\,p;\\ h^*(g,\varepsilon) &\leq h^*(g^p,\varepsilon)/p\,.\end{aligned}$$

By taking the limit when ε goes to zero we have $v_l^*(f^p)\leq v_l^*(f)/p$, $h_{\mathrm{loc}}(f)\leq h_{\mathrm{loc}}(f^p)/p$ and $h^*(f)\leq h^*(f^p)/p$. The equalities $v_l^*(f^p)=v_l^*(f)/p$, $h_{\mathrm{loc}}(f)=h_{\mathrm{loc}}(f^p)/p$ and $h^*(f)=h^*(f^p)/p$ hold also true but are not used here.

3.2. **Rate of convergence of the tail entropy for ultradifferentiable maps.** We will make an adapted choice of p and r together to give a precise rate of convergence of V-ultradifferentiable maps.

Proof of Theorem A. We prove Theorem A for the ε-local l-volume growth for some given $0\leq l\leq m$. The proof is analogous for the ε-tail entropy and is left to the reader.

Now we fix a logarithmic weight $(M_k)_k$ and we consider $f\in C_V^{\mathcal{M}}(M)$. Let $1>\gamma>0$. We choose r and then p such that

- $r = \lceil \frac{l}{\gamma} \rceil$;
- $p = \lfloor \frac{r \log(\tilde{C}_{r,l,m} r^{2l})}{Dl \log M_0} \rfloor$.

In particular we have

- $\frac{l}{r} \leq \gamma$;
- $\frac{\log(\tilde{C}_{r,l,m} r^{2l})}{p} \leq \frac{2Dl \log M_0}{r} \leq 2D\gamma \log M_0$.

Then we fix ε so that the assumptions on the derivatives of f^p of Lemma 3.3 is checked with $A_n = M_0^p$ for all n, that is for all $1 \leq k < r$:

$$\|D^{(k+1)} \varepsilon^{-1} f^p \varepsilon\| \quad \leq \quad M_0^p.$$

Note that ε must also satisfy

$$\varepsilon M_0^p < R_{inj}.$$

Lemma 3.5. *With the previous notations, we have for $k \geq 1$*

$$\|D^{(k+1)} f^p\| \leq (kp)^k M_0^{(k+1)(p-1)} \max_{k_i \geq 1, \sum_i k_i = k} \prod_i M_{k_i}.$$

Proof. Let $k \geq 1$. Clearly $D^{(k+1)} f^p$ is a polynomial in $D^{(n+1)} f \circ f^m$ with $0 \leq m \leq p-1$ and $0 \leq n \leq k$. By an easy induction the total degree of this polynomial is $(k+1)(p-1)+1$, the degree of the variables involving the first derivative of f is at most $(k+1)(p-1)$ and the number of monomials does not exceed $k!p^k$. Also if we denote l_n the degree in the derivative of order $n+1$ we have $\sum_n n l_n = k$. □

We continue the proof of Theorem A. It follows from the logarithmic convexity of the weight $(M_k)_k$ that

$$\max_{k_i \geq 1, \sum_i k_i = k} \prod_i M_{k_i} \leq M_0^k M_k.$$

According to the above lemma we get then :

$$\|D^{(k+1)} \varepsilon^{-1} f^p \varepsilon\| \quad \leq \quad \varepsilon^k (kp)^k M_0^{(k+1)p} (M_k / M_0).$$

Therefore as $((M_k/M_0)^{\frac{1}{k}})_k$ is nondecreasing, we may choose

$$\varepsilon = \frac{1}{rp} M_0^{-p} \min\left(R_{inj}, (M_r/M_0)^{-\frac{1}{r}} \right).$$

Apply now Proposition 3.1 with the previous datas. We get :

$$\begin{aligned}
\frac{1}{p} v_l^*(f^p, \varepsilon) \quad &\leq \quad \frac{l}{rp} \log \|Df^p\| + \frac{\log(\tilde{C}_{r,l,m} B_r^{\frac{2l}{r}})}{p} \\
&\leq \quad \frac{l}{r} \log M_0 + \frac{\log(\tilde{C}_{r,l,m} r^{2l})}{p} \\
&\leq \quad (2D+1)\gamma \log M_0.
\end{aligned}$$

According to Lemma 3.4 we have the following upper bound on the local l-volume growth of f,

$$v_l^*(f, \varepsilon) \leq (2D+1)\gamma \log M_0.$$

We explicit now the function $\varepsilon = \psi(\gamma)$. In fact we give a lower bound φ of ψ which is increasing. Then we inverse φ to get $\gamma \leq \varphi^{-1}(\varepsilon)$. We have :

$$\begin{aligned} \varepsilon &= \frac{1}{rp} M_0^{-p} \min\left(R_{inj}, (M_r/M_0)^{-\frac{1}{r}}\right); \\ -\log\varepsilon &= \log(pr) + p\log M_0 + \max\left(\log^+(1/R_{inj}), \frac{\log(M_r/M_0)}{r}\right). \end{aligned}$$

Now we have by (l, m, D)-admissibility of $\mathcal{M}$

$$\begin{aligned} \log(pr) + p\log M_0 &\leq \log r + (1+\log M_0)p, \\ &\leq \log r + (1+\log M_0)\frac{r\log\left(\tilde{C}_{r,l,m} r^{2l}\right)}{Dl\log M_0} \\ &\leq \log r + \frac{2r\log\left(\tilde{C}_{r,l,m} r^{2l}\right)}{Dl}, \\ &\leq \frac{\log(M_r/M_0)}{r}. \end{aligned}$$

It follows that

$$-\log\varepsilon \leq 2\max\left(\log^+(1/R_{inj}), \frac{\log(M_r/M_0)}{r}\right),$$

that is for all $0 < \varepsilon < \min\left(1, R_{inj}^2\right)$ we have

$$-\log\varepsilon \leq 2\max\left(\frac{\log(M_r/M_0)}{r}\right),$$

Therefore by definition of G_l and then by replacing r by its expression in terms of γ

$$\begin{aligned} r &\geq G_l\left(|\log\varepsilon|/2\right); \\ \gamma &\leq \frac{2l}{G_l(|\log\varepsilon|/2)}. \end{aligned}$$

□

Remark 3.6. We gave here estimates of the rate of convergence of the ε-local entropy of f through ε-local volume growth. But by using the same method one can deal directly with the measure quantity, $h_{loc}(\mu, \varepsilon)$, and determine the rate of convergence of $\left(h_{loc}(\mu, \varepsilon)\right)_\varepsilon$ in terms of the maximal positive Lyapunov exponent of μ instead of $\log M_0 \geq \log^+\|DT\|$. However we do not attempt to present such further measure theoretical estimates of the ε-local entropy in this paper.

3.3. **Estimate of $C_{r,1,m}$.** In this section we will give an estimate of the algebraic constant in dimension 1 :

Proposition 3.7. *There exists a constant C such that for all r,*

$$C_{r,1,m} \leq C\, m^3 r^8.$$

Remark 3.8. In higher dimensional cases the proof of Gromov's algebraic lemma is more complicated and we do not plan to discuss this case in the present paper. In [11] the author proves that for any dimension m there exists a constant A_m depending only on m such that $C_{r,m} \leq r^{A_m r^m}$. With this estimate, by applying Theorem A we can get that for any analytic map f on a compact smooth manifold M of dimension m, it holds that

$$h^*(f,\varepsilon) \leq B_m \left(\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\right)^{\frac{1}{m+1}}$$

for some constant B_m depending only on m.

From the point of view of Proposition 3.7, it seems reasonable to ask the following question concerning the polynomial growth of $C_{r,l,m}$ in r for any dimension:

Question. For any $m \in \mathbb{N}$, do there exist constants $A_{m,l}, B_{m,l}$ such that for all $r \in \mathbb{N}$,

$$C_{r,l,m} \leq A_{m,l}\, r^{B_{m,l}} \text{ ?}$$

We will not directly adopt the proof of Gromov but give here a new proof of Lemma 2.1.2 in the one dimensional case. In fact, by following straightforwardly Gromov's work we only manage to get the inequality $C_{r,1,m} \leq Cm^3 10^{r^2}$, which is a super-exponential growth upper bound.

Proof of Proposition 3.7. Let $(P_1, ..., P_m) \in \mathcal{R}[X]^m$ be a finite family of polynomials of degree less than or equal to r.

First step : $\|P_j \circ \phi\|_1 \leq 1$. To bound the first derivative, we consider one connected component of the following set

$$[0,1] \setminus \bigcup_{i,j} \{P_i = 0, P_i = 1, |P_i'| = |P_j'|, |P_i'| = 1\}.$$

Observe there are at most $4rm+2rm^2 \leq 6rm^2$ components. The set $\bigcap_j P_j^{-1}([0,1])$ is the closure of the union of some of these intervals. On such an interval I we have $P_j(I) \subset [0,1]$ for any $j = 1, ..., m$. Moreover there exists i such that $|P_i'(x)| = \max_j |P_j'(x)|$ for all $x \in I$ and we have either $|P_i'(x)| \leq 1$ for all $x \in I$ or $|P_i'(x)| \geq 1$ for all $x \in I$. In the first case we just reparametrize $I := [a,b]$ from $[0,1]$ by an affine contraction

$$\phi_I(t) = a + t(b-a)$$

while in the second case of we consider the inverse of P_i

$$\phi_I(t) := P_i^{-1}(P_i(a) + t(P_i(b) - P_i(a))).$$

One easily checks that $\|\phi\|_1 \leq 1, \quad \|P_j \circ \phi\|_1 \leq 1$ for any j.

Second step : $(P_j \circ \phi)^{(k)}$ have constant sign for $k = 2, ..., r+1$. We subdivide $[0,1]$ into subintervals where the derivatives $(P_j \circ \phi_I)^{(k)}$ for $k = 2, \cdots, r+1$ and for $j = 1, ..., m$ have constant sign and therefore where $(P_j \circ \phi_I)^{(k)}$ for $k = 1, \cdots, r$ are monotone. It is enough to consider one connected component of the set

$$[0,1] \setminus \{(P_j \circ \phi_I)'' = 0; \cdots ; P_j \circ \phi_I^{(r)} = 0\}.$$

When ϕ_I is just a linear contraction, $\phi_I(t) = a + t(b-a)$ for all $t \in [0,1]$, there are at most

$$1 + \sum_j deg(P_j'') + \cdots + deg(P_j^{(r)}) \leq mr^2$$

such components. As before we reparametrize them from $[0,1]$ by affine contraction $t \mapsto c + t(d-c)$.

For the second case, $\phi_I(t) := P_i^{-1}(P_i(a) + t(P_i(b) - P_i(a)))$ for all $t \in [0,1]$, we use the following lemma.

Lemma 3.9. *Let $k \geq 1$. Then there exists a polynomial $R \in \mathbb{R}[X_1, ..., X_k]$ of total degree $k-1$ such that*

$$\begin{aligned}&\left(P_i^{-1}\left[P_i(a) + .(P_i(b) - P_i(a))\right]\right)^{(k)} \\ = \quad & \frac{R(P_i' \circ P_i^{-1}(a + .(b-a)), \cdots, P_i^{(k)} \circ P_i^{-1}(P_i(a) + .(P_i(b) - P_i(a))}{\left(P_i' \circ P_i^{-1}(P_i(a) + .(P_i(b) - P_i(a))\right)^{2k-1}}.\end{aligned}$$

In particular the numerator in the above lemma is a polynomial of degree at most $k(r-1)$ in $P_i^{-1}(P_i(a) + .(P_i(b) - P_i(a)))$. By Faa di Bruno formula it follows that $\left(P_j \circ P_i^{-1}\left[P(a) + .(P(b) - P(a))\right]\right)^{(k)}$ may be written as a rational function with a polynomial numerator of degree at most $(k+1)(r-1)$ in $P_i^{-1}(P_i(a) + .(P_i(b) - P_i(a)))$ and therefore $\left(P_j \circ P_i^{-1}(P_i(a) + .(P_i(b) - P_i(a)))\right)^{(k)}$ has at most $(k+1)(r-1)$ zeroes in $[0,1]$. Thus up to subdivide $[a,b]$ into at most mr^3 intervals one can assume $\left(P_j \circ P_i^{-1}(P_i(a) + .(P_i(b) - P_i(a)))\right)^{(k)}$ for $k = 1, \cdots, r+1$ have constant sign. We reparametrize all these subintervals affinely from $[0,1]$. Note that after this first step we get at most Cm^3r^4 reparametrizations.

Third step : $\|P_j \circ \phi\|_r \leq 1$. We let $H : [0,1] \to \mathbb{R}$ be a C^{r+1} function such that the derivatives $(H^{(k)})_{k=2,\ldots,r+1}$ have constant signs and such that $\|H\|_1 \leq 1$. We will show that $\|(H \circ Q_r)^{(k)}\|_\infty \leq Cr^{4k}$ for the reparametrization $Q_r : [0,1] \to [0,1]$ defined in the following lemma. Then to conclude the proof of Proposition 3.7 one apply this result to the maps $H = P_j \circ \phi$ where ϕ are the reparametrizations obtained at the end of the second step.

Lemma 3.10. *There exists a unique polynomial Q_r of degree $2r-1$ such that $Q(0) = 0$, $Q(1) = 1$ and $Q^{(k)}(0) = Q^{(k)}(1) = 0$ for $k = 1, \cdots, r-1$.*

Moreover Q_r satisfies the following properties :

- *Q_r satisfies the functional equation $1 - Q(1-X) = Q(X)$;*
- *Q_r is an homeomorphism from $[0,1]$ onto itself;*
- *$Q_r'(X) = b_r X^{r-1}(1-X)^{r-1}$ where $1/b_r = \beta(r,r)$ where β is the usual β function;*
- *$Q_r(x) \geq b_r x^r (1-x)^r$.*

Proof. We only prove the last item (the other statements are easy to check). By the third item we have

$$Q_r(x) = \int_0^x b_r t^{r-1}(1-t)^{r-1} dt.$$

Then by considering the change of variable $t = ux$ we get

$$\begin{aligned} Q_r(x) &= x^r \int_0^1 b_r u^{r-1}(1-ux)^{r-1} du \\ &\geq x^r \int_0^1 b_r u^{r-1}\left((1-u)(1-x)\right)^{r-1} du \\ &\geq x^r(1-x)^r b_r. \end{aligned}$$

□

Fix $1 \leq k \leq r$, $1 \leq l \leq k$ and $\underline{j} := (j_1, j_2, \cdots, j_{k-l+1})$ as in Faa di bruno formula (Lemma 3.2) and consider the polynomial

$$T_{l,\underline{j}} := \left(\frac{Q_r'}{1!}\right)^{j_1} \left(\frac{Q_r''}{2!}\right)^{j_2} \cdots \left(\frac{Q_r^{(k-l+1)}}{(k-l+1)!}\right)^{j_{k-l+1}}.$$

We let S be the polynomial $S(X) := X(1-X)$. Recall $Q_r' = b_r S^r$ and $\|S\|_\infty = 1/4$.

Lemma 3.11. *Let $0 \leq i \leq k-1$. Then there exists a polynomial R_i with $\|R_i\|_\infty \leq (r/2)^i i!$ such that*

$$Q_r^{(i+1)} = b_r S^{r-i} R_i\,.$$

In particular as $b_r = \frac{(2r-1)!}{(r-1)!^2} \leq C\sqrt{r}2^{2r}$, we have

$$Q_r^{(i+1)} \leq C\sqrt{r}2^{2i}\|R_i\|_\infty \leq C\sqrt{r}(2r)^i i! \leq Cr^{2k}.$$

Proof. We argue by induction on i. Observe $R_0 = 1$. The polynomials R_i satisfies the following property :

$$R_{i+1} = (r-i)S'R_i + SR_i'\,.$$

In particular the degree of R_i is equal to i. Now by Markov inequality,

$$\|R_i'\|_\infty \leq 2i^2\|R_i\|_\infty$$

(the norm $\|\cdot\|_\infty$ is the classical supremum norm over $[0,1]$) and therefore

$$\begin{aligned} \|R_{i+1}\|_\infty &\leq (r-i)\|S'\|_\infty\|R_i\|_\infty + \|S\|_\infty\|R_i'\|_\infty \\ &\leq \|R_i\|_\infty(r-i+i^2/2) \\ &\leq \|R_i\|_\infty r(i+1)/2 \\ &\leq (r/2)^{i+1}(i+1)!. \end{aligned}$$

□

Let us bound from above the supremum norm of $H^{(l)} \circ Q_r \times T_{l,\underline{j}}$ over $[0,1]$.

$$\begin{aligned}\|H^{(l)}\circ Q_r\times T_{l,\underline{j}}\|_\infty &\leq b_r^l\|H^{(l)}\circ Q_r\times S^{(r+1)l-k}\|_\infty\times\prod_{i=1}^{k-l+1}(\|R_{i-1}\|_\infty/i!)^{j_i}\\ &\leq b_r^l(r/2)^k\|H^{(l)}\circ Q_r\times S^{(r+1)l-k}\|_\infty\\ &\leq b_r^l(r/2)^k\|H^{(l)}\circ Q_r\times S^{r(l-1)}\|_\infty\\ &\leq (r/2)^k\|H^{(l)}\circ Q_r\times Q_r^{l_1}Q_r(1-.)^{l_2}\|_\infty.\end{aligned}$$

where l_1 (resp. l_2) is the number of $2\leq m\leq l$ such that $|P^{(m)}|$ is non-increasing (resp. nondecreasing).

Consider finally the term $\|H^{(l)}\circ Q_r\times Q_r^{l_1}Q_r(1-.)^{l_2}\|_\infty$. Assume first that $|H^{(l)}|$ is non-increasing on $[0,1]$. Then we have for all $2\leq m\leq l$

$$|H^{(l)}(1-Q_r(1-x)m/l)|\leq|H^{(l)}(1-Q_r(1-x))|=|H^{(l)}\left(Q_r(x)\right)|$$

and

$$\begin{aligned}|H^{(l)}(Q_r(x)m/l)\times Q_r(x)/l| &\leq \left|\int_{Q_r(x)(m-1)/l}^{Q_r(x)m/l}H^{(l)}(t)dt\right|\\ &\leq \max(H^{(l-1)}(Q_r(x)m/l),H^{(l-1)}(Q_r(x)(m-1)/l).\end{aligned}$$

When $|H^{(l)}|$ is nondecreasing on $[0,1]$ we get symmetrically

$$|H^{(l)}(Q_r(x)m/l)|\leq|H^{(l)}(Q_r(x))|=|H^{(l)}(Q_r(1-x)|$$

and

$$\begin{aligned}&\|H^{(l)}(1-Q_r(1-x)m/l)\times Q_r(1-x)/l\|_\infty\\ \leq\ &\max\Big(H^{(l-1)}(1-Q_r(1-x)m/l),H^{(l-1)}(1-Q_r(x)(m-1)/l\Big).\end{aligned}$$

By an easy induction one obtains

$$\|H^{(l)}\circ Q_r\times Q_r^{l_1}Q_r(1-.)^{l_2}\|_\infty\leq l^{l-1}\|H'\|_\infty\leq l^l,$$

and then

$$\|H^{(l)}\circ Q_r\times T_{l,\underline{j}}\|_\infty\leq(\frac{r}{2})^k\cdot l^l.$$

Finally by the identity $B_{k,l}(1!,\cdots,(l-k+1)!)=C_k^lC_{k-1}^{l-1}(k-l)!$ we get

$$\begin{aligned}\|(H\circ Q_r)^{(k)}\|_\infty &\leq \sum_{l=1}^k(\frac{r}{2})^k\cdot l^l\cdot B_{k,l}(1!,\cdots,(l-k+1)!)\\ &\leq \sum_{l=1}^k r^k\cdot l^l\cdot C_k^lC_{k-1}^{l-1}(k-l)!\\ &\leq \sum_{l=1}^k r^kl^lk^{2k}/l!\\ &\leq Cr^{4k}.\end{aligned}$$

We have also for $1 \leq k \leq r$ by Lemma 3.11,

$$\|Q_r^{(k)}\| \leq Cr^{2k} \leq Cr^{4k}.$$

To conclude the proof of Lemma 2.1.2, subdivide the unit interval into at most $[Cr^4]+1$ intervals I of length $1/Cr^4$ and let ψ_I be the affine reparametrizations from $[0,1]$ of I. One easily checks that $\|Q_r \circ \psi_I\|_r, \|P \circ Q_r \circ \psi_I\|_r \leq 1$ so that the family of reparametrization $Q_r \circ \psi_I$ satisfy the conclusions of Lemma 2.1.2.

□

3.4. **Surface diffeomorphisms: Proof of Corollary C.** With the assumptions in Corollary C, by combining Theorem A with Proposition 3.7, we have for some universal constant C for all integers $k \neq 0$

$$k \log k \leq \frac{\log M_k}{k} + 2k \log\left(\tilde{C}_{k,1,2}k^2\right) \leq Ck \log k,$$

and thus G_1 may be bounded from above as follows :

$$\begin{aligned} l &= \frac{\log M_k}{k} + 2k \log\left(\tilde{C}_{k,1,2}k^2\right); \\ &\geq k \log k; \\ \log l &\geq \log k, \end{aligned}$$

and then the function G_1 satisfies :

$$\begin{aligned} G_1(l) &= k; \\ &\geq \frac{l}{C \log k}; \\ &\geq \frac{l}{C \log l}. \end{aligned}$$

Thus, with $C(f)$ the constant in Theorem A,

$$\begin{aligned} h_{loc}(f,\varepsilon) &\leq v_1^*(f, 2\varepsilon); \\ &\leq \frac{2C(f)}{G_1(|\log(2C(f)\varepsilon|))}; \\ &\leq 2CC(f)\frac{\log|\log(2C(f)\varepsilon)|}{|\log(2C(f)\varepsilon)|}; \\ &\leq \frac{\widetilde{C}(f)\log|\log\varepsilon|}{|\log\varepsilon|}, \end{aligned}$$

for some constant $\widetilde{C}(f)$.

Remark 3.12. Corollary C holds also true for local surface diffeomorphisms. In fact one has again in this this case $h_{loc}(f,\varepsilon) \leq v_1^*(f,2\varepsilon)$ for any ε small enough. Indeed it was proved for local diffeomorphisms in [10] (Theorem 5) that there exists $\varepsilon > 0$ such that any invariant measure μ with $h_{loc}(\mu,\varepsilon) > 0$ has at least one negative Lyapunov exponent.

4. The case of one dimensional multimodal maps

We prove in this section all the results related to one dimensional dynamics : Theorem D, Theorem E and Theorem G. We will make use of the following lemma of analysis.

Lemma 4.1. *Let $k \geq 1$ and f be a C^{k+1} map of the interval I. If the derivative f' of f vanish at $x_1 < x_2 < \cdots < x_k$ then for any $x \in I$ we have*

$$|f'(x)| \leq \|f^{(k+1)}\|_\infty |I|^k.$$

Proof. By the assumptions, for any $1 \leq l \leq k$, there exists $y_l \in [x_1, x_k]$ such that $f^{(l)}(y_l) = 0$. Therefore, for any $x \in I$,

$$\begin{aligned} |f^{(k)}(x)| &= |\int_{y_k}^x f^{(k+1)}(z)dz| \leq |I| \|f^{(k+1)}\|_\infty; \\ |f^{(k-1)}(x)| &= |\int_{y_{k-1}}^x f^{(k)}(z)dz| \leq |I|^2 \|f^{(k+1)}\|_\infty; \\ &\vdots \\ |f'(x)| &= |\int_{y_1}^x f^{(2)}(z)dz| \leq |I|^k \|f^{(k+1)}\|_\infty. \end{aligned}$$

□

Proof of Theorem E. Let f be a C^l l-multimodal map of the unit interval. Let $x \in [0,1]$, $n \in \mathbb{N}$ and $\varepsilon > \delta > 0$. It is easily seen that the maximal cardinality of an (n, δ) separated set in $B_n(f, x, \varepsilon)$ is not more than the n/δ time the number of monotonic branches of f^n intersecting $B_n(f, x, \varepsilon)$ (see for example [29]). But the number of such f^n-monotonic branches is less than $\prod_{k=0}^{n-1} M_{f^k x, \varepsilon}$ where $M_{y,\varepsilon} \leq l$ is the number of f-monotonic branches in the ε-ball at $y \in [0,1]$. Therefore we have for all $x \in [0,1]$, $n \in \mathbb{N}$ and $0 < \delta < \varepsilon$

$$r_n(f, B_n(f, x, \varepsilon), \delta) \leq \frac{n}{\delta} \prod_{k=0}^{n-1} M_{f^k x, \varepsilon}$$

and

$$h^*(f, \varepsilon) \leq \limsup_n \frac{1}{n} \sup_x \log \prod_{k=0}^{n-1} M_{f^k x, \varepsilon}. \tag{8}$$

We let $m^*(f, \varepsilon)$ be the right member in the above inequality. Note that f' has at least $M_{y,\varepsilon} - 1$ zeroes in $B(y, \varepsilon)$. By Lemma 4.1, we get

$$\|f'|_{B(y,\varepsilon)}\|_\infty \leq \|f^{(M_{y,\varepsilon})}\|_\infty \, \varepsilon^{M_{y,\varepsilon}-1}.$$

Therefore, we have clearly on the other hand

$$\begin{aligned} r_n(f,B_n(f,x,\varepsilon),\delta) &\leq \frac{\varepsilon}{\delta}\prod_{k=0}^{n-1}\|f'|_{B(f^kx,\varepsilon)}\|; \\ &\leq \frac{\varepsilon}{\delta}\prod_{k=0}^{n-1}\|f^{(M_{f^kx,\varepsilon})}\|\varepsilon^{M_{f^kx,\varepsilon}-1}; \\ &\leq \frac{\varepsilon}{\delta}\cdot\varepsilon^{\sum_{k=0}^{n-1}M_{f^kx,\varepsilon}}\varepsilon^{-n}\|f\|_l^n. \end{aligned}$$

By geometric-arithmetic mean inequality we get

$$\sum_k M_{f^kx,\varepsilon} \geq n\left(\prod_{k=0}^{n-1}M_{f^kx,\varepsilon}\right)^{1/n}.$$

Therefore,

$$\begin{aligned} &\log r_n(f,B_n(f,x,\varepsilon),\delta) \\ \leq\ &\log(\frac{\varepsilon}{\delta}) + \max\left(n\left(\left(\prod_{k=0}^{n-1}M_{f^kx,\varepsilon}\right)^{1/n}-1\right)\log\varepsilon + n\log^+\|f\|_l, 0\right). \end{aligned}$$

and then combining with (8),

$$h^*(f,\varepsilon) \leq \min\left(\max\left((e^{m^*(f,\varepsilon)}-1)\log\varepsilon+\log^+\|f\|_l,0\right), m^*(f,\varepsilon)\right).$$

Now we maximize this function in $m^*(f,\varepsilon)$. The right hand side is maximal when $m^*(f,\varepsilon)=a$ where a is the solution of $(e^a-1)\log\varepsilon+\log^+\|f\|_l = a$. We have therefore

$$\begin{aligned} a\log\varepsilon+\log^+\|f\|_l &\geq a; \\ a &\leq \frac{\log^+\|f\|_l}{1-\log\varepsilon} \leq \frac{\log^+\|f\|_l}{|\log\varepsilon|}. \end{aligned}$$

The proof of Theorem E is completed. □

Remark 4.2. The idea of the proof of Theorem E is similar with the strategy to prove the existence of symbolic extensions for C^r interval maps in [17]. The production of local entropy by monotonic branches is somehow counterbalanced by the decreasing of the Lyapunov exponents.

Proof of Theorem D. The proof is very similar to this of Theorem E. As we consider $\varepsilon < L(f)$, any ε ball meets at most two f-monotone branches. Therefore for any $x\in[0,1]$ we have with $N_x^n := \sharp\{0\leq k<n,\ f^kx\in B(\{f'=0\},\varepsilon)\}$

$$r_n(f,B_n(f,x,\varepsilon),\delta) \leq \frac{n}{\delta}2^{N_x^n}$$

and on the other hand

$$r_n(f,B_n(f,x,\varepsilon),\delta) \leq \|f'\|_\infty^{n-N_x^n}w(f',\varepsilon)^{N_x^n}.$$

Therefore with $m^*(f,\varepsilon) := \lim_n \frac{1}{n}\sup_x N_x^n\log 2$ we get

$$h^*(f,\varepsilon) \leq \min\left((1-\frac{m^*(f,\varepsilon)}{\log 2})\log\|f'\| + \frac{m^*(f,\varepsilon)}{\log 2}\log w(f,\varepsilon), m^*(f,\varepsilon)\right)$$

which leads after optimization to

$$h^*(f,\varepsilon) \leq \frac{\log\|f'\|}{|\log w(f',\varepsilon)|}.$$

Remark 4.3. We only state a rate of convergence for C^1 smooth maps in Section 2. For general continuous multimodal maps, the rate of convergence to zero of the ε-tail entropy may be bounded from above as follows,

$$h^*(f,\varepsilon) \leq \frac{\log 2}{p_\varepsilon}$$

where p_ε is the largest integer p such that the minimal length of f^p-monotone branches, $L(f^p)$, is larger than ε.

Indeed as in the previous proof of Theorem D, one easily sees that for any multimodal maps g we have $h^*(g,\varepsilon) \leq \log 2$ for all $\varepsilon < L(g)$. Then by applying this fact to f^{p_ε} and Lemma 3.4 we get

$$\begin{aligned} h^*(f,\varepsilon) &\leq h^*(f^{p_\varepsilon},\varepsilon)/p_\varepsilon \\ &\leq \log 2/p_\varepsilon. \end{aligned}$$

For the tent map, $T(x) = 2\max(x, 1-x)$, one easily gets that p_ε is the integer part of $|\log\varepsilon|/\log 2$ and therefore $h^*(T,\varepsilon) \leq 1/|\log\varepsilon|$ (one can also prove as in Theorem G that $h^*(T,\varepsilon) \geq \log 2/|\log\varepsilon|$). However it seems quite hard to estimate p_ε for general multimodal maps.

Proof of Theorem G. To simplify the exposition we assume f is a C^2 unimodal map with a nondegenerate critical point c (of order 2) and $\Lambda = P$ is a hyperbolic repelling fixed point. We call a 2-horseshoe for f^p a pair of two closed disjoint intervals J_0, J_1 such that $f^p(J_k) \supset J_0 \cup J_1$ for $k = 0, 1$. It is well known that the f^p-invariant set associated to $J_0 \cup J_1$ is conjugated to the 2-shift. In particular if $f^l(J_0), f^l(J_1)$ have diameter less than ε for all $l = 0, ..., p$ it will imply that $h_{\mathrm{loc}}(\mu,\varepsilon) \geq \log 2/p$ with μ a measure of maximal entropy of this horseshoe and therefore $h_{\mathrm{loc}}(f,\varepsilon) \geq \log 2/p$. We will prove for any $\varepsilon > 0$ the existence of such a 2-horseshoe for f^p with $p \leq C|\log\varepsilon|$. The presence of a horseshoe for interval maps with an homoclinic tangency has previously been studied by Block in [4].

We let I_ε be the maximal neighborhood of c in $[c-\varepsilon, c+\varepsilon]$ such that the two connected components of f are mapped by f on the same interval. Note that I_ε is of the form either $[c-\varepsilon', c+\varepsilon]$ or $[c-\varepsilon, c+\varepsilon']$ with $\varepsilon' \leq \varepsilon$. For ε small enough $f^k I_\varepsilon$ has P on its boundary (recall $f^k(c) = P$ and c is a local extremum of f) and its length is of order ε^2 as $f''(c) \neq 0$. As c belongs to the unstable manifold of P we may also choose ε so small that $I_\varepsilon \subset W^u(P)$ and then l large enough such that $f^{-l}(I_\varepsilon) \subset f^k I_\varepsilon$. For all integers n we have $f^{-n}(I_\varepsilon) \in B(P, C'e^{-n\lambda(P)/2})$ with $e^{\lambda(p)} = |f'(P)| > 1$ so it is enough to take $l = C''|\log\varepsilon|$ for some constant C'' independent of ε. Then one can take $\delta_0, \delta_1 > 0$ small enough such that the two connected components $I_\varepsilon \setminus [c-\delta_0, c+\delta_1]$ have the same image by f and $f^{-l}(I_\varepsilon) \subset f^k(I_\varepsilon \setminus [c-\delta_0, c+\delta_1])$. This defines a 2-horseshoe for f^{k+l}. For a general hyperbolic repeller one uses Lemma 6.1. It will be explained in details in the next section for surface diffeomorphisms. As the argument is the same we do not reproduce it here. $\square$

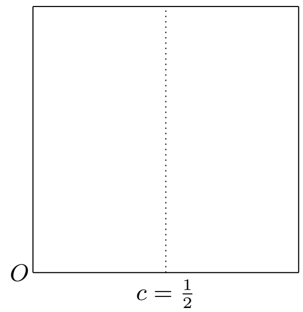

FIGURE 1. $f(x) = 4x(1-x)$ with a homoclinic tangency $c = \frac{1}{2}$

Proof of Proposition 2.2. We consider the quadratic map f_4, $f_4(x) = 4x(1-x)$. We assign to any f_4^n-monotone branch I_n an element $a(I_n)$ of $\{0,1\}^n$, as follows $(a(I_n))_k = 0$ if $f^k(I_n) \subset [0,1/2]$ and $(a(I_n))_k = 1$ if not. We also let $x(I_n)$ be the center of I_n. We consider the subshift Y_p of finite type of $\{0,1\}^{\mathbb{N}}$ where we have forbidden the word $\underbrace{010...0}_{p}$ which correspond to the f^p-monotone branch with the critical point $1/2$ on its right boundary. This f^p-monotone branch has length $\varepsilon := \varepsilon_p$ with $|\log \varepsilon_p| \underset{p}{\sim} p \log 2$: indeed the length of the f^{p-1}-monotone branch associated to $\underbrace{10...0}_{p-1}$ has length ε'_p with $|\log \varepsilon'_p| \underset{p}{\sim} p \log 4$ and the tangency at the critical point is quadratic. We also let $Y_p(n)$ be the set of words of length n in Y_p. Clearly $\{x(I_n),\ a(I_n) \in Y_p\}$ is (n, ε) separated. Therefore

$$\begin{aligned} h(f_4, \varepsilon) &\geq \limsup \frac{1}{n} \log \sharp Y_p(n); \\ &\geq h(\sigma, Y_p). \end{aligned}$$

Finally we have

$$\begin{aligned} h(\sigma, Y_p) &= h(\sigma^p, Y_p)/p; \\ &= \log(2^p - 1)/p; \\ &= \log 2 - \frac{1}{p2^p} + o(\frac{1}{p2^p}). \end{aligned}$$

We conclude that $h(f_4) - h(f_4, \varepsilon) = \log 2 - h(f_4, \varepsilon) = o\left(\frac{\varepsilon^\alpha}{|\log \varepsilon|}\right)$ for any $\alpha < 1$.

5. Modulus of continuity of the topological entropy : proof of Proposition 2.5 and some examples

*P*roof of Proposition 2.5: For any $\varepsilon > 0$ and any $g \in G$ we have

$$\begin{aligned} h(g) &\leq h(g,\varepsilon) + h_{\mathrm{loc}}(g,\varepsilon); \\ &\leq h(g,\varepsilon) + h^G_{\mathrm{loc}}(\varepsilon); \\ &\leq \frac{1}{p_\varepsilon} \log r_{p_\varepsilon}(g,\varepsilon/2) + h^G_{\mathrm{loc}}(\varepsilon). \end{aligned}$$

Now one easily checks by induction on k that $d(f^k,g^k) \leq d(f,g)\sum_{l=0}^{k-1} M_0^l \leq \varepsilon/4$ for any $k = 1,\cdots,p_\varepsilon$ once we have $d(f,g) \leq \frac{\varepsilon}{4} M_0^{-p_\varepsilon}$. Indeed for all $x \in M$ we have

$$d(f^k x, g^k x) \leq d(gf^{k-1}x, g^k x) + d(f^k x, gf^{k-1}x).$$

and then by induction hypothesis

$$\begin{aligned} d(f^k x, g^k x) &\leq M_0 d(f^{k-1}x, g^{k-1}x) + d(f,g); \\ &\leq d(f,g)\sum_{l=0}^{k-1} M_0^l; \\ &\leq \frac{\varepsilon}{4} M_0^{-p_\varepsilon} \frac{M_0^{p_\varepsilon}-1}{M_0-1} \leq \frac{\varepsilon}{4}. \end{aligned}$$

In this case we have then $r_{p_\varepsilon}(g,\varepsilon/2) \leq r_{p_\varepsilon}(f,\varepsilon/4)$ and finally we obtain according to the choice of p_ε :

$$\begin{aligned} h(g) &\leq \frac{1}{p_\varepsilon} \log r_{p_\varepsilon}(f,\varepsilon/4) + h^G_{\mathrm{loc}}(\varepsilon); \\ &\leq h(f,\varepsilon/4) + 2h^G_{\mathrm{loc}}(\varepsilon); \\ &\leq h(f) + 2h^G_{\mathrm{loc}}(\varepsilon). \end{aligned}$$

This concludes the proof of Proposition 2.5.

A continuous dynamical system f is said to satisfy the property (P) if for ε small enough we have

$$\frac{1}{n}\log r_n(f,\varepsilon) - h(f,\varepsilon) \simeq \frac{|\log\varepsilon|}{n},$$

i.e. there exists $C > 1$ and $\zeta(f) > 0$ such that for all $\zeta(f) > \varepsilon > 0$ and for all integers n we have

$$\frac{|\log\varepsilon|}{Cn} \leq \frac{1}{n}\log r_n(f,\varepsilon) - h(f,\varepsilon) \leq \frac{C|\log\varepsilon|}{n}.$$

One easily sees this is the case of the following zero topological entropy systems : the identity map, translation maps, interval and circles homeomorphisms,... Yomdin also proved in [45] that a polynomial of degree d on a compact invariant set of $\mathbb{R}^2$ of maximal entropy $\log d$ also satisfies this property.

Question. What are the dynamical systems satisfying property (P)? Does it contains a large class of systems?

We will study the modulus of continuity of the topological entropy for systems in $\mathcal{C}^{\mathcal{M}}$ C^0-close to a system satisfying the property (P). To simplify we will only consider surface V-ultradifferentiable maps and the limit case in Theorem C, $M_k = M_0 k^{k^2}$ for all integers k where M_0 is some fixed real number larger than e.

Corollary 5.1. *Let (f, M) be a continuous dynamical system satisfying property (P) with M a smooth compact surface. Then there exists a constant $C = C(f)$, such that for all $0 < \varepsilon < 1$ and for all $g \in C_V^{(M_0 k^{k^2})_k}(M)$ with $d_{C^0}(f, g) \leq \varepsilon$:*

$$h(g) \leq h(f) + C \log M_0 \sqrt{\frac{\log|\log \varepsilon|}{|\log \varepsilon|}}.$$

Proof. With the notation of Proposition 2.5 we have $p_\varepsilon \simeq |\log(\varepsilon/4)|/h^G_{loc}(\varepsilon)$. We assume now $G = C_V^{\mathcal{M}}(M)$ with $M_k = M_0 k^{k^2}$ for all $k \in \mathbb{N}$. Then by Theorem A, we can take $h^G_{loc}(\varepsilon) = C_1 \log M_0 \frac{\log|\log \varepsilon|}{|\log \varepsilon|}$ for some universal constant C_1. Thus, with $\delta_\varepsilon := \frac{\varepsilon}{4} M_0^{-p_\varepsilon}$, we have clearly $|\log \delta_\varepsilon| \simeq \frac{|\log \varepsilon|^2}{\log|\log \varepsilon|}$ and $\log|\log \delta_\varepsilon| \simeq \log|\log \varepsilon|$. It follows that

$$\begin{aligned}
h^G_{loc}(\varepsilon) &\leq C \log M_0 \frac{\log|\log \varepsilon|}{|\log \varepsilon|}; \\
&\leq C \log M_0 \frac{\log|\log \delta \varepsilon|}{\sqrt{|\log \delta_\varepsilon| \times \log|\log \delta_\varepsilon|}}; \\
&\leq C \log M_0 \sqrt{\frac{\log|\log \delta_\varepsilon|}{|\log \delta_\varepsilon|}},
\end{aligned}$$

for some $C = C(f)$. Therefore for $g \in C^{\mathcal{M}}(M)$ with $d(f, g) \leq \delta$ we get by applying Proposition 2.5

$$h(g) \leq 2C \log M_0 \sqrt{\frac{\log|\log \delta|}{|\log \delta|}}.$$

□

6. C^2 ROBUST EXAMPLES

In this section, we construct non h-expansive C^r ($r \geq 2$) open domains associated with homoclinic tangencies to prove Theorem H.

6.1. **Structure of hyperbolic sets.** We first make some definitions. Fix $f \in \mathrm{Diff}^r(M)$ with $r \geq 1$. Let $\Lambda \subset M$ be an f-invariant set. We call Λ a hyperbolic set for f if there exist $\lambda_0 \in (0, 1)$, $C > 0$, and a Df-invariant decomposition $T_\Lambda M = E^s \oplus E^u$ such that

$$\|D_x f^n v\| \leq C \lambda_0^n \|v\|, \quad \text{for any } n \geq 0,\ v \in E^s(x),\ x \in \Lambda;$$

$$\|D_x f^{-n} v\| \leq C \lambda_0^n \|v\|, \quad \text{for any } n \geq 0,\ v \in E^u(x),\ x \in \Lambda.$$

At most taking a suitable equivalent metric, we can assume $C = 1$ in above definition. Λ is further called a basic set if

- Λ is transitive: there exists $x \in \Lambda$ whose orbit is dense in Λ;
- Λ is isolated: there exists a neighborhood U of Λ such that
$$\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda.$$
Here U is called an adapted neighborhood of Λ.

For a hyperbolic set Λ, given a point $x \in \Lambda$, there exist C^r injectively immersed sub-manifolds $W^s(x)$ and $W^u(x)$ given by
$$W^s(x) = \{y \in M : d(f^n(y), f^n(x)) \to 0 \text{ as } n \to +\infty\}$$
and
$$W^u(x) = \{y \in M : d(f^{-n}(y), f^{-n}(x)) \to 0 \text{ as } n \to +\infty\},$$
see for example, Theorem 3.2 in [22]. Here $W^s(x)$, $W^u(x)$ are called the stable manifold and the unstable manifold at x, respectively. Furthermore, the stable manifold of size $\delta > 0$ is defined by
$$W^s_\delta(x) = \{y \in M : d(f^n(y), f^n(x)) \le \delta \text{ for all } n \ge 0\}.$$
Similarly, one can define the unstable manifold of size δ as $W^u_\delta(x)$ by considering f^{-1}.

For $x, y \in \Lambda$, and a point $z \in W^u(x) \cap W^s(y)$, we call z is a transversal intersection point if
$$T_z W^u(x) \oplus T_z W^s(x) = T_z M.$$
Conversely, a non transversal intersection point is called a tangency.

A periodic point p of f is a point such that there is a positive integer n with $f^n(p) = p$, where n is called a period of p. The periodic point p is hyperbolic if all eigenvalues of the derivative $Df^n(p)$ have modulus different from 1. In fact, a periodic point p is hyperbolic if and only if the orbit of p is a hyperbolic basic set. More generally, we say that an f-invariant set Λ is periodic if there exist a subset $\Lambda_1 \subset \Lambda$ and a positive integer n such that

- $f^n(\Lambda_1) = \Lambda_1$,
- $\Lambda = \bigcup_{0\le i<n} f^i(\Lambda_1)$.

In this case, we call n to be a period of Λ, and Λ_1 to be a base of Λ. Denote the diameter of Λ in the base Λ_1 by
$$\operatorname{diam}_{\Lambda_1}(\Lambda) = \max_{0\le i<n} \operatorname{diam}(f^i(\Lambda_1)).$$

By the uniform hyperbolicity of Λ, there exist $\varepsilon_0, \delta_0 > 0$, $\lambda \in (0,1)$ such that

- $$\begin{aligned} d(f^n(y), f^n(z)) &\le \lambda^n d(y,z), \quad \text{for all } n \ge 0,\ y, z \in W^s_{\varepsilon_0}(x),\, x \in \Lambda; \\ d(f^{-n}(y), f^{-n}(z)) &\le \lambda^n d(y,z), \quad \text{for all } n \ge 0,\ y, z \in W^u_{\varepsilon_0}(x),\, ,x \in \Lambda; \end{aligned}$$
- $W^s_{\varepsilon_0}(x) \cap W^u_{\varepsilon_0}(y)$ contains a single point $[x, y]$ whenever $d(x, y) < \delta_0$. Furthermore, the function
$$[\cdot,\cdot] : \{(x,y) \in M \times M \mid d(x,y) < \delta_0\} \to M$$
is continuous.

A rectangle R is understood by a subset of M with diameter smaller than ε_0 such that $[x,y]\in R$ whenever $x,y\in R$. For $x\in R$ let

$$W^s(x,R)=W^s_{\varepsilon_0}(x)\cap R \quad \text{and} \quad W^u(x,R)=W^u_{\varepsilon_0}(x)\cap R.$$

For a hyperbolic basic set Λ, one can obtain the following structure known as a Markov partition $\mathcal{R}=\{R_1,R_2,\cdots,R_l\}$ of Λ with properties:

(i) $\operatorname{Int} R_i\cap \operatorname{Int} R_j=\emptyset$ for $i\neq j$;

(ii) $fW^u(x,R_i)\supset W^u(fx,R_j)$ and
$fW^s(x,R_i)\subset W^s(fx,R_j)$ when $x\in \operatorname{Int} R_i,\ fx\in \operatorname{Int} R_j$,

See Bowen[7]. Using the Markov Partition $\mathcal{R}$ one can define the transition matrix $A=A(\mathcal{R})$ by

$$A_{i,j}=\begin{cases}1 & \text{if} \quad \operatorname{Int} R_i\cap f^{-1}(\operatorname{Int} R_j)\neq\emptyset;\\ 0 & \text{otherwise.}\end{cases}$$

The subshift (Σ_A,σ) associated with A is given by

$$\Sigma_A=\{\underline{q}\in\Sigma_l \mid A_{q_i,q_{i+1}}=1 \quad \forall i\in\mathbb{Z}\}.$$

For each $\underline{q}\in\Sigma_A$, the set $\cap_{i\in\mathbb{Z}}f^{-i}R_{q_i}$ contains of a single point, which we denote by $\pi_0(\underline{q})$. We define

$$\Sigma_A(i)=\{\underline{q}\in\Sigma_A\mid q_0=i\}.$$

The following properties hold for the map π_0 (see Theorem 28 of [7]):

(i) The map $\pi_0:\Sigma_A\to\Lambda$ is a continuous surjection satisfying $\pi_0\circ\sigma=f\circ\pi_0$;

(ii) $\pi_0(\Sigma_A(i))=R_i\cap\Lambda, \quad 1\le i\le l.$

Since Λ is a hyperbolic basic set, by Smale's Spectral Decomposition Theorem [40], there exists $n_0\in\mathbb{N}$ such that

$$\Lambda=\Lambda_1\cup\cdots\cup\Lambda_{n_0}, \quad \Lambda_i\cap\Lambda_j=\emptyset,\ 1\le i<j\le n_0,$$

$$f^i(\Lambda_1)=\Lambda_{1+i},\ 1\le i\le n_0-1, \quad f^{n_0}(\Lambda_1)=\Lambda_1.$$

Moreover, f^{n_0} is mixing in Λ_1, i.e., given pairs of open sets U_1,U_2 with nonempty intersections with Λ_1, $\exists\, n_1\in\mathbb{N}$, s.t. $f^{n_0n_1}(U_1)\cap U_2\neq\emptyset$, $\forall\, n\ge n_1$. Equivalently to say here, for the transition matrix B of a Markov partition $\mathcal{R}$ for $f^{n_0}|_{\Lambda_1}$, one can find $n_1\in\mathbb{N}$ such that all elements of the matrix B^{n_1} are positive.

Lemma 6.1. *There exists $\varepsilon_1>0$ such that for any $\varepsilon\in(0,\varepsilon_1)$ and $x_1,x_2\in\Lambda$, one can find a periodic point $p\in\Lambda$ with a period $\tau(p)\in[2|\log\varepsilon|/|\log\lambda|,9|\log\varepsilon|/|\log\lambda|]$ such that*

$$d(p,x_1)\le\varepsilon, \quad d(f^i(p),x_2)\le\varepsilon \quad \textit{for some} \ \ i\in[0,\tau(p)].$$

Proof. For $x_1,x_2\in\Lambda$, we can choose $m_1,m_2\in[0,n_0-1]$ such that

$$y_1:=f^{-m_1}(x_1)\in\Lambda_1, \quad y_2:=f^{-m_2}(x_2)\in\Lambda_1.$$

Let $g=f^{n_0}$. Take $\underline{q},\underline{q}'\in\Sigma_B$ with $y_1=\cap_{i\in\mathbb{Z}}g^i(R_{q_i})$, $y_2=\cap_{i\in\mathbb{Z}}g^i(R_{q'_i})$. Since all elements of the matrix B^{n_1} are positive, for any $n\ge n_1$ there exists a sequence $i_1,i_2,\cdots,i_{n_1-1},\, i'_1,i'_2,\cdots,i'_{n_1-1}$ such that

$$B_{q_n,i_1}B_{i_1,i_2}\cdots B_{i_{n_1-2},i_{n_1-1}}B_{i_{n_1-1},q'_{-n}}>0,\ B_{q'_n,i'_1}B_{i'_1,i'_2}\cdots B_{i'_{n_1-2},i'_{n_1-1}}B_{i'_{n_1-1}q_{-n}}>0$$

which imply the following periodic point is contained in Σ_B:

$$w := [q_{-n}, \cdots, q_{-1}, \overset{0}{q_0}; q_1, \cdots, q_n, i_1, \cdots, i_{n_1-1}, q'_{-n}, \cdots, q'_0, q'_1, \cdots, q'_n, i'_1, \cdots, i'_{n_1-1}].$$

Let $p = \cap_{i\in\mathbb{Z}} g^i(R_{w_i})$, which is a periodic point of g with a period $4n+2n_1$. Then for each $i \in [-n, n]$, $g^i(p), g^i(y_1)$ belong to the same rectangle in the Markov partition $\mathcal{R}$. Also, $g^i(g^{2n+n_1}(p)), g^i(y_2)$ belong to the same rectangle of $\mathcal{R}$. They imply

$$d(\pi^{s/u}_{g^i(y_1)}(g^i(p)), g^i(y_1)) \le \varepsilon_0, \quad d(\pi^{s/u}_{g^i(y_2)}(g^i(g^{2n+n_1}(p))), g^i(y_2)) \le \varepsilon_0,$$

for $i \in [-n, n]$, where $\pi^{s/u}_x(z)$ denotes the intersection point of $W^{u/s}_{\varepsilon_0}(z)$ and $W^{s/u}_{\varepsilon_0}(x)$. By the uniform hyperbolicity of Λ,

$$d(\pi^{s/u}_{y_1}(p), y_1) \le \varepsilon_0 \lambda^{nn_0}, \quad d(\pi^{s/u}_{y_2}(g^{2n+n_1}(p)), y_2) \le \varepsilon_0 \lambda^{nn_0}.$$

Note that there exists $C_0 > 0$ such that $d(x, z) \le C_0 \max(d(\pi^s_x(z), x), d(\pi^u_x(z), x))$ for any z with $d(z, x) \le \delta_0$, $x \in \Lambda$. We deduce

$$d(p, y_1) \le C_0\varepsilon_0\lambda^{nn_0}, \quad d(g^{2n+n_1}(p), y_2) \le C_0\varepsilon_0\lambda^{nn_0}.$$

Choose $\varepsilon_1 > 0$ such that $\max\{|\log(C_0\varepsilon_0)| + n_0 \log\|Df\|,\ n_0|\log\lambda|\} \le \frac{1}{4}|\log\varepsilon_1|$. For any $\varepsilon \in (0, \varepsilon_1)$, let

$$n = \Big\lfloor \frac{|\log\varepsilon - \log(C_0\varepsilon_0) - n_0\log\|Df\||}{n_0|\log\lambda|} \Big\rfloor + 1 \in \Big[\frac{|\log\varepsilon|}{2n_0|\log\lambda|}, \frac{3|\log\varepsilon|}{2n_0|\log\lambda|}\Big].$$

Observe that w has a period $4n + 2n_1 \in [4n, 6n]$. Then p as a periodic point of f has a period

$$\tau(p) \in [4n_0n, 6n_0n] \subset \Big[\frac{2|\log\varepsilon|}{|\log\lambda|}, \frac{9|\log\varepsilon|}{|\log\lambda|}\Big]$$

and

$$\max\Big\{d(p, y_1),\ d(f^{n_0(2n+n_1)}(p), y_2)\Big\} \le C_0\varepsilon_0\lambda^{nn_0} \le \varepsilon\|Df\|^{-n_0}.$$

Hence,

$$\begin{aligned} d(f^{m_1}(p), x_1) &= d(f^{m_1}(p), f^{m_1}(y_1)) \le \|Df\|^{m_1} d(p, y_1) \le \varepsilon \\ d(f^{m_2+n_0(2n+n_1)}(p), x_2) &= d(f^{m_2+n_0(2n+n_1)}(p), f^{m_2}(y_2)) \\ &\le \|Df\|^{m_2} d(f^{n_0(2n+n_1)}(p), y_2) \le \varepsilon. \end{aligned}$$

Moreover, $0 \le m_2 + n_0(2n + n_1) - m_1 \le 4n_0n \le \tau(p)$. The proof of Lemma 6.1 is completed. □

The following Proposition states that the uniformly hyperbolic structure holds in a persistent way.

Proposition 6.2. *Let $\Lambda = \Lambda(f)$ be a hyperbolic basic set for the C^1 diffeomorphism f on M with adapted neighborhood U. Given $C > 0$, there is a neighborhood $\mathcal{N}_C$ of f in $\mathrm{Diff}^1(M)$ such that if $g \in \mathcal{N}_C$, then $\Lambda(g) = \cap_{n\in\mathbb{Z}} g^n(U)$ is a hyperbolic basic set for g and there is a unique continuous embedding $h_g : \Lambda(f) \to M$ such that $h_g(\Lambda(f)) = \Lambda(g)$, $g \circ h_g = h_g \circ f$ and $d(h_g, \mathrm{id}) < C$. Moreover, $h_f = \mathrm{id}$.*

6.2. **Thickness of Cantor sets.** Let K be a cantor set, i.e., a compact perfect totally disconnected subset of $\mathbb{R}$. Let K_0 be the smallest closed interval containing K. Then $K_0 - K = \cup_{i=0}^{\infty} U_i$, where $U_i \cap U_j = \emptyset$ if $i \neq j$ and each U_i is a bounded open interval. Let U_{-2}, U_{-1} be the unbounded components of $\mathbb{R} \setminus K$. All U_i, $i \geq -2$, are called the gaps of K. For any $i \geq 1$, define $K_i = K_0 \setminus (\cup_{0 \leq j \leq i-1} U_j)$. Then

$$K_0 \supseteq K_1 \supseteq \cdots \supseteq \cdots .$$

Each K_i is a union of closed intervals and $K = \cap_{i \geq 0} K_i$. We call $\{K_i\}_{i \geq 0}$ to be a defining sequence for K. For $i \geq 1$, let K_i^* be the connected component of K_i containing U_i, then $K_i^* \setminus U_i$ is the union of two closed intervals I_i^l, I_i^r.

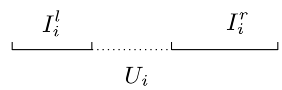

Figure 2. Remove open intervals

For an interval I, denote by $|I|$ the length of I. Set

$$\tau(\{K_i\}) = \inf_{i \geq 0} \Big\{ \min(\frac{|I_i^l|}{|U_i|}, \frac{|I_i^r|}{|U_i|}) \Big\}.$$

The thickness of K is defined by

$$\tau(K) = \sup\{\tau(\{K_i\}) : \{K_i\} \text{ is a defining sequence for } K\}.$$

Lemma 6.3 (Gap lemma, Lemma 4 of [32])**.** *Let K, F be two cantor sets with thicknesses τ_1, τ_2. If $\tau_1 \cdot \tau_2 > 1$, then one of the following alternatives occurs:*

- *K is contained in a gap closure of F;*
- *F is contained in a gap closure of K;*
- *$K \cap F \neq \emptyset$. In this case, for any defining sequences $\{K_i\}$ of K, $\{F_i\}$ of F with $\tau(\{K_i\}) \cdot \tau(\{F_i\}) > 1$, it holds that $\mathrm{Int}(K_i \cap F_i) \neq \emptyset$ for any i.*

Let Λ be a hyperbolic basic set of $f \in \mathrm{Diff}^2(M)$ and p be a periodic point of f. We can parameterize $W^s(p)$ and $W^u(p)$ such that $f|_{W^s(p)}$ and $f|_{W^u(p)}$ are linear, see [41]. We define the unstable thickness of (Λ, p) as $\tau^u(\Lambda, p) = \tau(W^s(p) \cap \Lambda)$, the stable thickness of (Λ, p) as $\tau^s(\Lambda, p) = \tau(W^u(p) \cap \Lambda)$. Observe that $W^s(p) \cap \Lambda$ is f-invariant, and $f|_{W^s(p)}$ is linear, there exist arbitrarily small compact neighborhoods K of p in $W^s(p) \cap \Lambda$ such that $\tau(K) = \tau(W^s(p) \cap \Lambda) = \tau^u(\Lambda, p)$. The same argument applies to $\tau^s(\Lambda, p)$. It can be shown that $\tau^{s/u}(\Lambda, p)$ is independent of p (Proposition 5 in [32]). We denote $\tau^{s/u}(\Lambda) = \tau^{s/u}(\Lambda, p)$. By Proposition 6.2, the persistence of Λ holds in a C^1 neighborhood $\mathcal{N}_1$ of f. Furthermore,

Proposition 6.4 (Proposition 6 in [32] or Theorem 2 of Chapter 4.3 in [35])**.** *There exists a C^2 neighborhood $\mathcal{N}_2 \subset \mathcal{N}_1$ of f such that the thicknesses $\tau^{s/u}(\Lambda(g))$ depend continuously for $g \in \mathcal{N}_2$.*

6.3. **Small Horseshoes.** Let Λ_0 be a hyperbolic basic set of $f \in \mathrm{Diff}^r(M)$ whose stable manifolds and unstable manifolds tangent at some point. Then by Lemma 7 and Lemma 8 of [32] we can at most by a C^r perturbation let f have a hyperbolic basic set Λ satisfying $\tau^s(\Lambda)\cdot\tau^u(\Lambda) > 1$ and containing a periodic point $p \in \Lambda$ with a tangency x_0 of $W^u_f(p)$ and $W^s_f(p)$. By Proposition 6.4, there exists a C^r neighborhood $\mathcal{N}_2$ of f such that $\tau^s(\Lambda(g))\cdot\tau^u(\Lambda(g)) > 1$ for $g \in \mathcal{N}_2$.

For each $g \in \mathcal{N}_2$, take a C^1 stable foliation $\mathcal{F}^s_g(U_1)$ in a neighborhood U_1 of $\Lambda(g)$ such that for $x \in \Lambda(g)$, the leave $\mathcal{F}^s_g(x)$ is a subset of $W^s_g(x)$. $\mathcal{F}^s_g(U_1)$ varies continuously with respect to $g \in \mathcal{N}_2$. Similarly, we have a C^1 unstable foliation $\mathcal{F}^u_g(U_1)$. See the constructions of stable and unstable foliations in Section 3, Chapter 2 of [35].

For the tangency point $x_0 \in W^u_f(p)\cap W^s_f(p)$, $T_{x_0}W^u_f(p) = T_{x_0}W^s_f(p)$. We let the tangency at x_0 is quadratic (like $y = ax^2$ near the tangency point). Otherwise we can obtain this with an arbitrarily C^r small perturbation. Denote

$$L = \max\left\{d_s(p, x_0), d_u(p, x_0)\right\}$$

where $d_{s/u}$ are the distances in the leaves of $\mathcal{F}^{s/u}$. So, for g C^r close to f, we can take a C^1 line $l(g)$ near x_0 consisting of tangencies of $\mathcal{F}^s_g(U)$ and $\mathcal{F}^u_g(U)$ with the transversal property:

$$T_x l(g) \oplus T_x\mathcal{F}^u_g(x) = T_x M, \quad \forall\, x \in l(g).$$

Now for small $\delta_2 > \delta_1 > 0$, and g C^r close to f, define projections

$$\begin{aligned} \pi_1(g) : W^s_{\delta_1}(p(g)) &\to l(g), \\ \pi_2(g) : W^u_{\delta_1}(p(g)) &\to l(g) \end{aligned}$$

which project along leaves of $\mathcal{F}^u_g(y, L+\delta_2)$ and $\mathcal{F}^s_g(y, L+\delta_2)$, $y \in W^{s/u}_{\delta_1}(p(g))$, where $\mathcal{F}^{u/s}_g(y, a)$ denote the a-disc centered at y in the leaves $\mathcal{F}^{u/s}_g(y)$. Here $\pi_1(g), \pi_2(g)$ are C^1 and continuous in g.

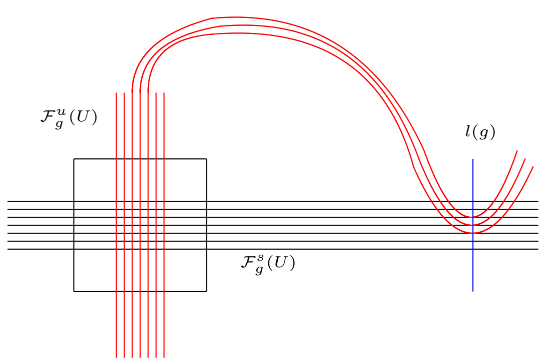

Figure 3. Interval of tangencies

Observing that Λ is uniformly hyperbolic, there exists $L_0 > 0$ and $\lambda \in (0,1)$, such that

$$d(g^n(x_1), g^n(x_2)) \le \lambda^n d(x_1, x_2), \quad \text{for all } n \ge 0,\ \forall\, x_1, x_2 \in W^s_{L_0}(x),\ x \in \Lambda(g),$$

$$d(g^{-n}(x_1), g^{-n}(x_2)) \le \lambda^n d(x_1, x_2), \quad \text{for all } n \ge 0,\ \forall\, x_1, x_2 \in W^u_{L_0}(x),\ x \in \Lambda(g).$$

Since L is fixed, we can take $N \in \mathbb{N}$ and $a_0 > 0$ such that for any $\delta \in (0, L + \delta_1)$,

- $\operatorname{diam}^s(g^n(B^s(x,\delta))) < a_0\delta g^N(B^s(x,\delta)) \subset W^s_{L_0}(g^N(y))$, for all $0 \le n \le N$, for all x in $W^s_{L+\delta_1}(y)$ and for all $y \in \Lambda(g)$,
- $\operatorname{diam}^u(g^{-n}(B^u(x,\delta))) < a_0\delta$ and $g^{-N}(B^u(x,\delta)) \subset W^u_{L_0}(g^{-N}(y))$, for all $0 \le n \le N$, for all x in $W^u_{L+\delta_1}(y)$ and for all $y \in \Lambda(g)$,

where $B^{s/u}(z,\delta)$ are the balls in $W^{s/u}(y)$ centered at z with radius δ; $\operatorname{diam}^{s/u}$ are the diameters along s/u-leaves. Consequently, we have

$$\operatorname{diam}^s(g^n(B^s(x,\delta))) < a_0\delta, \quad \text{for all } n \ge 0,\ \forall\, x \in W^s_{L+\delta_1}(y),\ y \in \Lambda(g),$$

$$\operatorname{diam}^u(g^{-n}(B^u(x,\delta))) < a_0\delta, \quad \text{for all } n \ge 0,\ \forall\, x \in W^u_{L+\delta_1}(y),\ y \in \Lambda(g).$$

We give $l(g)$ an orientation so that we can say up-side and below-side in $l(g)$. Without loss of generality, we suppose the leaves of $\mathcal{F}^s_g$ near $l(g)$ are horizontal. Noting that the tangency x_0 is quadratic, we can see all leaves of $\mathcal{F}^u$ bent upwardly nearby $l(g)$. Thus, there is $a_1 > 0$ such that for any $z_1 \in l(g)$ and $z_2 \in l(g)$ below z_1, the nearby two intersections of $\mathcal{F}^s(z_1)$ and $\mathcal{F}^u(z_2)$ are contained in a ball with radius $a_1\sqrt{d(z_1, z_2)}$.

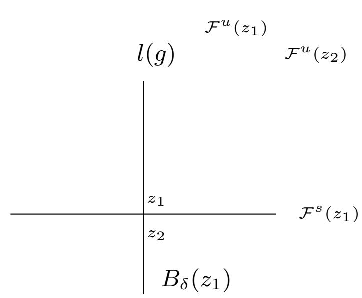

FIGURE 4. The size of the transversal intersection

Let $K_1(g)$, $K_2(g)$ be small compact one side neighborhoods of $p(g)$ in $W^s(p(g)) \cap \Lambda(g)$ and $W^u(p(g)) \cap \Lambda(g)$, depending continuously on g, and such that

$$\tau(K_1(g)) = \tau^u(\Lambda(g)), \quad \tau(K_2(g)) = \tau^s(\Lambda(g)).$$

Define

$$l_i(g) = (\pi_i(g))(K_i(g)), \quad i = 1, 2.$$

We can take $K_1(g)$ and $K_2(g)$ small so that

$$\frac{\|D_x\pi_i(g)\|}{\|D_y\pi_i(g)\|} \text{ close to } 1, \quad \text{for } x, y \in K_i,\ i = 1, 2,$$

which implies that

$$\tau(l_i(g)) \text{ close to } \tau(K_i(g)), \quad \text{for } i = 1, 2.$$

Hence, together with $\tau(K_1(g)) \cdot \tau(K_2(g)) > 1$, we have

$$\tau(l_1(g)) \cdot \tau(l_2(g)) > 1.$$

For two Cantor sets Y_1, Y_2, let $I_1 = [s_1, s_2], I_2 = [t_1, t_2]$ be minimal closed intervals such that $I_1 \supseteq Y_1, I_2 \supseteq Y_2$. We say Y_1, Y_2 are linked if I_1, I_2 are linked, i.e., $s_1 < t_1 < s_2 < t_2$ or $t_1 < s_1 < t_2 < s_2$. Since $l_1(f)$ and $l_2(f)$ has a boundary point in common, so taking a small perturbation, there exists a C^2 open set $\mathcal{N} \subset \mathcal{N}_2$ such that $l_1(g)$ and $l_2(g)$ are linked and $\tau(l_1(g)) \cdot \tau(l_2(g)) > 1$, $\forall\, g \in \mathcal{N}$. By Lemma 6.3, the third case of Lemma 6.3 is satisfied, which implies the existence of a tangency $z_0 \in l(g)$ of $\mathcal{F}^u_g(x_0, L + \delta_1)$ and $\mathcal{F}^s_g(y_0, L + \delta_1)$ for some $x_0 \in K_1(g)$, $y_0 \in K_2(g)$. Moreover, one of the following two cases happens:

(i) there exist $u_i \in l_1(g)$ below z_0 with $u_i \to z_0$ as $i \to +\infty$;

(ii) there exist $v_i \in l_2(g)$ above z_0 with $v_i \to z_0$ as $i \to +\infty$.

Otherwise, z_0 is a boundary point of both $l_1(g)$ and $l_2(g)$, contradicting that $\mathrm{Int}(F_j \cap G_j) \neq \emptyset$, $\forall\, j \in \mathbb{N}$, where $\{F_j\}, \{G_j\}$ are defining sequences of $l_1(g), l_2(g)$ with $\tau(\{F_j\})\tau(\{G_j\}) > 1$.

Lemma 6.5. *Given $\delta > 0$, there are $x_i \in K_i$, $i = 1, 2$, such that $W^u_{L+\delta_1}(x_1)$ intersects $W^s_{L+\delta_1}(x_2)$ transversally in a δ-neighborhood of $z_0 \in l(g)$ and, the two nearby intersections are contained in $B(z_0, \delta)$.*

Proof. We have assumed that $\mathcal{F}^u(z_0)$ stays on the up-side of the horizontal $\mathcal{F}^s(z_0)$ in a small neighborhood of z_0.

Corresponding to (i), there is $u_i \in l_1$ on the below-side of z_0 with $d(u_i, z_0) < (a_1^{-1}\delta)^2$, then we can take $x_1 \in K_1$ such that $u_i \in W^u_{L+\delta_1}(x_1) \cap l(g)$ and, $W^u_{L+\delta_1}(x_1)$ transversally intersects $W^s_{L+\delta_1}(y_0)$ in a δ-neighborhood of some $\widetilde{z} \in l(g)$; Let $x_2 = y_0$.

Corresponding to (ii), there is $v_i \in l_2$ on the above-side of z_0 with $d(v_i, z_0) < (a_1^{-1}\delta)^2$. The argument is similar by taking $x_1 = x_0$. □

Given $\delta > 0$, let x_1, x_2 as in Lemma 6.5, and $z_1 \in W^u_{L+\delta_1}(x_1) \cap l(g)$, $z_2 \in W^s_{L+\delta_1}(x_2) \cap l(g)$, $d(z_1, z_2) < (a_1^{-1}\delta)^2$. Since g is C^2, the two maps

$$x \in \Lambda(g) \quad \to \quad x^u \in W^u_{L+\delta_1}(x) \cap l(g),$$

$$x \in \Lambda(g) \quad \to \quad x^s \in W^s_{L+\delta_1}(x) \cap l(g)$$

are C^1 smooth and well defined in a neighborhood of x_1 and a neighborhood of x_2, respectively. We can take $a_2 > 0$ as the Lipschitz constant for the above two maps. Applying Lemma 6.1 for $\varepsilon = a_2^{-1} d(z_1, z_2)/3$, we can find a periodic point $q \in \Lambda(g)$ satisfies

- $\tau(q) \in [2|\log(a_2^{-1} d(z_1, z_2)/3)|/|\log \lambda|,\ 9|\log(a_2^{-1} d(z_1, z_2)/3)|/|\log \lambda|]$
- $d(q, x_1) \le a_2^{-1} d(z_1, z_2)/3, \quad d(g^{i_0} q, x_2) \le a_2^{-1} d(z_1, z_2)/3$ for some $i_0 \in (0, \tau(q))$.

Furthermore,

$$d(q^u, z_1) \le d(z_1, z_2)/3, \quad d((g^{i_0} q)^s, z_2) \le d(z_1, z_2)/3.$$

Hence, $W^u_{L+\delta_1}(q)$ transversally intersects $W^s_{L+\delta_1}(g^{i_0} q)$ at two points y_1, y_2 with $d(y_1, y_2) \le a_1 \sqrt{\frac{5}{3} d(z_1, z_2)}$.

We choose a rectangle centered at the origin $O := (f^i(q))^s$ as follows

$$L_{z_1,z_2} = \big\{(e_1, e_2) \mid |e_1|_s \le a_1 \sqrt{\frac{5}{3} d(z_1, z_2)},\ |e_2|_u \le \frac{d(z_1, z_2)}{10}\big\}.$$

where $|\cdot|_s$, $|\cdot|_u$ denote the distances in the horizontal axis (s-direction) and the vertical axis, respectively.

By iterations, $g^n(L_{z_1,z_2})$ will become longer along u-foliation, and narrower along s-foliation. Observe that for $d(z_1, z_2)$ sufficiently small, $\tau(q)$ will be large enough. In order to make $g^{k\tau(q)+i_0}(L_{z_1,z_2})$ as u-foliation intersect L_{z_1,z_2} as s-foliation transversally near O, we take k such that the length of the u-leaves of $g^{k\tau(q)}(L_{z_1,z_2})$ is at least $L + \delta_1$, i.e.,

$$\frac{d(z_1, z_2)}{10} \lambda^{-\tau(q)k} \ge L + \delta_1.$$

So,

$$k \le \frac{\log \frac{d(z_1,z_2)}{10} - \log(L + \delta_1)}{\tau(q) \log \lambda} \le \frac{\log \frac{d(z_1,z_2)}{10} - \log(L + \delta_1)}{2|\log(a_2^{-1} d(z_1, z_2)/3)|/|\log \lambda| \cdot \log \lambda}.$$

We can take a constant $T_1 \in \mathbb{N}$ independent of $d(z_1, z_2)$, such that $k \le T_1$. For $t = T_1\tau(q) + i_0 \in [T_1\tau(q), (T_1 + 1)\tau(q)]$, $g^t(L_{z_1,z_2})$ will intersect L_{z_1,z_2} transversally near O. Here we need to further cut the unnecessary parts outside the foliation $\mathcal{F}^u$. This is equivalent to take a sub-rectangle $L'_{z_1,z_2} \subset L_{z_1,z_2}$ with the height in the vertical direction of L_{z_1,z_2} smaller but no change on the length in the horizontal direction. To make $g^t(L'_{z_1,z_2})$ also intersect L'_{z_1,z_2} transversally near O, it is sufficient to let the length of u-leaves of $g^t(L'_{z_1,z_2})$ be $C_0\sqrt{d(z_1, z_2)}$ for some constant C_0 independent of $d(z_1, z_2)$. Therefore,

$$\begin{aligned}
\operatorname{diam}^s(g^n(L'_{z_1,z_2})) &\le a_0 \cdot \operatorname{diam}^s(L'_{z_1,z_2}) \le a_0 \cdot \sqrt{\frac{5}{3} a_1 d(z_1, z_2)},\ 0 \le n \le t; \\
\operatorname{diam}^u(g^n(L'_{z_1,z_2})) &= \operatorname{diam}^u(g^{-(t-n)} \circ g^t(L'_{z_1,z_2})) \\
&\le a_0 \operatorname{diam}^u(g^t(L'_{z_1,z_2})) \le a_0 \cdot C_0\sqrt{d(z_1, z_2)},\ 0 \le n \le t.
\end{aligned}$$

Therefore, $\operatorname{diam}(g^n(L'_{z_1,z_2})) \le C_1\sqrt{d(z_1, z_2)}$, $0 \le n \le t$, for some constant C_1 independent of $d(z_1, z_2)$.

Let

$$\Gamma_g(z_1, z_2) := \bigcap_{n\in\mathbb{Z}} g^n(L'_{z_1,z_2}).$$

Then $\Gamma_g(z_1, z_2)$ is a periodic hyperbolic basic set with period t and with diameter no more than $C_1\sqrt{d(z_1, z_2)}$. Let $\eta = C_1\sqrt{d(z_1, z_2)}$. Since $L'_{z_1,z_2} \cap g^t(L'_{z_1,z_2})$ contains two strips,

$$h(g^t, \Gamma_g(z_1, z_2)) \ge \log 2,$$

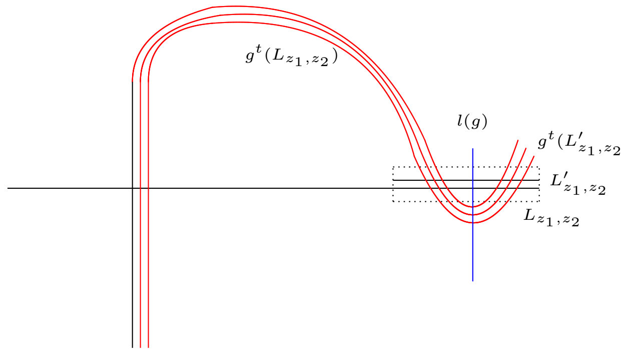

FIGURE 5. Transversal intersections

which implies for the maximal entropy measure μ supporting on $\Gamma_g(z_1, z_2)$,

$$\begin{aligned} h_{\mathrm{loc}}(g, \mu, \eta) &\geq h(g, \Gamma_g(z_1, z_2)) \geq \frac{h(g^t, \Gamma_g(z_1, z_2))}{t} \\ &\geq \frac{\log 2}{(T_1 + 1)\tau(q)} \geq \frac{\log 2}{9(T_1 + 1)|\log(a_2^{-1} d(z_1, z_2)/3)|/|\log \lambda|} \\ &\geq \frac{C_2}{|\log \eta|} \end{aligned}$$

for some constant C_2 independent of η. Hence, $h_{\mathrm{loc}}(g, \eta) \geq C_2/|\log \eta|$. Note that a_i, C_i, $i = 0, 1, 2$, can be chosen uniformly for $g \in \mathcal{N}$. The proof of Theorem H is completed.

Remark 6.6. The presence of horseshoes for surface diffeomorphisms with homoclinic tangencies has previously been studied in a qualitative way by Homburg and Weiss in [21].

□

7. C^∞ EXAMPLES WITH ARBITRARILY SLOW CONVERGENCE

We start with an analytic diffeomorphism $T : M^2 \to M^2$ on a surface M^2 with an interval I of homoclinic tangencies associated with a hyperbolic fixed point p (unstable and stable manifold at p in fact coincide by analyticity[2]). Such a T can be taken, for example, as the time one map of an analytic flow X^t on M^2 with a homoclinic orbit Γ for some hyperbolic fixed point p. One can also choose T with $\|DT(p)\| = \|DT\|_\infty$ and $\|DT\|_\infty > 1$ arbitrarily close to one (in the following we will take T with $2B \log \|DT\| < 1$ for some constant B precised later on).

Assume that in a local chart $U \supset I$ the interval I may be written as $I = [0, 1]$ and $U \supset [-3, 3]^2$. For any positive real function a with $\lim_{\varepsilon \to 0} a(\varepsilon) = 0$ (that we can assume to be nondecreasing and convex) we will construct a C^∞ map $f_a : [-1, 1] \to [0, 1]$ such that if θ_a is a C^∞ diffeomorphism of M satisfying in local coordinates

[2]One can prove by using Irwin Method [23] and the implicit function theorem in [24] that the stable and unstable manifold are in $C^{\mathcal{M}}$ once T is in $C^{\mathcal{M}}$

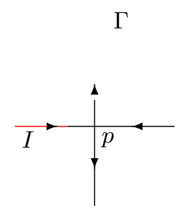

FIGURE 6. Interval I of tangencies from a homoclinic orbit Γ

- $\theta_a = Id$ outside $[-2,2]^2$;
- $\theta_a(x,y) = (x, y + f_a(x))$ in $[-1,1]^2$,

then the diffeomorphism $F_a := \theta_a \circ T$ satisfies

$$h_{loc}(F_a, \varepsilon) \geq a(\varepsilon)$$

for all $\varepsilon \leq \zeta(f_a)$ with some constant $\zeta(f_a) > 0$. The map f_a is chosen to be C^∞ flat at 0 so that F_a has a homoclinic tangency of infinite order at $(0,0)$. Moreover we may take θ_a given in the local chart by $\theta_a(x,y) = (x, y + \chi(x)\chi(|y|)f_a(x))$ where χ is a non negative C^∞ bump function, such that $\chi(t) = 1$ if $0 \leq t \leq 1$ and $\chi(t) = 0$ if $t > 2$ or $t < -1$. Such a bump function χ may be chosen in any non quasi analytic U-ultradifferentiable class. Here we choose χ in $U^{(k^{2k})_k}$. In particular χ is in $V^{(AB^k k^{2k})_k}$ for some constants A and B.

The idea introduced by Misiurewicz in [30] and developed later by Downarowicz-Newhouse [18] and Buzzi [13] and in other recent works [9],[15],[2] consists in creating arbitrarily small horseshoes accumulating at the fixed point p by choosing the graph of f looking like small snakes closer and closer to the tangency.

We describe now the main properties of the map f_a. We produce snakes only on the intervals of the form $[c_n, d_n] := \left[\frac{1}{4n+1}, \frac{1}{4n}\right]$ for all integers n. More precisely we put with $R_{a,n} > M_{a,n} > 0$ and $N_{a,n} \in \mathbb{N}$ (which we precise later on),

$$f_a = \sum_n f_n, \text{ with } f_n := \chi\left(\frac{x - c_n}{d_n - c_n}\right)\left(R_{a,n} + M_{a,n}\sin\left(N_{a,n}\frac{x - c_n}{d_n - c_n}\right)\right).$$

This sum is zero on $\mathbb{R}^-$ and it defines a C^∞ function on $\mathbb{R}^+ \setminus \{0\}$ as the terms of the sum are C^∞ function with compact disjoint supports accumulating only at 0. We let $\varepsilon = \varepsilon_n := d_n - c_n = \frac{1}{4n(4n+1)}$ and we denote $R_{a,\varepsilon} := R_{a,n}$, $M_{a,\varepsilon} := M_{a,n}$ and $f_\varepsilon := f_n$ for the integer n giving ε. We may choose $R_{a,\varepsilon}$ and $M_{a,\varepsilon}$ so that any branch of the sinusoidal in the graph of f_a crosses all the branches after a time $P_{a,\varepsilon}$ with

$$M_{a,\varepsilon} e^{\lambda_u(p) P_{a,\varepsilon}} = \varepsilon,$$

$$R_{a,\varepsilon} e^{\lambda_u(p) P_{a,\varepsilon}} \leq C,$$

where $\lambda_u(p) > 0$ is the positive Lyapunov exponent of T at p and $C = C(T)$ depends only on T.

We consider a rectangle $L_{a,\varepsilon}$ as in the proof of Theorem H p.34. Here the intersection of $L_{a,\varepsilon}$ with $f^{P_{a,\varepsilon}}L_{a,\varepsilon}$ consists in $N_{a,\varepsilon}$ strips so that the entropy of the associated horseshoe $H_{a,\varepsilon}$ is given by

$$h(H_{a,\varepsilon}) = \frac{\log N_{a,\varepsilon}}{P_{a,\varepsilon}}.$$

Note also that $H_{a,\varepsilon}$ is contained in a infinite dynamical ball of size ε. We explain now how to choose $P_{a,\varepsilon}$ and $N_{a,\varepsilon}$ with respect to ε and a. Firstly we can take $P_{a,\varepsilon} = -\log\varepsilon/a(\varepsilon)$ and then $N_{a,\varepsilon}$ as follows

$$N_{a,\varepsilon} := \lceil e^{P_{a,\varepsilon}a(\varepsilon)}\rceil = \lceil 1/\varepsilon\rceil.$$

It will imply by considering a measure of maximal entropy of $H_{a,\varepsilon}$ that

$$h_{loc}(F_a,\varepsilon) \geq \frac{\log N_{a,\varepsilon}}{P_{a,\varepsilon}} \geq a(\varepsilon).$$

The only thing we need to check is that the resulting map may be C^∞ extended at 0. It is enough to prove that $\|f_\varepsilon\|_r$ goes to zero when ε goes to zero for any given integer r. Fix $r \geq 1$. We have for some constants $C_r = C_r(T)$

$$\begin{aligned}\|f_\varepsilon\|_r &\leq C_r M_{a,\varepsilon}\left(N_{a,\varepsilon}/\varepsilon\right)^r \\ &\leq C_r\varepsilon e^{-\lambda_u(p)P_{a,\varepsilon}}\left(e^{a(\varepsilon)P_{a,\varepsilon}}/\varepsilon\right)^r \\ &= C_r(1/\varepsilon)^{-\frac{\lambda_u(p)}{a(\varepsilon)}+2r-1}.\end{aligned}$$

This last term goes to zero when ε goes to zero because $\lambda_u(p) > 0$ and $\lim_{\varepsilon\to 0} a(\varepsilon) = 0$. This proves F_a may be C^∞ extended. Moreover, if when defining f_a we only consider the rest of the series from N, i.e.

$$f_a^N := \sum_{n\geq N} f_n,$$

then the resulting diffeomorphisms $(F_a^N)_N$ converge to T in the C^∞ topology when N goes to infinity.

Let us now see how such examples are related with Theorem A : we prove Theorem L. We fix some $\varepsilon > 0$ and we consider the map F_a^ε obtained as in the previous construction but where we change f_a by considering only the n_ε-term of the series, i.e.

$$f_a := f_\varepsilon.$$

Let $\mathcal{M} = (M_k)_k$ be the weight defined by

$$M_0 = \|DT\|$$

and for all integers $k > 0$

$$M_k := M_0\left(\frac{1}{a^{-1}(\log\|DT\|/k)}\right)^k.$$

As the sequence $(1/k)_k$ is convex and a is convex and non decreasing, the function $(a^{-1}(\log\|DT\|/k))_k$ is concave. Finally as $-\log$ is convex non increasing, we get that $(M_k)_k$ is a logarithmic convex weight. The condition $a(\varepsilon) \geq \varepsilon^{1/6} \geq 2B\log\|DT\|\varepsilon^{1/6}$ for all ε implies that $(M_k/M_0)^{\frac{1}{k}} = \frac{1}{a^{-1}(\log\|DT\|/k)} \geq (2Bk)^6$ for all k. In particular $(\log(M_k/M_0)/k)_k$ is not bounded. We also consider the weight $\tilde{\mathcal{M}} = (\tilde{M}_k)_k$ defined for all integers k by

$$\tilde{M}_k := (2Bk)^{6k} M_k^2/M_0.$$

The weight $\tilde{\mathcal{M}}$ is also clearly logarithmic convex. Observe also that $\tilde{M}_k/\tilde{M}_0 \leq (M_k/M_0)^3$.

We have for all $x > 0$ and for all $0 < \varepsilon < 1$

$$\begin{aligned}
G_{\tilde{\mathcal{M}}}(x) &\geq G_{\mathcal{M}}(x/3), \\
&\geq \frac{\log \|DT\|}{a(1/e^{x/3})}, \\
&\geq \frac{\log \|DT\|}{a(1/e^{x/3})}, \\
G_{\tilde{\mathcal{M}}}(|\log \varepsilon|/3) &\geq \frac{\log \|DT\|}{a(\varepsilon)}.
\end{aligned}$$

We check now for any $\varepsilon > 0$ that F_a^ε belongs to $C_V^{\tilde{\mathcal{M}}}$. Let us compute $\|F_a^\varepsilon\|_r$ for any r. We put $r_\varepsilon := \lambda_u(p)/a(\varepsilon)$. By applying Faa di Bruno formula we have for any $r \in \mathbb{N}$:

$$\|F_a^\varepsilon\|_r \leq B_r \|T\|_r \max_{\sum i j_i = r} \|\theta_a^\varepsilon\|_i^{j_i}.$$

Now by derivation of a product we have

$$\|\theta_a^\varepsilon\|_i \leq 2^i \|\chi\|_i \|f_\varepsilon\|_i.$$

As we took $\chi \in V^{(AB^k k^{2k})_k}$ we have for some constants $C = C(T)$

$$\begin{aligned}
\|f_\varepsilon\|_i &\leq C 2^i M_{a,\varepsilon} \|\chi\|_i (N_{a,\varepsilon}/\varepsilon)^i, \\
&\leq C(2B)^i i^{2i} M_{a,\varepsilon} (N_{a,\varepsilon}/\varepsilon)^i, \\
\|\theta_a^\varepsilon\|_i &\leq C(2Bi)^{4i} M_{a,\varepsilon} (N_{a,\varepsilon}/\varepsilon)^i.
\end{aligned}$$

and then, by analyticity of T,

$$\begin{aligned}
\|F_a^\varepsilon\|_r &\leq C r^{2r} \max_{\sum i j_i = r} \|\theta_a^\varepsilon\|_i^{j_i}, \\
&\leq C(2Br)^{6r} M_{a,\varepsilon} (N_{a,\varepsilon}/\varepsilon)^r, \\
&\leq C(2Br)^{6r} (1/\varepsilon)^{-\frac{\lambda_u(p)}{a(\varepsilon)} + 2r - 1}, \\
&\leq C(2Br)^{6r} (1/\varepsilon)^{-r_\varepsilon + 2r - 1}.
\end{aligned}$$

and thus for any r and ε small enough ($C\varepsilon \leq 1$) we get

$$\begin{aligned}
\log \|F_a^\varepsilon\|_r / r &\leq 6\log(2Br) + 2\log(1/\varepsilon), \\
&\leq 6\log(2Br) + 2\log(1/a^{-1}(\log \|DT\|/r_\varepsilon)), \\
&\leq 6\log(2Br) + 2\log(1/a^{-1}(\log \|DT\|/r)), \\
&\leq 6\log(2Br) + 2\frac{\log(M_r/M_0)}{r}.
\end{aligned}$$

This implies $F_a^\varepsilon \in C_V^{\tilde{\mathcal{M}}}(M)$. Finally we have again

$$\begin{aligned} h_{loc}(F_a^\varepsilon) &\geq h(H_{a,\varepsilon}), \\ &\geq a(\varepsilon), \\ &\geq \frac{\log \|DT\|}{H_{\tilde{\mathcal{M}}}(|\log \varepsilon|/3)}. \end{aligned}$$

Remark 7.1. The construction presented here may be adapted to interval maps to produce examples with the same properties. One only needs to repeat the previous construction near a flat homoclinic tangency at a hyperbolic repulsing fixed point of a C^∞ interval map (See for example [9] and [38] for similar constructions of horseshoes accumulating near a tangency).

Remark 7.2. The perturbation θ_a may be chosen volume preserving by using the pasting lemma of Arbieto and Matheus (see Lemma 3.9 of [3]) as in [15].

Remark 7.3. The previous construction may be embedded in any manifold of dimension larger than two such that f_a and f_a^ε are diffeomorphisms on this manifold.

Remark 7.4. In the above proof the local entropy is produced by small horseshoes, i.e. horseshoes included in some infinite ε-Bowen ball, which are persistent under C^1 (even C^0 for interval maps) small perturbation. In particular if f_a is as in the above example there exists for any ε a polynomial map P_ε with $h^*(P_\varepsilon, 2\varepsilon) \geq a(\varepsilon)$. That's why one can not expect to have a lower bound in $C/|\log \varepsilon|$ with C independent from the degree in Theorem E or Corollary F.

Remark 7.5. In [18], [15], [2], the persistence of homoclinic tangencies and of small horseshoes allow to use a Baire argument to build generic examples with no principal symbolic extensions, in particular non asymptotically h-expansive. Here we do not know if Corollary K holds for a C^∞ generic subset of Newhouse domains. Indeed we only are able to show that $\{f,\ h^*(f,\varepsilon) \geq a(\varepsilon)\}$ are $\tilde{a}(\varepsilon)$-dense in Newhouse domains for some function $\tilde{a}$ depending on a.

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