MAXIMAL MEASURE AND ENTROPIC CONTINUITY OF LYAPUNOV EXPONENTS FOR $\mathcal{C}^r$ SURFACE DIFFEOMORPHISMS WITH LARGE ENTROPY

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ABSTRACT. We prove a finite smooth version of the entropic continuity of Lyapunov exponents proved recently by Buzzi, Crovisier and Sarig for $\mathcal{C}^\infty$ surface diffeomorphisms [10]. As a consequence we show that any $\mathcal{C}^r$, $r > 1$, smooth surface diffeomorphism f with $h_{top}(f) > \frac{1}{r} \limsup_n \frac{1}{n} \log^+ \|df^n\|_\infty$ admits a measure of maximal entropy. We also prove the $\mathcal{C}^r$ continuity of the topological entropy at f.

INTRODUCTION

The entropy of a dynamical system quantifies the dynamical complexity by counting distinct orbits. There are topological and measure theoretical versions which are related by a variational principle : the topological entropy of a continuous map on a compact space is equal to the supremum of the entropy of the invariant (probability) measures. An invariant measure is said to be of maximal entropy (or a maximal measure) when its entropy is equal to the topological entropy, i.e. this measure realizes the supremum in the variational principle. In general a topological system may not admit a measure of maximal entropy. But such a measure exists for dynamical systems satisfying some expansiveness properties. In particular Newhouse [15] has proved their existence for C^∞ systems by using Yomdin's theory. In the present paper we show the existence of a measure of maximal entropy for $\mathcal{C}^r$, $1 < r < +\infty$, smooth surface diffeomorphisms with large entropy.

Other important dynamical quantities for smooth systems are given by the Lyapunov exponents which estimate the exponential growth of the derivative. For $\mathcal{C}^\infty$ surface diffeomorphisms, J. Buzzi, S. Crovisier and O. Sarig proved recently a property of continuity in the entropy of the Lyapunov exponents with many statistical applications [10]. More precisely, they showed that for a $\mathcal{C}^\infty$ surface diffeomorphism f, if ν_k is a converging sequence of ergodic measures with $\lim_k h(\nu_k) = h_{top}(f)$, then the Lyapunov exponents of ν_k are going to the (average) Lyapunov exponents of the limit (which is a measure of maximal entropy). We prove a $\mathcal{C}^r$ version of this fact for $1 < r < +\infty$.

1. STATEMENTS

We define now some notations to state our main results. For a $\mathcal{C}^r$, $r \geq 1$, diffeomorphism f on a compact Riemannian surface $(\mathbf{M}, \|\cdot\|)$ we let $F : \mathbb{P}T\mathbf{M} \circlearrowleft$ be the induced map on the projective tangent bundle $\mathbb{P}T\mathbf{M} = T^1\mathbf{M}/\pm 1$ and we denote by $\phi, \psi : \mathbb{P}T\mathbf{M} \to \mathbb{R}$ the continuous observables on $\mathbb{P}T\mathbf{M}$ given respectively by $\phi : (x, v) \mapsto \log \|d_x f(v)\|$ and $\psi : (x, v) \mapsto \log \|d_x f(v)\| - \frac{1}{r} \log^+ \|d_x f\|$ with $\|d_x f\| = \sup_{v \in T_x\mathbf{M} \setminus \{0\}} \frac{\|d_x f(v)\|}{\|v\|}$. For $k \in$

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$\mathbb{N}^*$ we define more generally $\phi_k : (x,v) \mapsto \log \|d_x f^k(v)\|$ and $\psi_k : (x,v) \mapsto \phi_k(x,v) - \frac{1}{r}\sum_{l=0}^{k-1} \log^+ \|d_{f^k x} f\|$. Then we let $\lambda^+(x)$ and $\lambda^-(x)$ be the pointwise Lyapunov exponents given by $\lambda^+(x) = \limsup_{n\to+\infty} \frac{1}{n}\log\|d_x f^n\|$ and $\lambda^-(x) = \liminf_{n\to-\infty} \frac{1}{n}\log\|d_x f^n\|$ for any $x \in \mathbf{M}$ and $\lambda^+(\mu) = \int \lambda^+(x)\, d\mu(x)$, $\lambda^-(\mu) = \int \lambda^-(x)\, d\mu(x)$, for any f-invariant measure μ.

Also we put $\lambda^+(f) := \lim_n \frac{1}{n}\log^+\|df^n\|_\infty$ with $\|df^n\|_\infty = \sup_{x\in\mathbf{M}}\|d_x f^n\|$. The function $f \mapsto \lambda^+(f)$ is upper semi-continuous in the $\mathcal{C}^1$ topology on the set of $\mathcal{C}^1$ diffeomorphisms on $\mathbf{M}$. For an f-invariant measure μ with $\lambda^+(x) > 0 \geq \lambda^-(x)$ for μ a.e. x, there are by Oseledets* theorem one-dimensional invariant vector spaces $\mathcal{E}_+(x)$ and $\mathcal{E}_-(x)$, resp. called the unstable and stable Oseledets bundle, such that

$$\forall\, \mu \text{ a.e. } x\ \forall v \in \mathcal{E}_\pm(x)\setminus\{0\},\ \lim_{n\to\pm\infty}\frac{1}{n}\log\|d_x f^n(v)\| = \lambda^\pm(x).$$

Then we let $\hat{\mu}^+$ be the F-invariant measure given by the lift of μ on $\mathbb{P}T\mathbf{M}$ with $\hat{\mu}^+(\mathcal{E}_+) = 1$. When writing $\hat{\mu}^+$ we assume implicitly that the push-forward measure μ on $\mathbf{M}$ satisfies $\lambda^+(x) > 0 \geq \lambda^-(x)$ for μ a.e. x.

A sequence of $\mathcal{C}^r$, with $r > 1$, surface diffeomorphisms $(f_k)_k$ on $\mathbf{M}$ is said to converge $\mathcal{C}^r$ weakly to a diffeomorphism f, when f_k goes to f in the $\mathcal{C}^1$ topology and the sequence $(f_k)_k$ is $\mathcal{C}^r$ bounded. In particular f is $\mathcal{C}^{r-1}$.

Theorem (Buzzi-Crovisier-Sarig, Theorem C [10])**.** *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of $\mathcal{C}^r$, with $r > 1$, surface diffeomorphisms converging $\mathcal{C}^r$ weakly to a diffeomorphism f. Let $(F_k)_{k\in\mathbb{N}}$ and F be the lifts of $(f_k)_{k\in\mathbb{N}}$ and f to $\mathbb{P}T\mathbf{M}$. Assume there is a sequence $(\hat{\nu}_k^+)_k$ of ergodic F_k-invariant measures converging to $\hat{\mu}$.*

Then there are $\beta \in [0,1]$ and F-invariant measures $\hat{\mu}_0$ and $\hat{\mu}_1^+$ with $\hat{\mu} = (1-\beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$, such that:

$$\limsup_{k\to+\infty} h(\nu_k) \leq \beta h(\mu_1) + \frac{\lambda^+(f) + \lambda^+(f^{-1})}{r-1}.$$

In particular when f ($= f_k$ for all k) is $\mathcal{C}^\infty$ and $h(\nu_k)$ goes to the topological entropy of f, then β is equal to 1 and therefore $\lambda^+(\nu_k)$ goes to $\lambda^+(\mu)$:

Corollary (Entropic continuity of Lyapunov exponents [10])**.** *Let f be a $\mathcal{C}^\infty$ surface diffeomorphism with $h_{top}(f) > 0$.*

Then if $(\nu_k)_k$ is a sequence of ergodic measures converging to μ with $\lim_k h(\nu_k) = h_{top}(f)$, then

- $h(\mu) = h_{top}(f)$ †,
- $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$.

We state an improved version of Buzzi-Crovisier-Sarig Theorem, which allows to prove the same entropy continuity of Lyapunov exponents for $\mathcal{C}^r$, $1 < r < +\infty$, surface diffeomorphisms with large enough entropy (see Corollary 1).

Main Theorem. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of $\mathcal{C}^r$, with $r > 1$, surface diffeomorphisms converging $\mathcal{C}^r$ weakly to a diffeomorphism f. Let $(F_k)_{k\in\mathbb{N}}$ and F be the lifts of $(f_k)_{k\in\mathbb{N}}$ and f*

*We refer to [16] for background on Lyapunov exponents and Pesin theory.

†This follows from the upper semi-continuity of the entropy function h on the set of f-invariant probability measures for a $\mathcal{C}^\infty$ diffeomorphism f (in any dimension), which was first proved by Newhouse in [15].

to $\mathbb{P}T\mathbf{M}$. Assume there is a sequence $(\hat{\nu}_k^+)_k$ of ergodic F_k-invariant measures converging to $\hat{\mu}$.

Then for any $\alpha > \frac{\lambda^+(f)}{r}$, there are $\beta = \beta_\alpha \in [0,1]$ and F-invariant measures $\hat{\mu}_0 = \hat{\mu}_{0,\alpha}$ and $\hat{\mu}_1^+ = \hat{\mu}_{1,\alpha}^+$ with $\hat{\mu} = (1-\beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$, such that:

$$\limsup_{k\to+\infty} h(\nu_k) \leq \beta h(\mu_1) + (1-\beta)\alpha.$$

The Main Theorem implies Buzzi-Crovisier-Sarig statement. Indeed, either $\lim_k \lambda^+(\nu_k) = \int \phi\, d\hat{\mu} \leq \frac{\lambda^+(f)}{r}$ and we get by Ruelle inequality, $\limsup_k h(\nu_k) \leq \frac{\lambda^+(f)}{r}$ or there exists $\alpha \in \left]\frac{\lambda^+(f)}{r}, \min\left(\int \phi\, d\hat{\mu}, \frac{\lambda^+(f)}{r-1}\right)\right[$. By applying our Main Theorem with respect to α, there is a decomposition $\hat{\mu} = (1-\beta_\alpha)\hat{\mu}_{0,\alpha} + \beta_\alpha\hat{\mu}_{1,\alpha}^+$ satisfying $\limsup_{k\to+\infty} h(\nu_k) \leq \beta_\alpha h(\mu_{1,\alpha}) + (1-\beta_\alpha)\alpha$. But it follows from the proofs that $\beta_\alpha\mu_{1,\alpha}$ is a component of $\beta\mu_1$ with β and μ_1 being as in Buzzi-Crovisier-Sarig's statement (see Remark 6). In particular $\beta_\alpha h(\mu_{1,\alpha}) \leq \beta h(\mu_1)$, therefore $\limsup_{k\to+\infty} h(\nu_k) \leq \beta h(\mu_1) + \frac{\lambda^+(f)+\lambda^+(f^{-1})}{r-1}$. In Theorem C [10], the authors also proved $\int \phi\, d\hat{\mu}_0 = 0$ whenever $\beta \neq 1$. Therefore we get here $(1-\beta_\alpha)\int \phi\, d\hat{\mu}_{0,\alpha} \geq (1-\beta)\int \phi\, d\hat{\mu}_0 = 0$, then $\int \phi\, d\hat{\mu}_{0,\alpha} \geq 0$. But maybe we could have $\int \phi\, d\hat{\mu}_{0,\alpha} > 0$.

Corollary 1 (Existence of maximal measures and entropic continuity of Lyapunov exponents)**.** *Let f be a $\mathcal{C}^r$, with $r > 1$, surface diffeomorphism satisfying $h_{top}(f) > \frac{\lambda^+(f)}{r}$.*

Then f admits a measure of maximal entropy. More precisely, if $(\nu_k)_k$ is a sequence of ergodic measures converging to μ with $\lim_k h(\nu_k) = h_{top}(f)$, then

- $h(\mu) = h_{top}(f)$,
- $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$.

It was proved in [9] that any $\mathcal{C}^r$ surface diffeomorphism satisfying $h_{top}(f) > \frac{\lambda^+(f)}{r}$ admits at most finitely many ergodic measures of maximal entropy. On the other hand, J. Buzzi has built examples of $\mathcal{C}^r$ surface diffeomorphisms for any $+\infty > r > 1$ with $\frac{h_{top}(f)}{\lambda^+(f)}$ arbitrarily close to $1/r$ without a measure of maximal entropy [7]. Such results were already known for interval maps [3, 6, 8].

Proof. We consider the constant sequence of diffeomorphisms equal to f. By taking a subsequence, we can assume that $(\hat{\nu}_k^+)_k$ is converging to a lift $\hat{\mu}$ of μ. By using the notations of the Main Theorem with $h_{top}(f) > \alpha > \frac{\lambda^+(f)}{r}$, we have

$$\begin{aligned} h_{top}(f) &= \lim_{k\to+\infty} h(\nu_k), \\ &\leq \beta h(\mu_1) + (1-\beta)\alpha, \\ &\leq \beta h_{top}(f) + (1-\beta)\alpha, \\ (1-\beta)h_{top}(f) &\leq (1-\beta)\alpha. \end{aligned}$$

But $h_{top}(f) > \alpha$, therefore $\beta = 1$, i.e. $\hat{\mu}_1^+ = \hat{\mu}$ and $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$. Moreover $h_{top}(f) = \lim_{k\to+\infty} h(\nu_k) \leq \beta h(\mu_1) + (1-\beta)\alpha = h(\mu)$. Consequently μ is a measure of maximal entropy of f.

□

Corollary 2 (Continuity of topological entropy and maximal measures)**.** *Let $(f_k)_k$ be a sequence of $\mathcal{C}^r$, with $r > 1$, surface diffeomorphisms converging $\mathcal{C}^r$ weakly to a diffeomorphism*

f with $h_{top}(f) \geq \frac{\lambda^+(f)}{r}$.

Then

$$h_{top}(f) = \lim_k h_{top}(f_k).$$

Moreover if $h_{top}(f) > \frac{\lambda^+(f)}{r}$ and ν_k is a maximal measure of f_k for large k, then any limit measure of $(\nu_k)_k$ for the weak-$*$ topology is a maximal measure of f.

Proof. By Katok's horseshoes theorem [14], the topological entropy is lower semi-continuous for the $\mathcal{C}^1$ topology on the set of $\mathcal{C}^r$ surface diffeomorphisms. Therefore it is enough to show the upper semi-continuity.

By the variational principle there is a sequence of probability measures $(\nu_k)_{k\in K}$, $K \subset \mathbb{N}$ with $\sharp K = \infty$, such that :

- ν_k is an ergodic f_k-invariant measure for each k,
- $\lim_{k\in K} h(\nu_k) = \limsup_{k\in\mathbb{N}} h_{top}(f_k)$.

By extracting a subsequence we can assume $(\hat{\nu}_k^+)_k$ is converging to a F-invariant measure $\hat{\mu}$ in the weak-$*$ topology. We can then apply the Main Theorem for any $\alpha > \frac{\lambda^+(f)}{r}$ to get for some f-invariant measures μ_1, μ_0 and $\beta \in [0,1]$ (depending on α) with $\mu = (1-\beta)\mu_0 + \beta\mu_1$:

$$\begin{aligned} \limsup_k h_{top}(f_k) &= \lim_k h(\nu_k), \\ &\leq \beta h(\mu_1) + (1-\beta)\alpha, \\ &\leq \beta h_{top}(f) + (1-\beta)\alpha, \\ &\leq \max(h_{top}(f), \alpha). \end{aligned} \tag{1.1}$$

By letting α go to $\frac{\lambda^+(f)}{r}$ we get

$$\limsup_k h_{top}(f_k) \leq h_{top}(f).$$

If $h_{top}(f) > \frac{\lambda^+(f)}{r}$, we can fix $\alpha \in \left]\frac{\lambda^+(f)}{r}, h_{top}(f)\right[$ and the inequalities (1.1) may be then rewritten as follows :

$$\begin{aligned} \limsup_k h_{top}(f_k) &\leq \beta h(\mu_1) + (1-\beta)\alpha, \\ &\leq h_{top}(f). \end{aligned}$$

By the lower semi-continuity of the topological entropy, we have $h_{top}(f) \leq \limsup_k h_{top}(f_k)$ and therefore these inequalities are equalities, which implies $\beta = 1$, then $\mu_1 = \mu$, and $h(\mu) = h_{top}(f)$. $\square$

The corresponding result was proved for interval maps in [5] by using a different method. We also refer to [5] for counterexamples of the upper semi-continuity property for interval maps f with $h_{top}(f) < \frac{\lambda^+(f)}{r}$. Finally, in [7], the author built, for any $r > 1$, a $\mathcal{C}^r$ surface diffeomorphism f with $\limsup_{g \xrightarrow{\mathcal{C}^r} f} h_{top}(g) = \frac{\lambda^+(f)}{r} > h_{top}(f) = 0$. We recall also that upper semi-continuity of the topological entropy in the $\mathcal{C}^\infty$ topology was established in any dimension by Y. Yomdin in [18].

Newhouse proved that for a $\mathcal{C}^\infty$ system $(\mathbf{M}, f)$, the entropy function $h : \mathcal{M}(\mathbf{M}, f) \to \mathbb{R}^+$ is an upper semi-continuous function on the set $\mathcal{M}(\mathbf{M}, f)$ of f-invariant probability measure.

It follows from our Main Thereom, that the entropy function is upper semi-continuous at ergodic measures with entropy larger than $\frac{\lambda^+(f)}{r}$ for a $\mathcal{C}^r$, $r>1$, surface diffeomorphism f.

Corollary 3 (Upper semi-continuity of the entropy function at ergodic measures with large entropy)**.** *Let $f:\mathbf{M}\circlearrowleft$ be a $\mathcal{C}^r$, $r>1$, surface diffeomorphism.*

Then for any ergodic measure μ with $h(\mu)\geq\frac{\lambda^+(f)}{r}$, we have

$$\limsup_{\nu\to\mu} h(\nu)\leq h(\mu).$$

Proof. By continuity of the ergodic decomposition at ergodic measures and by harmonicity of the entropy function, we have for any ergodic measure μ (see e.g. Lemma 8.2.13 in [12]):

$$\limsup_{\nu \text{ ergodic}, \nu\to\mu} h(\nu)=\limsup_{\nu\to\mu} h(\mu).$$

Let $(\nu_k)_{k\in\mathbb{N}}$ be a sequence of ergodic f-invariant measures with $\lim_k h(\nu_k)=\limsup_{\nu\to\mu} h(\nu)$. By extracting a subsequence we can assume that the sequence $(\hat{\nu}_k^+)_k$ is converging to some lift $\hat{\mu}$ of μ. Take α with $\alpha>\frac{\lambda^+(f)}{r}$. Then, in the decomposition $\hat{\mu}=(1-\beta)\hat{\mu}_0+\beta\hat{\mu}_1^+$ given by the Main Theorem, we have $\mu_1=\mu_0$ by ergodicity of μ. Therefore

$$\begin{aligned}\lim_k h(\nu_k) &\leq \beta h(\mu)+(1-\beta)\alpha,\\ &\leq \max\left(h(\mu),\alpha\right).\end{aligned}$$

By letting α go to $\frac{\lambda^+(f)}{r}$ we get

$$\lim_k h(\nu_k)\leq h(\mu).$$

□

2. Main steps of the proof

We follow the strategy of the proof of [10]. We point out below the main differences:

- *Geometric and neutral empirical component.* For $\lambda^+(\nu_k)>\frac{\lambda^+(f)}{r}$ we split the orbit of a ν_k-typical point x into two parts. We consider the empirical measures from x at times lying between to M-close consecutive times where the unstable manifold has a "bounded geometry". We take their limit in k, then in M. In this way we get an invariant component of $\hat{\mu}$. In [10] the authors consider rather such empirical measures for α-hyperbolic times and then take the limit when α go to zero.
- *Entropy computations.* To compute the asymptotic entropy of the ν_k's, we use the static entropy w.r.t. partitions and its conditional version. Instead the authors in [10] used Katok's like formulas.
- *$\mathcal{C}^r$ Reparametrizations.* Finally we use here reparametrization methods from [4] and [2] respectively rather than Yomdin's reparametrizations of the projective action F as done in [10]. This is the principal difference with [10].

2.1. **Empirical measures.** Let (X,T) be a topological system. For a fixed Borel measurable subset G of X we let $E(x)=E_G(x)$ be the set of times of visits in G from x:

$$E(x)=\{n\in\mathbb{Z},\ T^nx\in G\}.$$

When $a<b$ are two consecutive times in $E(x)$, then $[a,b[$ is called a *neutral block* (by following the terminology of [9]). For all M we let then

$$E^M(x)=\bigcup_{a<b\in E(x),\ |a-b|\leq M}[a,b[.$$

The complement of $E^M(x)$ is made of disjoint neutral blocks of length larger than M. We consider the associated empirical measures :

$$\forall n,\ \mu^M_{x,n}=\frac{1}{n}\sum_{k\in E^M(x)\cap[0,n[}\delta_{T^kx}.$$

Let ν be an ergodic measure. We denote by χ^M the indicator function of $\{x, 0\in E^M(x)\}$. By the Birkhoff ergodic theorem, there is a set $\mathtt{G}$ of full ν-measure such that the empirical measures $\left(\mu^M_{x,n}\right)_n$ are converging for any $x\in\mathtt{G}$ and any $M\in\mathbb{N}^*$ to $\xi^M:=\chi^M\nu$ in the weak-$*$ topology. We also let $\eta^M=\nu-\xi^M$. Moreover we put $\beta_M=\int\chi^M\,d\nu$, then $\xi^M=\beta_M\cdot\underline{\xi}^M$ when $\beta_M\neq 0$ and $\eta^M=(1-\beta_M)\cdot\underline{\eta}^M$ when $\beta_M\neq 1$ with $\underline{\xi}^M$, $\underline{\eta}^M$ being thus probability measures. Following partially [10], the measures ξ^M and η^M are respectively called here the *geometric and neutral components* of ν. In general these measures are not T-invariant. From the definition one easily checks that $\xi^M\geq\xi^N$ for $M\geq N$.

2.2. **Pesin unstable manifolds.** We consider a smooth compact riemannian manifold $(\mathbf{M},\|\cdot\|)$. Let $\exp_x$ be the exponential map at x and let R_{inj} be the radius of injectivity of $(\mathbf{M},\|\cdot\|)$. We consider the distance d on $\mathbf{M}$ induced by the Riemannian structure. Let $f:\mathbf{M}\circlearrowleft$ be a $\mathcal{C}^r$, $r>1$, surface diffeomorphism. We denote by $\mathcal{R}$ the set of Lyapunov regular points with $\lambda^+(x)>0>\lambda^-(x)$. For $x\in\mathbf{M}$ we let $W^u(x)$ denote the unstable manifold at x :

$$W^u(x):=\left\{y\in\mathbf{M},\ \lim_n\frac{1}{n}\log\mathrm{d}(f^nx,f^ny)<0\right\}.$$

By Pesin unstable manifold theorem, the set $W^u(x)$ for $x\in\mathcal{R}$ is a $\mathcal{C}^r$ submanifold tangent to $\mathcal{E}_+(x)$ at x.

For $x\in\mathcal{R}$, we let $\hat{x}$ be the vector in $\mathbb{P}T\mathbf{M}$ associated to the unstable Oseledets bundle $\mathcal{E}_+(x)$. For $\delta>0$ the point x is said *δ-hyperbolic* with respect to ϕ (resp. ψ) when we have $\phi_l(F^{-l}\hat{x})\geq\delta l$ (resp. $\psi_l(F^{-l}\hat{x})\geq\delta l$) for all $l>0$. Note that if x is δ-hyperbolic with respect to ψ then it is δ-hyperbolic with respect to ϕ.

Let ν be an ergodic measure with $\lambda^+(\nu)-\frac{\log^+\|df\|_\infty}{r}>\delta>0>\lambda^-(\nu)$. By applying the Ergodic Maximal Inequality (see e.g. Theorem 1.1 in [1]) to the measure preserving system $(F^{-1},\hat{\nu}^+)$ with the observable $\psi^\delta=\delta-\psi\circ F^{-1}$, we get with $A_\delta=\{\hat{x}\in\mathbb{P}T\mathbf{M},\ \exists k\geq 0 \text{ s.t. } \sum_{l=0}^k\psi^\delta(F^{-l}\hat{x})>0\}$:

$$\int_{A_\delta}\psi^\delta\,d\hat{\nu}^+\geq 0.$$

But the set $H_\delta:=\left\{\hat{x}\in\mathbb{P}T\mathbf{M},\ \forall l>0\ \psi_l(F^{-l}\hat{x})\geq\delta l\right\}$ of δ-hyperbolic points w.r.t. ψ is just the complement set $\mathbb{P}T\mathbf{M}\setminus A_\delta$ of A_δ. Therefore $\int_{H_\delta}(\delta-\psi\circ F^{-1})\,d\hat{\nu}^+\leq\int(\delta-\psi\circ F^{-1})\,d\hat{\nu}^+=\delta-\lambda^+(\nu)+\frac{1}{r}\int\frac{\log^+\|df\|}{r}\,d\nu<0$. In particular we have $\hat{\nu}^+(H_\delta)>0$.

A point $x \in \mathcal{R}$ is said to have κ-bounded geometry for $\kappa > 0$ when $\exp_x^{-1} W^u(x)$ contains the graph of an κ-*admissible* map at x, which is defined as a 1-Lipschitz map $f : I \to \mathcal{E}_+(x)^\perp \subset T_x\mathbf{M}$, with I being an interval of $\mathcal{E}_+(x)$ containing 0 with length κ. We let G_κ be the subset of points in $\mathcal{R}$ with κ-bounded geometry.

Lemma 1. *The set G_κ is Borel measurable.*

Proof. For $x \in \mathcal{R}$ we have $W^u(x) = \bigcup_{n\in\mathbb{N}} f^n W^u_{loc}(f^{-n}x)$ with W^u_{loc} being the Pesin unstable local manifold at x. The sequence $\left(f^{-n}W^u_{loc}(f^n x)\right)_n$ is increasing in n for the inclusion. Therefore, if we let G^n_κ be the subset of points x in G_κ, such that $\exp_x^{-1} f^n W^u_{loc}(f^{-n}x)$ contains the graph of a κ-admissible map, then we have

$$G_\kappa = \bigcup_n G^n_\kappa.$$

There are closed subsets, $(\mathcal{R}_l)_{l\in\mathbb{N}}$, called the Pesin blocks, such that $\mathcal{R} = \bigcup_l \mathcal{R}_l$ and $x \mapsto W^u_{loc}(x)$ is continuous on $\mathcal{R}_l$ for each l (see e.g. [16]). Let $(x_p)_p$ be sequence in $G^n_\kappa \cap \mathcal{R}_l$ which converges to $x \in \mathcal{R}_l$. By extracting a subsequence we can assume that the associated sequence of κ-admissible maps f_p at x_p is converging pointwisely to a κ-admissible map at x, when p goes to infinity. In particular $G^n_\kappa \cap \mathcal{R}_l$ is a closed set and therefore $G_\kappa = \bigcup_{l,n} (G^n_\kappa \cap \mathcal{R}_l)$ is Borel measurable.

□

2.3. **Entropy of conditional measures.** We consider an ergodic hyperbolic measure ν, i.e an ergodic measure with $\nu(\mathcal{R}) = 1$. A measurable partition ς is *subordinated* to the Pesin unstable local lamination W^u_{loc} of ν if the atom $\varsigma(x)$ of ς containing x is a neighborhood of x inside the curve $W^u_{loc}(x)$ and $f^{-1}\varsigma \succ \varsigma$. By Rokhlin's disintegration theorem, there are a measurable set $\mathtt{Z}$ of full ν-measure and probability measures ν_x on $\varsigma(x)$ for $x \in \mathtt{Z}$, called the *conditional measures* on unstable manifolds, satisfying $\nu = \int \nu_x \, d\nu(x)$. Moreover $\nu_y = \nu_x$ for $x, y \in \mathtt{Z}$ in the same atom of ς. Ledrappier and Young [13] proved the existence of such subordinated measurable partitions and showed that for ν-a.e. x, we have with $B_n(x,\rho)$ being the Bowen ball $B_n(x,\rho) := \bigcap_{0\le k<n} f^{-k}B(f^k x, \rho)$ (where $B(f^k x, \rho)$ denotes the ball for d at $f^k x$ with radius ρ):

$$\lim_{\rho\to 0} \liminf_n -\frac{1}{n} \log \nu_x \left(B_n(x,\rho)\right) = h(\nu). \tag{2.1}$$

Fix an error term $\iota > 0$ depending[‡] on ν. There is $\rho > 0$ and a measurable set $\mathtt{F} \subset \mathtt{Z} \cap \mathcal{R}$ with $\nu(\mathtt{F}) > 0$ such that

$$\forall x \in \mathtt{F}, \liminf_n -\frac{1}{n} \log \nu_x \left(B_n(x,\rho)\right) \ge h(\nu) - \iota.$$

We fix $x_* \in \mathtt{F}$ with $\nu_{x_*}(\mathtt{F}) > 0$ and we let $\zeta = \frac{\nu_{x_*}(\cdot)}{\nu_{x_*}(\mathtt{F})}$ be the probability measure induced by ν_{x_*} on $\mathtt{F}$. Observe that $\nu_x = \nu_{x_*}$ for ζ a.e. x. We let D be the $\mathcal{C}^r$ curve given by the Pesin local unstable manifold $W^u_{loc}(x_*)$ at x_*. For a finite measurable partition P and a Borel probability measure μ we let $H_\mu(P)$ be the static entropy, $H_\mu(P) = -\sum_{A\in P} \mu(A) \log \mu(A)$. Moreover we let $P^n = \bigvee_{k=0}^{n-1} f^{-k}P$ be the n-iterated partition, $n \in \mathbb{N}$. We also denote by P^n_x the atom of P^n containing the point $x \in \mathbf{M}$.

[‡]In the proof of the Main Theorem we will take $\iota = \iota(\nu_k) \xrightarrow{k} 0$ for the converging sequence of ergodic measures $(\nu_k)_k$.

Lemma 2. *For any (finite measurable) partition P with diameter less than ρ, we have*

$$\liminf_n \frac{1}{n} H_\zeta(P^n) \geq h(\nu) - \iota.$$

Proof.

$$\begin{aligned}\liminf_n \frac{1}{n} H_\zeta(P^n) &= \liminf_n \int -\frac{1}{n} \log \zeta(P^n_x)\, d\zeta(x), \text{ by the definition of } H_\zeta,\\ &\geq \int \liminf_n -\frac{1}{n} \log \zeta(P^n_x)\, d\zeta(x), \text{ by Fatou's Lemma,}\\ &\geq \int \liminf_n -\frac{1}{n} \log \nu_{x_*}(P^n_x)\, d\zeta(x), \text{ by the definition of } \zeta,\\ &\geq \int \liminf_n -\frac{1}{n} \log \nu_x(P^n_x)\, d\zeta(x), \text{ as } \nu_x = \nu_{x_*} \text{ for } \zeta \text{ a.e. } x,\\ &\geq \int \liminf_n -\frac{1}{n} \log \nu_x(B_n(x,\rho))\, d\zeta(x), \text{ as } \operatorname{diam}(P) < \rho,\\ &\geq h(\nu) - \iota, \text{ by the choice of } \mathtt{F}.\end{aligned}$$

□

2.4. **Entropy splitting of the neutral and the geometric component.** The natural projection from $\mathbb{P}T\mathbf{M}$ to $\mathbf{M}$ is denoted by π. We consider a distance $\hat{\mathrm{d}}$ on the projective tangent bundle $\mathbb{P}T\mathbf{M}$, such that $\hat{\mathrm{d}}(\hat{x},\hat{y}) \geq \mathrm{d}(\pi\hat{x},\pi\hat{y})$ for all $\hat{x},\hat{y} \in \mathbb{P}T\mathbf{M}$. In this section we split the entropy contribution of the neutral and geometric components $\hat{\eta}^M$ and $\hat{\xi}^M$ of the ergodic F-invariant measure $\hat{\nu}^+$ associated to $G = H_\delta \cap \pi^{-1} G_\kappa \subset \mathbb{P}T\mathbf{M}$, where the parameters δ and κ will be fixed later on. We also consider their projections η^M and ξ^M on $\mathbf{M}$. Let $\mathtt{F}$ and P as in the previous subsection. Without loss of generality we can assume

- $\{\hat{x},\ x \in \mathtt{F}\} \subset \mathtt{G}$ with $\mathtt{G}$ being the set of full $\hat{\nu}^+$-measure of points $\hat{x}$ such that the empirical measures $\mu^M_{\hat{x},n}$ are converging to $\hat{\xi}^M$ for any M (see Subsection 2.1),
- the boundary of P has zero ν-measure,
- for any $M \in \mathbb{N}$ and for any continuous function $\varphi : \mathbb{P}T\mathbf{M} \to \mathbb{R}$,

$$\frac{1}{n} \sum_{k \in E^M(x) \cap [1,n[} \varphi(F^k \hat{x}) \xrightarrow{n} \int \varphi\, d\hat{\xi}^M \text{ uniformly in } x \in \mathtt{F}. \tag{2.2}$$

- for any continuous function $\vartheta : \mathbf{M} \to \mathbb{R}$,

$$\frac{1}{n} \sum_{k \in [1,n[} \vartheta(f^k x) \xrightarrow{n} \int \vartheta\, d\nu \text{ uniformly in } x \in \mathtt{F}. \tag{2.3}$$

Let us detail the proof of the third item. If $\mathcal{F} = (\varphi_k)_{k\in\mathbb{N}}$ is a dense countable family in the set $\mathcal{C}^0(\mathbb{P}T\mathbf{M},\mathbb{R})$ of real continuous functions on $\mathbb{P}T\mathbf{M}$ endowed with the supremum norm $\|\cdot\|_\infty$, then for all k, M, by Egorov's theorem applied to the pointwise converging sequence $(f_n : \mathtt{F} \to \mathbb{R})_n = \left(x \mapsto \int \varphi_k\, d\mu^M_{\hat{x},n}\right)_n$, there is a subset $\mathtt{F}^M_k$ of $\mathtt{F}$ with $\nu(\mathtt{F}^M_k) > \nu(\mathtt{F})\left(1 - \frac{1}{2^{k+M+3}}\right)$ such that $\int \varphi_k\, d\mu^M_{\hat{x},n}$ converges to $\int \varphi_k\, d\xi^M$ uniformly in $x \in \mathtt{F}^M_k$. Let $\mathtt{F}' = \bigcap_{k,M} \mathtt{F}^M_k$. We have $\nu(\mathtt{F}') \geq \frac{\nu(\mathtt{F})}{2}$. Then, if $\varphi \in \mathcal{C}^0(\mathbb{P}T\mathbf{M},\mathbb{R})$, we may find for any $\epsilon > 0$ a function $\varphi_k \in \mathcal{F}$

with $\|\varphi-\varphi_k\|_\infty<\epsilon$. Let $M\in\mathbb{N}$. Take $N=N_\epsilon^{k,M}$ such that $|\int\varphi_k\,d\mu^M_{\hat{x},n}-\int\varphi_k\,d\xi^M|<\epsilon$ for $n>N$ and for all $x\in \mathtt{F}_k^M$. In particular for all $x\in\mathtt{F}'$ we have for $n>N$

$$\begin{aligned}\left|\int\varphi\,d\mu^M_{\hat{x},n}-\int\varphi\,d\xi^M\right|\leq&\left|\int\varphi_k\,d\mu^M_{\hat{x},n}-\int\varphi\,d\mu^M_{\hat{x},n}\right|+\left|\int\varphi_k\,d\mu^M_{\hat{x},n}-\int\varphi_k\,d\xi^M\right|\\&+\left|\int\varphi_k\,d\xi^M-\int\varphi\,d\xi^M\right|,\\ \leq&2\|\varphi-\varphi_k\|_\infty+\left|\int\varphi_k\,d\mu^M_{\hat{x},n}-\int\varphi_k\,d\xi^M\right|,\\ <&3\epsilon.\end{aligned}$$

This proves (2.2) by taking $\mathtt{F}'$ in the place of $\mathtt{F}$. One proves similarly (2.3).

Fix now M. For each $n\in\mathbb{N}$ and $x\in\mathtt{F}$ we let $E_n(x)=E(\hat{x})\cap[0,n[$ and $E_n^M(x)=E^M(\hat{x})\cap[0,n[$. We also let $\mathtt{E}_n^M$ be the partition of $\mathtt{F}$ with atoms $A_E:=\{x\in D,\ E_n^M(x)=E\}$ for $E\subset[0,n[$. Given a partition Q of $\mathbb{P}T\mathbf{M}$, we also let $Q^{\mathtt{E}_n^M}$ be the partition of $\hat{\mathtt{F}}:=\{\hat{x},\ x\in\mathtt{F}\cap D\}$ finer than $\pi^{-1}\mathtt{E}_n^M$ with atoms $\left\{\hat{x}\in\hat{\mathtt{F}},\ E_n^M(x)=E \text{ and } \forall k\in E,\ F^k\hat{x}\in Q_k\right\}$ for $E\subset[0,n[$ and $(Q_k)_{k\in E}\in Q^E$. We let ∂Q be the boundary of the partition Q, which is the union of the boundaries of its atoms. For a measure η and a subset A of $\mathbf{M}$ with $\eta(A)>0$ we denote by $\eta_A=\frac{\eta(A\cap\cdot)}{\eta(A)}$ the induced probability measure on A. Moreover, for two sets A,B we let $A\Delta B$ denote the symmetric difference of A and B, i.e. $A\Delta B=(A\setminus B)\cup(B\setminus A)$. Finally, let $H:]0,1[\to\mathbb{R}^+$ be the map $t\mapsto -t\log t-(1-t)\log(1-t)$. Recall that $\hat{\zeta}^+$ is the lift of ζ on $\mathbb{P}T\mathbf{M}$ to the unstable Oseledets bundle (with ζ as in Subsection 2.3).

Lemma 3. *For any finite partition Q and any $m\in\mathbb{N}^*$ with $\hat{\xi}^M(\partial Q^m)=0$ we have*

$$h(\nu)\leq\beta_M\frac{1}{m}H_{\underline{\hat{\xi}}^M}(Q^m)+\limsup_n\frac{1}{n}H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathtt{E}_n^M})+H(2/M)+\frac{12\log\sharp Q}{M}+\iota. \tag{2.4}$$

Before the proof of Lemma 3, we first recall a technical lemma from [2].

Lemma 4 (Lemma 6 in [2]). *Let (X,T) be a topological system. Let μ be a Borel probability measure on X and let E be a finite subset of $\mathbb{N}$. For any finite partition Q of X, we have with $\mu^E:=\frac{1}{\sharp E}\sum_{k\in E}T^k_*\mu$ and $Q^E:=\bigvee_{k\in E}T^{-k}Q$:*

$$\frac{1}{\sharp E}H_\mu(Q^E)\leq\frac{1}{m}H_{\mu_E}(Q^m)+6m\frac{\sharp(E+1)\Delta E}{\sharp E}\log\sharp Q.$$

Proof of Lemma 3. As the complement of $E_n^M(x)$ is the disjoint union of neutral blocks with length larger than M, there are at most $A_n^M=\sum_{k=0}^{[2n/M]+1}\binom{n}{k}$ possible values for $E_n^M(x)$ so that

$$\begin{aligned}\frac{1}{n}H_\zeta(P^n)&=\frac{1}{n}H_\zeta(P^n|\mathtt{E}_n^M)+H_\zeta(\mathtt{E}_n^M),\\&\leq\frac{1}{n}H_\zeta(P^n|\mathtt{E}_n^M)+\log A_n^M,\end{aligned}$$

$$\liminf_n\frac{1}{n}H_\zeta(P^n)\leq\limsup_n\frac{1}{n}H_\zeta(P^n|\mathtt{E}_n^M)+H(2/M) \text{ by using Stirling's formula.}$$

Moreover

$$\frac{1}{n}H_\zeta(P^n|\mathtt{E}_n^M) = \frac{1}{n}H_{\hat{\zeta}^+}(\pi^{-1}P^n|\pi^{-1}\mathtt{E}_n^M),$$
$$\leq \frac{1}{n}H_{\hat{\zeta}^+}(Q^{\mathtt{E}_n^M}|\pi^{-1}\mathtt{E}_n^M) + \frac{1}{n}H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathtt{E}_n^M}).$$

For $E \subset [0, n[$ we let $\hat{\zeta}^+_{E,n} = \frac{n}{\sharp E}\int \mu^M_{\hat{x},n}\, d\zeta_{A_E}(x)$, which may be also written as $\left(\hat{\zeta}^+_{\pi^{-1}A_E}\right)^E$ by using the notations of Lemma 4. By Lemma 4 applied to the system $(\mathbb{P}T\mathbf{M}, F)$ and the measures $\mu := \hat{\zeta}^+_{\pi^{-1}A_E}$ for $A_E \in \mathtt{E}_n^M$ we have for all $n > m \in \mathbb{N}^*$:

$$H_{\hat{\zeta}^+}\left(Q^{\mathtt{E}_n^M}|\pi^{-1}\mathtt{E}_n^M\right) = \sum_E \zeta(A_E)H_{\hat{\zeta}^+_{\pi^{-1}A_E}}(Q^E),$$
$$\leq \sum_E \zeta(A_E)\sharp E\left(\frac{1}{m}H_{\hat{\zeta}^+_{E,n}}(Q^m) + 6m\frac{\sharp(E+1)\Delta E}{\sharp E}\log\sharp Q\right).$$

Recall again that if $E = E_n^M(x)$ for some x then the complement set of E in $[1, n[$ is made of neutral blocks of length larger than M, therefore $\sharp(E+1)\Delta E \leq \frac{2M}{n}$. Moreover it follows from $\xi^M(\partial Q^m) = 0$ and (2.2), that $\mu^M_{\hat{x},n}(A^m)$ for $A^m \in Q^m$ and $\sharp E_n^M(x)/n$ are converging to $\underline{\hat{\xi}}^M(A^m)$ and β_M respectively uniformly in $x \in \mathtt{F}$ when n goes to infinity. Then we get by taking the limit in n:

$$\limsup_n \frac{1}{n}H_{\hat{\zeta}^+}\left(Q^{\mathtt{E}_n^M}|\pi^{-1}\mathtt{E}_n^M\right) \leq \beta_M\frac{1}{m}H_{\underline{\hat{\xi}}^M}(Q^m) + \frac{12m\log\sharp Q}{M},$$
$$h(\nu) - \iota \leq \liminf_n \frac{1}{n}H_\zeta(P^n) \leq \beta_M\frac{1}{m}H_{\underline{\hat{\xi}}^M}(Q^m) + \limsup_n \frac{1}{n}H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathtt{E}_n^M})$$
$$+ H(2/M) + \frac{12m\log\sharp Q}{M}.$$

□

2.5. **Bounding the entropy of the neutral component.** For a $\mathcal{C}^1$ diffeomorphism f on $\mathbf{M}$ we put $C(f) := 2A_f H(A_f^{-1}) + \frac{\log^+\|df\|_\infty}{r} + B_r$ with $A_f = \log^+\|df\|_\infty + \log^+\|df^{-1}\|_\infty + 1$ and a universal constant B_r depending only r precised later on. Clearly $f \mapsto C(f)$ is continuous in the $\mathcal{C}^1$ topology and $\frac{\lambda^+(f)}{r} = \lim_{\mathbb{N}\ni p\to+\infty}\frac{C(f^p)}{p}$ whenever $\lambda^+(f) > 0$ (indeed $A_{f^p} \xrightarrow{p} +\infty$, therefore $H(A_{f^p}^{-1}) \xrightarrow{p} 0$). In particular, if $\frac{\lambda^+(f)}{r} < \alpha$ and $f_k \xrightarrow{k} f$ in the $\mathcal{C}^1$ topology, then there is p with $\lim_k \frac{C(f_k^p)}{p} < \alpha$.

In this section we consider the empirical measures associated to an ergodic hyperbolic measure ν with $\lambda^+(\nu) > \frac{\log\|df\|_\infty}{r} + \delta$, $\delta > 0$. Without loss of generality we can assume $\delta < \frac{r-1}{r}\log 2$. Then as observed in Subsection 2.2 we have $\hat{\nu}^+(H_\delta) > 0$. For $x \in \mathcal{R}$ we let $m_n(x) = \max\{k < n,\ F^k\hat{x} \in H_\delta\}$. By a standard application of the ergodic theorem we have

$$\frac{m_n(x)}{n} \xrightarrow{n} 1 \text{ for } \nu \text{ a.e. } x.$$

By taking a smaller subset $\mathtt{F}$, we can assume the above convergence of m_n is uniform on $\mathtt{F}$ and that $\sup_{x\in\mathtt{F}}\min\{k \leq n,\ F^k\hat{x} \in H_\delta\} \leq N$ for some positive integer N.

We bound the term $\limsup_n \frac{1}{n} H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathtt{E}_n^M})$ in the right member of (2.4) Lemma 3, which corresponds to the local entropy contribution plus the entropy in the neutral part.

Lemma 5. *There is $\kappa > 0$ such that the empirical measures associated to $G := \pi^{-1}G_\kappa \cap H_\delta$ satisfy the following properties. For all $q, M \in \mathbb{N}^*$, there are $\epsilon_q > 0$ (depending only on $\|d^k(f^q)\|_\infty$, $2 \leq q \leq r$ [§]) and $\gamma_{q,M}(f) > 0$ with*

$$\forall K > 0 \;\; \limsup_q \limsup_M \left(\sup_f \left\{ \gamma_{q,M}(f) \mid \|df\|_\infty \vee \|df^{-1}\|_\infty < K \right\} \right) = 0 \tag{2.5}$$

such that for any partition Q of $\mathbb{P}T\mathbf{M}$ with diameter less than ϵ_q, we have:

$$\begin{aligned}\limsup_n \frac{1}{n} H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathtt{E}_n^M}) \leq & (1-\beta_M)C(f) \\ & + \left(\log 2 + \frac{1}{r-1}\right)\left(\int \frac{\log^+ \|df^q\|}{q} d\xi^M - \int \phi \, d\hat{\xi}^M\right) \\ & + \gamma_{q,M}(f).\end{aligned}$$

The proof of Lemma 5 appears after the statement of Proposition 4, which is a *semi-local Reparametrization Lemma.*

Proposition 4. *There is $\kappa > 0$ such that the empirical measures associated to $G := \pi^{-1}G_\kappa \cap H_\delta$ satisfy the following properties. For all $q \in \mathbb{N}^*$ there are $\epsilon_q > 0$ (depending only on $\|d^k(f^q)\|_\infty$, $2 \leq q \leq r$) and $\gamma_{q,M}(f) > 0$ with*

$$\forall K > 0 \;\; \limsup_q \limsup_M \left(\sup_f \left\{ \gamma_{q,M}(f) \mid \|df\|_\infty \vee \|df^{-1}\|_\infty < K \right\} \right) = 0$$

such that for any partition Q with diameter less than $\epsilon < \epsilon_q$, the following property holds for n large enough.

Any atom F_n of the partition $Q^{\mathtt{E}_n^M}$ may be covered by a family Ψ_{F_n} of $\mathcal{C}^r$ curves $\psi : [-1,1] \to \mathbf{M}$ satisfying $\|d(f^k \circ \psi)\|_\infty \leq 1$ for any $k = 0, \cdots, n-1$, such that

$$\begin{aligned}\frac{1}{n}\log \sharp \Psi_{F_n} \leq & \left(1 - \frac{\sharp E_n^M}{n}\right) C(f) \\ & + \left(\log 2 + \frac{1}{r-1}\right)\left(\int \frac{\log^+ \|d_x f^q\|_\epsilon}{q} \, d\zeta^M_{F_n}(x) - \int \phi \, d\hat{\zeta}^M_{F_n}\right) \\ & + \gamma_{q,M}(f) + \tau_n,\end{aligned}$$

where $\lim_n \tau_n = 0$, $E_n^M = E_n^M(x)$ for $x \in F_n$, $\hat{\zeta}^M_{F_n} = \int \mu^M_{\hat{x},n} \, d\zeta_{F_n}(x)$ and $\zeta^M_{F_n} = \pi_* \hat{\zeta}^M_{F_n}$ its push-forward on $\mathbf{M}$.

The proof of Proposition 4 is given in the last section. Proposition 4 is very similar to the Reparametrization Lemma in [4]. Here we reparametrize an atom F_n of $Q^{\mathtt{E}_n^M}$ instead of Q^n in [4].

[§]Here

$$\|d^k(f^q)\|_\infty = \sup_{\alpha \in \mathbb{N}^2, \, |\alpha| = k} \sup_{x,y} \left\| \partial_y^\alpha \left(\exp^{-1}_{f(x)} \circ f \circ \exp_x \right)(\cdot) \right\|_\infty$$

Proof of Lemma 5 assuming Proposition 4. We take $\kappa > 0$ and $\epsilon_q > 0$ as in Proposition 4. Observe that

$$H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathtt{E}_n^M}) \leq \sum_{F_n \in Q^{\mathtt{E}_n^M}} \hat{\zeta}^+(F_n) \log \sharp\{A^n \in P^n,\ \pi^{-1}(A^n) \cap \hat{\mathtt{F}} \cap F_n \neq \emptyset\}.$$

As $\nu(\partial P) = 0$, for all $\gamma > 0$, there is $\chi > 0$ and a continuous function $\vartheta : \mathbf{M} \to \mathbb{R}^+$ equal to 1 on the χ-neighborhood ∂P^χ of ∂P satisfying $\int \vartheta \, d\nu < \gamma$. Then we have uniformly in $x \in \mathtt{F}$ by (2.3):

$$\limsup_n \frac{1}{n} \sharp\{0 \leq k < n,\ f^k x \in \partial P^\chi\} \leq \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \vartheta(f^k x) = \int \vartheta \, d\nu < \gamma. \tag{2.6}$$

Assume that for arbitrarily large n there is $F_n \in Q^{\mathtt{E}_n^M}$ and $\psi \in \Psi_{F_n}$ with $\sharp\{A^n \in P^n,\ A^n \cap \psi([-1,1]) \cap \mathtt{F} \neq \emptyset\} > ([\chi^{-1}]+1)\sharp P^{\gamma n}$. We reparametrize ψ on $\mathtt{F}$ by $[\chi^{-1}]+1$ affine contractions θ so that the length of $f^k \circ \psi \circ \theta$ is less than χ for all $0 \leq k < n$ and $(\psi \circ \theta)([-1,1]) \cap \mathtt{F} \neq \emptyset$. Then we have $\sharp\{0 \leq k < n,\ \partial P \cap f^k \circ \psi \circ \theta([-1,1]) \neq \emptyset\} > \gamma n$ for some θ. In particular we get $\sharp\{0 \leq k < n,\ f^k x \in \partial P^\chi\} > \gamma n$ for any $x \in \psi \circ \theta([-1,1])$, which contradicts (2.6). Therefore we have

$$\limsup_n \sup_{F_n,\, \psi \in \Psi_{F_n}} \frac{1}{n} \log \{A^n \in P^n,\ A^n \cap \psi([-1,1]) \cap \mathtt{F} \neq \emptyset\} = 0.$$

Together with Proposition 4 we get

$$\begin{aligned}
\limsup_n \frac{1}{n} H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathtt{E}_n^M}) &\leq \limsup_n \sum_{F_n \in Q^{\mathtt{E}_n^M}} \hat{\zeta}^+(F_n) \frac{1}{n} \log \sharp\Psi_{F_n}, \\
&\leq \limsup_n \sum_{F_n \in Q^{\mathtt{E}_n^M}} \hat{\zeta}^+(F_n) \left(1 - \frac{\sharp E_n^M}{n}\right) C(f) + \\
&+ \limsup_n \sum_{F_n \in Q^{\mathtt{E}_n^M}} \hat{\zeta}^+(F_n) \left(\log 2 + \frac{1}{r-1}\right) \left(\int \frac{\log^+ \|df^q\|}{q} \, d\zeta_{F_n}^M - \int \phi \, d\hat{\zeta}_{F_n}^M\right) \\
&+ \gamma_{q,M}(f), \\
&\leq (1 - \beta_M) C(f) + \left(\log 2 + \frac{1}{r-1}\right) \left(\int \frac{\log^+ \|df^q\|}{q} d\xi^M - \int \phi \, d\hat{\xi}^M\right) + \gamma_{q,M}(f).
\end{aligned}$$

This concludes the proof of Lemma 5.

□

2.6. **Proof of the Main Theorem.** We first reduce the Main Theorem to the following statement.

Proposition 5. *Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of $\mathcal{C}^r$, with $r > 1$, surface diffeomorphisms converging $\mathcal{C}^r$ weakly to a diffeomorphism f. Assume there is a sequence $(\hat{\nu}_k^+)_k$ of ergodic F_k-invariant measures converging to $\hat{\mu}$ with $\lim_k \lambda^+(\nu_k) > \frac{\log^+ \|df\|_\infty}{r}$.*

Then, there are F-invariant measures $\hat{\mu}_0$ and $\hat{\mu}_1^+$ with $\hat{\mu} = (1-\beta)\hat{\mu}_0 + \beta \hat{\mu}_1^+$, $\beta \in [0,1]$, such that:

$$\limsup_{k \to +\infty} h(\nu_k) \leq \beta h(\mu_1) + (1-\beta) C(f).$$

Proof of the Main Theorem assuming Proposition 5. Let $(\hat{\nu}_k^+)_k$ be a sequence of ergodic F_k-invariant measures converging to $\hat{\mu}$.

As previously mentionned, for any $\alpha > \lambda^+(f)/r$ there is $p \in \mathbb{N}^*$ with $\alpha > \frac{C(f^p)}{p}$. We can also assume $\frac{\log \|df^p\|_\infty}{pr} < \alpha$. Let $\hat{\nu}_k^{+,p}$ be an ergodic component of $\hat{\nu}_k^+$ for F_k^p and let us denote by ν_k^p its push forward on $\mathbf{M}$. We have $h_{f_k^p}(\nu_k^p) = ph_{f_k}(\nu_k)$ for all k. By taking a subsequence we can assume that $(\hat{\nu}_k^{+,p})_k$ is converging. Its limit $\hat{\mu}^p$ satisfies $\frac{1}{p}\sum_{0\leq l<p} F_*^k \hat{\mu}^p = \hat{\mu}$. If $\lim_k \lambda^+(\nu_k^p) < \frac{\log^+ \|df^p\|_\infty}{r} < p\alpha$, then by Ruelle's inequality we get

$$\begin{aligned}
\limsup_{k\to+\infty} h_{f_k}(\nu_k) &= \limsup_{k\to+\infty} \frac{1}{p} h_{f_k^p}(\nu_k^p),\\
&\leq \lim_{k\to+\infty} \frac{1}{p}\lambda^+(\nu_k^p),\\
&\leq \alpha.
\end{aligned}$$

This proves the Main Theorem with $\beta = 1$.

We consider then the case $\lim_k \lambda^+(\nu_k^p) > \frac{\log^+ \|df^p\|_\infty}{r}$. By applying Proposition 4 to the p-power systems, we get F^p-invariant measure $\hat{\mu}_0^p$ and $\hat{\mu}_1^{+,p}$ with $\hat{\mu}^p = (1-\beta)\hat{\mu}_0^p + \beta\hat{\mu}_1^{+,p}$, $\beta \in [0,1]$, such that we have with $\mu_1^p = \pi_*\hat{\mu}_1^{+,p}$:

$$\limsup_{k\to+\infty} h_{f_k^p}(\nu_k^p) \leq \beta h_{f^p}(\mu_1^p) + (1-\beta)C(f^p).$$

But $h_{f^p}(\mu_1^p) = ph_f(\mu_1)$ with $\mu_1 = \frac{1}{p}\sum_{0\leq l<p} f^k \mu_1^p$. One easily checks that $\hat{\mu}_1^+ = \frac{1}{p}\sum_{0\leq l<p} F^k \hat{\mu}_1^{+,p}$. Moreover we have :

$$\begin{aligned}
\limsup_{k\to+\infty} h_{f_k}(\nu_k) &= \limsup_{k\to+\infty} \frac{1}{p} h_{f_k^p}(\nu_k^p),\\
&\leq \beta\frac{1}{p}h_{f^p}(\mu_1^p) + (1-\beta)\frac{C(f^p)}{p},\\
&\leq \beta h_f(\mu_1) + (1-\beta)\alpha.
\end{aligned}$$

□

We show now Proposition 5 by using Lemma 5. Without loss of generality we can assume $\liminf_k h(\nu_k) > 0$. For μ a.e. x, we have $\lambda^-(x) \leq 0$. If not, some ergodic component $\tilde{\mu}$ of μ would have two positive Lyapunov exponents and therefore should be the periodic measure at a source S (see e.g. Proposition 4.4 in [17]). But then for large k the probability ν_k would give positive measure to the basin of attraction of the sink S for f^{-1} and therefore ν_k would be equal to $\tilde{\mu}$ contradicting $\liminf_k h(\nu_k) > 0$.

Let $\delta > 0$ with $\lim_k \lambda^+(\nu_k) > \frac{\log \|df\|_\infty}{r} + \delta$. Then take κ as in Lemma 5. We consider the empirical measures associated to $G = \pi^{-1}G_\kappa \cap H_\delta$. By a diagonal argument, there is a subsequence in k such that the geometric component $\hat{\xi}_k^M$ of $\hat{\nu}_k^+$ is converging to some $\hat{\xi}_\infty^M$ for all $M \in \mathbb{N}$. Let us also denote by β_M^∞ the limit in k of β_M^k. Then consider a subsequence in M such that $\hat{\xi}_\infty^M$ is converging to $\beta\hat{\mu}_1$ with $\beta = \lim_M \beta_M^\infty$. We also let $(1-\beta)\hat{\mu}_0 = \hat{\mu} - \beta\hat{\mu}_1$. In this way, $\hat{\mu}_0$ and $\hat{\mu}_1$ are both probability measures.

Lemma 6. *The measures $\hat{\mu}_0$ and $\hat{\mu}_1$ satisfy the following properties:*

- *$\hat{\mu}_1$ and $\hat{\mu}_0$ are F-invariant,*

- *$\lambda^+(x) \geq \delta$ for μ_1-a.e. x and $\hat{\mu}_1 = \hat{\mu}_1^+$.*

Proof. The neutral blocks in the complement set of $E^M(x)$ have length larger than M. Therefore for any continuous function $\varphi : \mathbb{P}T\mathbf{M} \to \mathbb{R}$ and for any k, we have

$$\left| \int \varphi \, d\hat{\xi}_k^M - \int \varphi \circ F \, d\hat{\xi}_k^M \right| \leq \frac{2 \sup_{\hat{x}} |\varphi(\hat{x})|}{M}.$$

Letting k, then M go to infinity, we get $\int \varphi \, d\hat{\mu}_1 = \int \varphi \circ F \, d\hat{\mu}_1$, i.e. $\hat{\mu}_1$ is F-invariant.

We let K_M be the compact subset of $\mathbb{P}T\mathbf{M}$ given by $K_M = \{\hat{x} \in \mathbb{P}T\mathbf{M}, \ \exists 1 \leq m \leq M \ \phi_m(\hat{x}) \geq m\delta\}$. Let $\hat{x} \in \mathtt{G}_k$, where $\mathtt{G}_k$ is the set where the empirical measures are converging to $\hat{\xi}_k^M$ (see Subsection 2.1). Observe that

$$\lim_n \mu_{\hat{x},n}^M(K_M) = \hat{\xi}_k^M(K_M) = \hat{\xi}_k^M(\mathbb{P}T\mathbf{M}). \tag{2.7}$$

Indeed for any $k \in E^M(\hat{x})$ there is $1 \leq m \leq M$ with $F^m(F^k\hat{x}) \in G \subset H_\delta$. Moreover, as already mentioned, δ-hyperbolic points w.r.t. ψ are δ-hyperbolic w.r.t. ϕ. Therefore $\phi_m(F^k\hat{x}) \geq m\delta$. Consequently we have $\lim_n \mu_{\hat{x},n}^M(K_M) = \lim_n \mu_{\hat{x},n}^M(\mathbb{P}T\mathbf{M}) = \xi_k^M(\mathbb{P}T\mathbf{M})$. The set K_M being compact in $\mathbb{P}T\mathbf{M}$, we get $\xi_k^M(K_M) \geq \lim_n \mu_{\hat{x},n}^M(K_M)$ and (2.7) follows.

Also we have $\hat{\xi}_\infty^M(K_M) \geq \limsup_k \hat{\xi}_k^M(K_M) = \limsup_k \hat{\xi}_k^M(\mathbb{P}T\mathbf{M}) = \beta_M^\infty$. Therefore we have $\hat{\mu}_1(\bigcup_M K_M) = 1$ as $\hat{\xi}_\infty^M$ goes increasingly in M to $\beta\hat{\mu}_1$. The F-invariant set $\bigcap_{k\in\mathbb{Z}} F^{-k}\left(\bigcup_M K_M\right)$ has also full $\hat{\mu}_1$-measure and for all $\hat{x} = (x, v)$ in this set we have $\limsup_n \frac{1}{n} \log \|d_x f^n(v)\| \geq \delta$. Consequently the measure $\hat{\mu}_1$ is supported on the unstable bundle $\mathcal{E}_+(x)$ and $\lambda^+(x) \geq \delta$ for μ_1-a.e. x. □

Remark 6. *In Theorem C of* [10]*, the measure $\beta\hat{\mu}_1^+$ is obtained as the limit when δ goes to zero of the component associated to the set $G^\delta := \{x, \ \forall l > 0 \ \phi_l(\hat{x}) \geq \delta l\} \supset \pi^{-1}G_\kappa \cap H_\delta$.*

We pursue now the proof of Proposition 5. Let $q, M \in \mathbb{N}^*$. Fix a sequence $(\iota_k)_k$ of positive numbers with $\iota_k \xrightarrow{k} 0$. We consider a partition Q satisfying $\mathrm{diam}(Q) < \epsilon_q$ with ϵ_q as in Lemma 5. The sequence $(f_k)_k$ being $\mathcal{C}^r$ bounded, one can choose ϵ_q independently of f_k, $k \in \mathbb{N}$.

By a standard argument of countability we may assume that for all $m \in \mathbb{N}^*$ the boundary of Q^m has zero-measure for $\hat{\mu}_1$ and all the measures $\hat{\xi}_k^M$, $M \in \mathbb{N}^*$ and $k \in \mathbb{N} \cup \{\infty\}$. Combining Lemma 5 and Lemma 3 we get with $\gamma_{q,Q,M}(f) = \gamma_{q,M}(f) + H\left(\frac{2}{M}\right) + \frac{12 \log \sharp Q}{M}$:

$$\begin{aligned} h(\nu_k) \leq & \beta_M^k \frac{1}{m} H_{\underline{\hat{\xi}_k}^M}(Q^m) + (1 - \beta_M^k) C(f_k) \\ & + \left(\log 2 + \frac{1}{r-1}\right) \left(\int \frac{\log^+ \|df_k^q\|}{q} d\xi_k^M - \int \phi \, d\hat{\xi_k}^M \right) \\ & + \gamma_{q,Q,M}(f_k) + \iota_k. \end{aligned}$$

By letting k, then M go to infinity, we obtain for all m:

$$\begin{aligned} \limsup_k h(\nu_k) \leq & \beta \frac{1}{m} H_{\hat{\mu}_1^+}(Q^m) + (1 - \beta) C(f) \\ & + \left(\log 2 + \frac{1}{r-1}\right) \left(\int \frac{\log^+ \|df^q\|}{q} d\mu_1 - \int \phi \, d\hat{\mu}_1^+ \right) \\ & + \limsup_M \sup_k \gamma_{q,Q,M}(f_k). \end{aligned}$$

By letting m go to infinity, we get:

$$\begin{aligned}\limsup_k h(\nu_k) \leq & \beta h(\hat{\mu}_1^+) + (1-\beta)C(f) \\ & + \left(\log 2 + \frac{1}{r-1}\right)\left(\int \frac{\log^+ \|df^q\|}{q} d\mu_1 - \int \phi \, d\hat{\mu}_1^+\right) \\ & + \limsup_M \sup_k \gamma_{q,M}(f_k).\end{aligned}$$

But $h(\hat{\mu}_1^+) = h(\mu_1)$ (see e.g. Corollary 4.2 in [10]) and $\int \phi \, d\hat{\mu}_1^+ = \lambda^+(\mu_1) = \lim_q \int \frac{\log^+ \|df^q\|}{q} d\mu_1$. Therefore by letting q go to infinity we finally obtain with the asymptotic property (2.5) of $\gamma_{q,M}$:

$$\limsup_k h(\nu_k) \leq \beta h(\mu_1) + (1-\beta)C(f).$$

3. Semi-local Reparametrization Lemma

In this section we prove the semi-local *Reparametrization Lemma* stated in Proposition 4.

3.1. **Strongly bounded curves.** To simplify the exposition (by avoiding irrelevant technical details involving the exponential map) we assume that $\mathbf{M}$ is the two-torus $\mathbb{T}^2$ with the usual Riemannian structure inherited from $\mathbb{R}^2$. Borrowing from [2] we first make the following definitions.

A $\mathcal{C}^r$ embedded curve $\sigma : [-1,1] \to \mathbf{M}$ is said *bounded* when $\max_{k=2,\cdots,r} \|d^k\sigma\|_\infty \leq \frac{\|d\sigma\|_\infty}{6}$.

Lemma 7. *Assume σ is a bounded curve. Then for any $x \in \sigma([-1,1])$, the curve σ contains the graph of a κ-admissible map at x with $\kappa = \frac{\|d\sigma\|_\infty}{6}$.*

Proof. Let $x = \sigma(s)$, $s \in [-1,1]$. One checks easily (see Lemma 7 in [4] for further details) that for all $t \in [-1,1]$ the angle $\angle \sigma'(s), \sigma'(t) < \frac{\pi}{6} \leq 1$ and therefore $\int_0^1 \sigma'(t) \cdot \frac{\sigma'(s)}{\|\sigma'(s)\|} \, dt \geq \frac{\|d\sigma\|_\infty}{6}$. Therefore, as $\sigma'(s) \in \mathcal{E}_+(x)$, the image of σ contains the graph of an $\frac{\|d\sigma\|_\infty}{6}$-admissible map at x. □

A $\mathcal{C}^r$ bounded curve $\sigma : [-1,1] \to \mathbf{M}$ is said *strongly ϵ-bounded* for $\epsilon > 0$ if $\|d\sigma\|_\infty \leq \epsilon$. For $n \in \mathbb{N}^*$ and $\epsilon > 0$ a curve is said *strongly (n,ϵ)-bounded* when $f^k \circ \sigma$ is strongly ϵ-bounded for all $k = 0, \cdots, n-1$.

We consider a $\mathcal{C}^r$ smooth diffeomorphism $g : \mathbf{M} \circlearrowleft$ with $\mathbb{N} \ni r \geq 2$. For $\hat{x} = (x,v) \in \mathbb{P}T\mathbf{M}$ with $\pi(\hat{x}) = x$, we let $k_g(x) \geq k'_g(\hat{x})$ be the following integers:

$$k_g(x) := [\log \|d_x g\|],$$

$$k'_g(\hat{x}) := [\log \|d_x g(v)\|] = [\phi_g(\hat{x})].$$

In the next lemma, we reparametrize the image by g of a bounded curve. The proof of this lemma is mostly contained in the proof of the Reparametrization Lemma [2], but we reproduce it for the sake of completeness.

Lemma 8. *Let* $\frac{R_{inj}}{2} > \epsilon = \epsilon_g > 0$ *satisfying* $\|d^s g^x_{2\epsilon}\|_\infty \leq 3\epsilon \|d_x g\|$ *for all* $s = 1, \cdots, r$ *and all* $x \in \mathbf{M}$*, where* $g^x_{2\epsilon} = g \circ \exp_x(2\epsilon \cdot) = g(x + 2\epsilon \cdot) : \{w_x \in T_x\mathbf{M},\ \|w_x\| \leq 1\} \to \mathbf{M}$*. We assume* $\sigma : [-1,1] \to \mathbf{M}$ *is a strongly* ϵ*-bounded* $\mathcal{C}^r$ *curve and we let* $\hat{\sigma} : [-1,1] \to \mathbb{P}T\mathbf{M}$ *be the associated induced map.*

Then for some universal constant $C_r > 0$ *depending only on* r *and for any pair of integers* (k, k') *there is a family* Θ *of affine maps from* $[-1,1]$ *to itself satisfying:*

- $\hat{\sigma}^{-1}\left(\left\{\hat{x} \in \mathbb{P}T\mathbf{M},\ k_g(x) = k \text{ and } k'_g(\hat{x}) = k'\right\}\right) \subset \bigcup_{\theta \in \Theta} \theta([-1,1])$,
- $\forall \theta \in \Theta$*, the curve* $g \circ \sigma \circ \theta$ *is bounded,*
- $\forall \theta \in \Theta,\ |\theta'| \leq e^{\frac{k'-k-1}{r-1}}/4$,
- $\sharp\Theta \leq C_r e^{\frac{k-k'}{r-1}}$.

Proof. First step : **Taylor polynomial approximation.** One computes for an affine map $\theta : [-1,1] \circlearrowleft$ with contraction rate b precised later and with $y = \sigma(t)$, $k_g(y) = k$, $k'_g(y) = k'$, $t \in \theta([-1,1])$:

$$\begin{aligned}\|d^r(g \circ \sigma \circ \theta)\|_\infty &\leq b^r \left\|d^r\left(g^y_{2\epsilon} \circ \sigma^y_{2\epsilon}\right)\right\|_\infty, \text{with } \sigma^y_{2\epsilon} := (2\epsilon)^{-1}\exp_y^{-1} \circ \sigma = 2\epsilon^{-1}(\sigma(\cdot) - y),\\ &\leq b^r \left\|d^{r-1}\left(d_{\sigma^y_{2\epsilon}} g^y_{2\epsilon} \circ d\sigma^y_{2\epsilon}\right)\right\|_\infty,\\ &\leq b^r 2^r \max_{s=0,\cdots,r-1} \left\|d^s\left(d_{\sigma^y_{2\epsilon}} g^y_{2\epsilon}\right)\right\|_\infty \max_{k=1,\cdots,r} \|d^k \sigma^y_{2\epsilon}\|_\infty.\end{aligned}$$

By assumption on ϵ, we have $\|d^s g^y_{2\epsilon}\|_\infty \leq 3\epsilon\|d_y g\|$ for any $r \geq s \geq 1$. Moreover $\max_{k=1,\cdots,r} \|d^k \sigma^y_{2\epsilon}\|_\infty \leq 1$ as σ is strongly ϵ-bounded. Therefore by Faá di Bruno's formula, we get for some¶ constants $C_r > 0$ depending only on r:

$$\max_{s=0,\cdots,r-1} \|d^s\left(d_{\sigma^y_{2\epsilon}} g^y_{2\epsilon}\right)\|_\infty \leq \epsilon C_r \|d_y g\|,$$

then ,

$$\begin{aligned}\|d^r(g \circ \sigma \circ \theta)\|_\infty &\leq \epsilon C_r b^r \|d_y g\| \max_{k=1,\cdots,r} \|d^k \sigma^y_{2\epsilon}\|_\infty,\\ &\leq C_r b^r \|d_y g\| \|d\sigma\|_\infty,\\ &\leq (C_r b^{r-1}\|d_y g\|)\|d(\sigma \circ \theta)\|_\infty,\\ &\leq (C_r b^{r-1} e^k)\|d(\sigma \circ \theta)\|_\infty, \text{ because } k(y) = k\ ,\\ &\leq e^{k'-4}\|d(\sigma \circ \theta)\|_\infty, \text{ by taking } b = \left(C_r e^{k-k'+4}\right)^{-\frac{1}{r-1}}.\end{aligned}$$

Therefore the Taylor polynomial P at 0 of degree $r-1$ of $d(g \circ \sigma \circ \theta)$ satisfies on $[-1,1]$:

$$\|P - d(g \circ \sigma \circ \theta)\|_\infty \leq e^{k'-4}\|d(\sigma \circ \theta)\|_\infty.$$

We may cover $[-1,1]$ by at most $b^{-1} + 1$ such affine maps θ.

Second step : **Bezout theorem.** Let $a = e^{k'}\|d(\sigma \circ \theta)\|_\infty$. Note that for $s \in [-1,1]$ with $k(\sigma \circ \theta(s)) = k$ and $k'(\sigma \circ \theta(s)) = k'$ we have $\|d(g \circ \sigma \circ \theta)(s)\| \in [ae^{-2}, ae^2]$, therefore $\|P(s)\| \in [ae^{-3}, ae^3]$. Moreover if we have now $\|P(s)\| \in [ae^{-3}, ae^3]$ for some $s \in [-1,1]$ we get also $\|d(g \circ \sigma \circ \theta)(s)\| \in [ae^{-4}, ae^4]$.

¶Although these constants may differ at each step, they are all denoted by C_r.

By Bezout theorem the semi-algebraic set $\{s \in [-1,1],\ \|P(s)\| \in [e^{-3}a, e^3 a]\}$ is the disjoint union of closed intervals $(J_i)_{i\in I}$ with $\sharp I$ depending only on r. Let θ_i be the composition of θ with an affine reparametrization from $[-1,1]$ onto J_i.

Third step : **Landau-Kolmogorov inequality.** By the Landau-Kolmogorov inequality on the interval (see Lemma 6 in [2]), we have for some constants $C_r \in \mathbb{N}^*$ and for all $1 \leq s \leq r$:

$$\begin{aligned}\|d^s(g\circ\sigma\circ\theta_i)\|_\infty &\leq C_r\left(\|d^r(g\circ\sigma\circ\theta_i)\|_\infty + \|d(g\circ\sigma\circ\theta_i)\|_\infty\right),\\ &\leq C_r\frac{|J_i|}{2}\left(\|d^r(g\circ\sigma\circ\theta)\|_\infty + \sup_{t\in J_i}\|d(g\circ\sigma\circ\theta)(t)\|\right),\\ &\leq C_r a\frac{|J_i|}{2}.\end{aligned}$$

We cut again each J_i into $1000C_r$ intervals $\tilde{J}_i$ of the same length with

$$\theta(\tilde{J}_i)\cap\sigma^{-1}\left\{x,\ k_g(x)=k \text{ and } k'_g(x)=k'\right\}\neq\emptyset.$$

Let $\tilde{\theta}_i$ be the affine reparametrization from $[-1,1]$ onto $\theta(\tilde{J}_i)$. We check that $g\circ\sigma\circ\tilde{\theta}_i$ is bounded:

$$\begin{aligned}\forall s=2,\cdots,r,\ \|d^s(g\circ\sigma\circ\tilde{\theta}_i)\|_\infty &\leq (1000C_r)^{-2}\|d^s(g\circ\sigma\circ\theta_i)\|_\infty,\\ &\leq \frac{1}{6}(1000C_r)^{-1}\frac{|J_i|}{2}a_n e^{-4},\\ &\leq \frac{1}{6}(1000C_r)^{-1}\frac{|J_i|}{2}\min_{s\in J_i}\|d(g\circ\sigma\circ\theta)(s)\|,\\ &\leq \frac{1}{6}(1000C_r)^{-1}\frac{|J_i|}{2}\min_{s\in\tilde{J}_i}\|d(g\circ\sigma\circ\theta)(s)\|,\\ &\leq \frac{1}{6}\|d(g\circ\sigma\circ\tilde{\theta}_i)\|_\infty.\end{aligned}$$

This conclude the proof with Θ being the family of all $\tilde{\theta}_i$'s. □

We recall now a useful property of bounded curve (see Lemma 7 in [4] for a proof).

Lemma 9. *Let $\sigma : [-1,1]\to\mathbf{M}$ be a $\mathcal{C}^r$ bounded curve and let B be a ball of radius less than ϵ. Then there exists an affine map $\theta : [-1,1]\circlearrowleft$ such that :*

- *$\sigma\circ\theta$ is strongly 3ϵ-bounded,*
- *$\theta([-1,1])\supset\sigma^{-1}B$.*

3.2. **Choice of the parameters κ and ϵ_q.** For a diffeomorphism $f : \mathbf{M}\circlearrowleft$ the scale ϵ_f in Lemma 8 may be chosen such that $\epsilon_{f^k}\leq\epsilon_{f^l}\leq\max(1,\|df\|_\infty)^{-k}$ for any $q\geq k\geq l\geq 1$. We take $\kappa=\frac{\epsilon_f}{36}$ and we choose $\epsilon_q<\frac{\epsilon_{f^q}}{3}$ such that for any $\hat{x},\hat{y}\in\mathbb{P}T\mathbf{M}$ which are ϵ_q-close and for any $0\leq l\leq q$:

$$\begin{aligned}\left|k_{f^l}(x)-k_{f^l}(y)\right| &\leq 1,\\ \left|k'_{f^l}(\hat{x})-k'_{f^l}(\hat{y})\right| &\leq 1.\end{aligned}\tag{3.1}$$

Without loss of generality we can assume the local unstable curve D (defined in Subsection 2.3) is reparametrized by a $\mathcal{C}^r$ strongly ϵ_q-bounded map $\sigma : [-1,1]\to D$.

Let F_n be an atom of the partition $Q^{\mathsf{E}_n^M}$ and let $E_n^M = E_n^M(x)$ for any $\hat{x} \in F_n$. Recall that the diameter of Q is less than ϵ_q. It follows from (3.1) that for any $\hat{x} \in F_n$ we have with $\hat{\zeta}_{F_n}^M = \int \mu_{\hat{x},n}^M \, d\zeta_{F_n}(x)$:

$$\sum_{l \in E_n^M} \left| k_{f^q}(f^l x) - k'_{f^q}(F^l \hat{x}) \right| \leq 10 \sharp E_n^M + \int \log^+ \|d_y f^q\| \, d\zeta_{F_n}^M(y) - \int \phi_q \, d\hat{\zeta}_{F_n}^M.$$

Therefore we may fix some $0 \leq c < q$, such that for any $x \in F_n$

$$\sum_{l \in (c+q\mathbb{N}) \cap E_n^M} \left| k_{f^q}(f^l x) - k'_{f^q}(F^l \hat{x}) \right| \leq 10\frac{n}{q} + \frac{1}{q}\left(\int \log^+ \|d_y f^q\| \, d\zeta_{F_n}^M(y) - \int \phi_q \, d\hat{\zeta}_{F_n}^M \right),$$

$$\leq 10\frac{n}{q} + 2A_f\frac{qn}{M} + \frac{1}{q}\int \log^+ \|d_y f^q\| \, d\zeta_{F_n}^M(y) - \int \phi \, d\hat{\zeta}_{F_n}^M.$$

3.3. **Combinatorial aspects.** We put $\partial_l E_n^M := \{a \in E_n^M \text{ with } a-1 \notin E_n^M\}$. Then we let $\mathcal{A}_n := \{0 = a_1 < a_2 < \cdots a_m\}$ be the union of $\partial_l E_n^M$, $[0, n[\setminus E_n^M$ and $(c + q\mathbb{N}) \cap [0, n[$. We also let $b_i = a_{i+1} - a_i$ for $i = 1, \cdots, m-1$ and $b_m = n - a_m$.

For a sequence $\mathbf{k} = (k_l, k'_l)_{l \in \mathcal{A}_n}$ of integers, a positive integer m_n and a subset $\overline{E}$ of $[0, n[$, we let $F_n^{\mathbf{k}, \overline{E}, m_n}$ be the subset of points $\hat{x} \in F_n$ satisfying:

- $\overline{E} = E_n(x) \setminus E_n^M(x)$,
- $k_{a_i} = k_{f^{b_i}}(f^{a_i} x)$ and $k'_{a_i} = k'_{f^{b_i}}(F^{a_i}\hat{x})$ for $i = 1, \cdots, m$,
- $m_n(x) = m_n$.

Lemma 10.

$$\sharp \left\{ (\mathbf{k}, \overline{E}, m_n), \ F_n^{\mathbf{k}, \overline{E}, m_n} \neq \emptyset \right\} \leq n e^{2nA_f H(A_f^{-1})} 3^{n(1/q + 1/M)} e^{nH(1/M)}.$$

Proof. Firstly observe that if $a_i \notin E_n^M$ then $b_i = 1$. In particular $\sum_{i, \ a_i \notin E_n^M} k_{a_i} \leq (n - \sharp E_n^M) \log^+ \|df\|_\infty \leq (n - \sharp E_n^M)(A_f - 1)$. The number of such sequences $(k_{a_i})_{i, \ a_i \notin E_n^M}$ is therefore bounded above by $\binom{r_n A_f}{r_n}$ with $r_n = n - \sharp E_n^M$and its logarithm is dominated by $r_n A_f H(A_f^{-1}) + 1 \leq nA_f H(A_f^{-1}) + 1$. Similarly the number of sequence $(k'_{a_i})_{i, \ a_i \notin E_n^M}$ is less than $nA_f H(A_f^{-1}) + 1$.

Then from the choice of ϵ_q in (3.1) there are at most three possible values of $k_{a_i}(x)$ for $a_i \in E_n^M$ and $x \in F_n$.

Finally as $\sharp \overline{E} \leq n/M$, the number of admissible sets $\overline{E}$ is less than $\binom{n}{[n/M]}$ and thus its logarithm is bounded above by $nH(1/M) + 1$. Clearly we can also fix the value of m_n up to a factor n.

□

3.4. **The induction.** We fix $\mathbf{k}$, m_n and $\overline{E}$ and we reparametrize appropriately the set $F_n^{\mathbf{k}, \overline{E}, m_n}$.

Lemma 11. *With the above notations there are families $(\Theta_i)_{i \leq m}$ of affine maps from $[-1, 1]$ into itself such that :*

- $\forall \theta \in \Theta_i \ \forall j \leq i$ *the curve* $f^{a_i} \circ \sigma \circ \theta$ *is strongly* $\epsilon_{f^{b_i}}$*-bounded,*
- $\hat{\sigma}^{-1}\left(F_n^{\mathbf{k}, \overline{E}, m_n}\right) \subset \bigcup_{\theta \in \Theta_i} \theta([-1, 1])$,

- $\forall \theta_i \in \Theta_i\ \forall j < i, \exists \theta^i_j \in \Theta_j,\ \frac{|\theta'_i|}{|(\theta^i_j)'|} \leq \prod_{j\leq l<i} e^{\frac{k'_{a_l}-k_{a_l}-1}{r-1}}/4,$
- $\sharp \Theta_i \leq C\max\left(1, \|df\|_\infty\right)^{\sharp \overline{E}\cap[1,a_i]} \prod_{j<i} C_r e^{\frac{k_{a_j}-k'_{a_j}}{r-1}}.$

Proof. We argue by induction on $i \leq m$. By changing the constant C, it is enough to consider i with $a_i > N$. Recall that the integer N was chosen in such a way that for any $x \in \mathtt{F}$ there is $0 \leq k \leq N$ with $F^k\hat{x} \in H_\delta$. We assume the family Θ_i for $i < m$ already built and we will define Θ_{i+1}. Let $\theta_i \in \Theta_i$. We apply Lemma 8 to the strongly $\epsilon_{f^{b_i}}$-bounded curve $f^{a_i}\circ\sigma\circ\theta_i$ with $g = f^{b_i}$. Let Θ be the family of affine reparametrizations of $[-1,1]$ satisfying the conclusions of Lemma 8, in particular $f^{a_{i+1}}\circ\sigma\circ\theta_i\circ\theta$ is bounded, $|\theta'| \leq e^{\frac{k'_{a_i}-k_{a_i}-1}{r-1}}/4$ for all $\theta \in \Theta$ and $\sharp\Theta \leq C_r e^{\frac{k_{a_i}-k'_{a_i}}{r-1}}$. We distinguish three cases:

- $\underline{a_{i+1} \in E^M_n}$. The diameter of $F^{a_{i+1}}F_n$ is less than $\epsilon_q \leq \frac{\epsilon_{f^{b_{i+1}}}}{3}$. By Lemma 9 there is an affine map $\psi : [-1,1] \circlearrowleft$ such that $f^{a_{i+1}}\circ\sigma\circ\theta_i\circ\theta\circ\psi$ is strongly $\epsilon_{f^{b_{i+1}}}$-bounded and its image contains the intersection of the bounded curve $f^{a_{i+1}}\circ\sigma\circ\theta_i\circ\theta$ with $f^{a_{i+1}}F_n$. We let then $\theta_{i+1} = \theta_i\circ\theta\circ\psi \in \Theta_{i+1}$.
- $\underline{a_{i+1} \in E\setminus E^M_n}$. Observe that $b_{i+1} = 1$, therefore $\epsilon_{f^{b_i}} \leq \epsilon_{f^{b_{i+1}}}$. Then the length of the curve $f^{a_{i+1}}\circ\sigma\circ\theta_i\circ\theta$ is less than $3\|df\|_\infty\epsilon_{f^{b_i}}$, thus may be covered by $[3\|df\|_\infty]+1$ balls of radius less than $\epsilon_{f^{b_{i+1}}}$. We then use Lemma 9 as in the previous case to reparametrize the intersection of this curve with each ball by a strongly $\epsilon_{f^{b_{i+1}}}$-bounded curve. We define in this way the associated parametrizations of Θ_{i+1}.
- $\underline{a_{i+1} \notin E \text{ and } a_{i+1} \notin E^M_n}$. We claim that $\|d(f^{a_{i+1}}\circ\sigma\circ\theta_i\circ\theta\| \leq \epsilon_f/6$. Take $\hat{x} \in F_n^{\mathbf{k},\overline{E},m_n}$ with $x = \pi(\hat{x}) = \sigma\circ\theta_i\circ\theta(s)$. Let $K_x = \max\{k < a_{i+1},\ F^k\hat{x} \in H_\delta\} \geq N$. Observe that $[K_x, a_{i+1}] \cap E^M_n = \emptyset$, therefore for $K_x \leq a_l < a_{i+1}$, we have $b_l = 1$, then $a_l = a_{i+1} - i - 1 + l$. We argue by contradiction by assuming :

$$\|d(f^{a_{i+1}}\circ\sigma\circ\theta_i\circ\theta\| \geq \epsilon_f/6 = 6\kappa \tag{3.2}$$

By Lemma 7, the point $f^{a_{i+1}}x$ belongs to G_κ. We will show $F^{a_{i+1}}\hat{x} \in H_\delta$. Therefore we will get $F^{a_{i+1}}\hat{x} \in G = \pi^{-1}G_\kappa \cap H_\delta$ contradicting $a_{i+1} \notin E$. To prove $F^{a_{i+1}}\hat{x} \in H_\delta$ it is enough to show $\sum_{j\leq l<a_{i+1}} \psi(F^l\hat{x}) \geq (a_{i+1}-j)\delta$ for any $K_x \leq j < a_{i+1}$ because $F^{K_x}(\hat{x})$ belongs to H_δ. For any $K_x \leq j < a_{i+1}$ we have :

$$\begin{aligned}
\|d(f^{a_{i+1}}\circ\sigma\circ\theta_i\circ\theta\|_\infty &\leq 2\|d_s(f^{a_{i+1}}\circ\sigma\circ\theta_i\circ\theta\|, \text{ because } f^{a_{i+1}}\circ\sigma\circ\theta_i\circ\theta \text{ is bounded,}\\
&\leq 2\|d_{f^jx}f^{a_{i+1}-j}(\hat{x})\| \times \|d_s(f^{a_{\overline{j}}}\circ\sigma\circ\theta^i_{\overline{j}})\| \times \frac{|\theta'_i|\times|\theta'|}{|(\theta^i_{\overline{j}})'|}, \text{ with } a_{\overline{j}} = j,\\
&\leq \frac{\epsilon_f}{3}\|d_{f^jx}f^{a_{i+1}-j}(\hat{x})\| \prod_{\overline{j}\leq l\leq i} e^{\frac{k'_{a_l}-k_{a_l}-1}{r-1}}/4 \text{ by induction hypothesis,}
\end{aligned}$$

$$\frac{1}{2} \leq \|d_{f^jx}f^{a_{i+1}-j}(\hat{x})\| \prod_{\overline{j}\leq l\leq i} e^{\frac{k'_{a_l}-k_{a_l}-1}{r-1}}/4 \text{ by assumption (3.2).} \tag{3.3}$$

Recall again that for $\overline{j} \leq l \leq i$, we have $b_l = 1$, thus

$$|k_{a_l} - \log\|d_{f^{a_l}x}f\|| \leq 1$$

and

$$k'_{a_l} \leq \phi(F^{a_l}\hat{x}).$$

Therefore we get for any $K_x \leq j < a_{i+1}$ from (3.3):

$$\begin{aligned}
2^{a_{i+1}-j} &\leq e^{\frac{r}{r-1}\sum_{j\leq l<a_{i+1}} \phi(F^l\hat{x})} e^{-\frac{1}{r-1}\sum_{j\leq l<a_{i+1}} \log^+ \|d_{f^l x} f\|}, \\
(a_{i+1}-j)\log 2 &\leq \frac{r}{r-1}\sum_{j\leq l<a_{i+1}} \psi(F^l\hat{x}), \text{ by definition of } \psi, \\
(a_{i+1}-j)\delta &\leq \sum_{j\leq l<a_{i+1}} \psi(F^l\hat{x}), \text{ as } \delta \text{ was chosen less than } \frac{r-1}{r}\log 2.
\end{aligned}$$

□

Lemma 12.

$$\sum_{i,\ m_n>a_i\notin E_n^M} \frac{k_{a_i}-k'_{a_i}}{r-1} \leq \left(n-\sharp E_n^M\right)\left(\frac{\log^+\|df\|_\infty}{r}+\frac{1}{r-1}\right).$$

Proof. The intersection of $[0, m_n[$ with the complement set of E_n^M is the disjoint union of neutral blocks and possibly an interval of integers of the form $[l, m_n[$. In any case $F^{\mathtt{j}}\hat{x}$ belongs to H_δ for such an interval $[\mathtt{i},\mathtt{j}[$ for any $x \in F_n^{\mathbf{k},\overline{E},m_n}$. In particular, we have

$$\sum_{l, a_l\in[\mathtt{i},\mathtt{j}[} k'_{a_i} - \frac{k_{a_i}}{r} \geq (\delta-1)(\mathtt{j}-\mathtt{i})$$

therefore

$$\begin{aligned}
\sum_{i,\ m_n>a_i\notin E_n^M} k'_{a_i} - \frac{k_{a_i}}{r} &\geq -(n-\sharp E_n^M), \\
\sum_{i,\ m_n>a_i\notin E_n^M} \frac{k_{a_i}-k'_{a_i}}{r-1} &\leq \frac{n-\sharp E_n^M}{r-1} + \frac{\sum_{i,\ m_n>a_i\notin E_n^M} k_{a_i}}{r}, \\
&\leq \left(n-\sharp E_n^M\right)\left(\frac{\log^+\|df\|_\infty}{r}+\frac{1}{r-1}\right).
\end{aligned}$$

□

3.5. **Conclusion.** We let Ψ_n be the family of $\mathcal{C}^r$ curves $\sigma\circ\theta$ for $\theta \in \Theta_m = \Theta_m(\mathbf{k}, \overline{E}, m_n)$ with Θ_m as in Lemma 11 over all admissible parameters $\mathbf{k}, \overline{E}, m_n$. For $\theta \in \Theta_m$ the curve $f^{a_i}\circ\sigma\circ\theta$ is strongly $\epsilon_{f^{b_i}}$-bounded for any $i = 1, \cdots, m$, in particular

$$\forall i = 1,\cdots,m,\ \|d(f^{a_i}\circ\sigma\circ\theta)\|_\infty \leq \epsilon_{f^{b_i}} \leq \max(1, \|df\|_\infty)^{-b_i},$$

therefore

$$\forall j = 0,\cdots,n,\ \|d(f^j\circ\sigma\circ\theta)\|_\infty \leq 1.$$

By combining the previous estimates, we get moreover:

$$\begin{aligned}
\sharp\Psi_n &\leq \sharp\left\{(\mathbf{k},\overline{E},m_n),\ F_n^{\mathbf{k},\overline{E},m_n}\neq\emptyset\right\}\times\sup_{\mathbf{k},\overline{E},m_n}\sharp\Theta_n(\mathbf{k},\overline{E},m_n),\\
&\leq ne^{2(n-\sharp E_n^M)A_fH(A_f)}3^{n(1/q+1/M)}e^{nH(1/M)}\sup_{\mathbf{k},\overline{E},m_n}\sharp\Theta_n(\mathbf{k},\overline{E},m_n),\ \text{by Lemma 10},\\
&\leq ne^{2(n-\sharp E_n^M)A_fH(A_f)}3^{n(1/q+1/M)}e^{nH(1/M)}\max(1,\|df\|_\infty)^{\sharp\overline{E}}\prod_{j\leq m}C_re^{\frac{k_{a_j}-k'_{a_j}}{r-1}},\ \text{by Lemma 11}.
\end{aligned}$$

Then we decompose the product into four terms :

- $\sum_{i,\ m_n>a_i\notin E_n^M}\frac{k_{a_i}-k'_{a_i}}{r-1}\leq(n-\sharp E_n^M)\left(\frac{\log^+\|df\|_\infty}{r}+\frac{1}{r-1}\right)$ by Lemma 12,
- $\sum_{i,\ m_n\leq a_i}\frac{k_{a_i}-k'_{a_i}}{r-1}\leq(n-m_n)\frac{A_f}{r-1}$,
- $\sum_{i,a_i\in E_n^M\cap(c+q\mathbb{N})}\frac{k_{a_i}-k'_{a_i}}{r-1}\leq 10\frac{n}{q}+2A_f\frac{qn}{M}+\frac{1}{r-1}\left(\int\frac{\log^+\|d_yf^q\|}{q}\,d\zeta_{F_n}^M(y)-\int\phi\,d\hat{\zeta}_{F_n}^M\right)$,
- $\sum_{i,a_i\in E_n^M\setminus(c+q\mathbb{N})}\frac{k_{a_i}-k'_{a_i}}{r-1}\leq 2A_f\frac{qn}{M}$.

By letting

$$B_r=\frac{1}{r-1}+\log C_r,$$

$$\gamma_{q,M}(f):=2\left(\frac{1}{q}+\frac{1}{M}\right)\log C_r+H(1/M)+\frac{10+\log 3}{q}+\frac{4qA_f+\log 3}{M},$$

$$\tau_n=\sup_{x\in\mathsf{F}}\left(1-\frac{m_n(x)}{n}\right)\frac{A_f}{r-1}+\frac{\log(nC)}{n},$$

we get with $C(f):=2A_fH(A_f^{-1})+\frac{\log^+\|df\|_\infty}{r}+B_r$:

$$\begin{aligned}
\frac{1}{n}\log\sharp\Psi_{F_n}\leq&\left(1-\frac{\sharp E_n^M}{n}\right)C(f)\\
&+\left(\log 2+\frac{1}{r-1}\right)\left(\int\frac{\log^+\|d_xf^q\|}{q}\,d\zeta_{F_n}^M(x)-\int\phi\,d\hat{\zeta}_{F_n}^M\right)\\
&+\gamma_{q,M}(f)+\tau_n,
\end{aligned}$$

This concludes the proof of Proposition 4.

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