EMBEDDING ASYMPTOTICALLY EXPANSIVE SYSTEMS

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Abstract. A topological dynamical system is said asymptotically expansive when entropy and periodic points grow subexponentially at arbitrarily small scales. We prove a Krieger like embedding theorem for asymptotically expansive systems with the small boundary property. We show that such a system (X,T) embeds in the K-full shift with $h_{top}(T) < \log K$ and $\sharp Per_n(X,T) \leq K^n$ for any integer n. The embedding is in general not continuous (unless the system is expansive and X is zero-dimensional) but the induced map on the set of invariant measures is a topological embedding. It is shown that this property implies asymptotical expansiveness. We prove also that the inverse of the embedding map may be continuously extended to a faithful principal symbolic extension.

1. Introduction

Symbolic dynamics play since the pioneer work of Hadamard [Had] a crucial role in the theory of dynamical systems. Here we investigate the problem of embedding a dynamical system in a shift with a finite alphabet.

For measure preserving ergodic systems the celebrated Krieger generator theorem gives a complete answer in terms of measure theoretical entropy :

Theorem 1.1. *(Theorem 2.1 and 4.3 in* [Kri70]*) Let T be an ergodic automorphism of a standard probability space $(X,\mathcal{U},\mu)$ then T embeds measure theoretically in the shift with K letters, i.e. there exists a measurable injective map $\psi : X \to \{1,...,K\}^{\mathbb{Z}}$ with $\sigma \circ \psi = \psi \circ T$ if and only if:*

either $h_\mu(T) < \log K$,
or $h_\mu(T) = \log K$ and T is Bernoulli.

For topological dynamical systems (i.e. continuous maps on a compact metrizable space) this question was also solved by Krieger. The obstructions are now of three kinds : topological (X is zero-dimensional), set theoretical (the number of n-periodic points of (X,T) is less than or equal to the number of n-periodic points of the shift) and dynamical ((X,T) is expansive and its topological entropy is less than the entropy of the shift). For any integer n we let $Per_n(X,T)$ be the set of periodic points of (X,T) with least period equal to n.

Theorem 1.2. *(Theorem 3 in* [Kri82]*) Let (X,T) be a topological dynamical system, then (X,T) embeds topologically in the shift σ with K letters, i.e. there exists a continuous injective map $\psi : X \to \{1,...,K\}^{\mathbb{Z}}$ with $\sigma \circ \psi = \psi \circ T$ if and only if:*

either we have

- *X is zero-dimensional,*
- *$\frac{1}{n}\log \sharp Per_n(X,T) \leq \log K$ for any integer n,*

- *$h_{top}(T) < \log K$,*
- *T is expansive,*

or (X,T) is topologically conjugated to $(\{1,...,K\}^{\mathbb{Z}},\sigma)$ (and thus $h_{top}(T) = \log K$).

Krieger only deals with the case $h_{top}(T) < \log K$, however the case of equality may be proved as in Theorem 1.1 (See the Appendix).

To embed topologically more general systems (in particular of arbitrarily topological dimension) into shift spaces one has to consider the shift with alphabet in $[0,1]^k$, $k \in \mathbb{N} \cup \{\mathbb{N}\}$. The existence of an embedding in the shift over $[0,1]^k$ with finite k is one important question in the theory of mean dimension, see e.g. [GT15] and references therein (for $k = \mathbb{N}$ it always exists by considering an embedding of X into the Hilbert cube $[0,1]^{\mathbb{N}}$). If we want still work with the shift with a finite alphabet we have to consider a larger class of embedding maps. Recently Hochman gives a Borel version of Krieger theorem for Borel dynamical systems, where the embedding is a Borel map. In this setting, after forgetting periodic points, the only constraint is the supremum of the entropy of Borel ergodic invariant probability measures:

Theorem 1.3. *(Theorem 1.5 in* [Hoch13]*) Let T be a measurable automorphism of a standard Borel space $(X,\mathcal{U})$ then (X,T) embeds almost Borel in the shift with K letters, i.e. there exists a Borel subset E of X of full measure with respect to any ergodic T-invariant aperiodic measure and a Borel injective map $\psi : E \to \{1,...,K\}^{\mathbb{Z}}$ with $\sigma \circ \psi = \psi \circ T$, if and only if:*

either we have $h_\mu(T) < \log K$ for any Borel ergodic T-invariant probability measure μ,

or (X,T) admits a unique measure μ of maximal entropy, μ is Bernoulli and $h_\mu(T) = \log K$.

In the present paper we give a version of Krieger theorem for asymptotically expansive topological dynamical system with the small boundary property. A dynamical system is asymptotically expansive when entropy and periodic points grow subexponentially at arbitrarily small scales. The small boundary property is satisfied by many aperiodic systems, i.e. dynamical systems without periodic points. In particular any aperiodic system in finite dimension satisfies this property. Precise definitions and further known facts related to these two properties are given in Section 2.

Our embedding is not continuous but it is in some sense precised later a limit of essentially continuous functions. However the map induced by the embedding on the set of invariant measures is continuous. Conversely we prove that a topological dynamical system embedding in such a way in a shift with a finite alphabet is asymptotically expansive.

Finally we relate for asymptotically expansive systems the present Krieger embedding theorem with the theory of symbolic extensions. More precisely the inverse of the Krieger embedding defines a principal faithful symbolic extension. In this way we build a bridge between the theory of symbolic extensions developed by M. Boyle and T. Downarowicz [BFF02][BD04][Dow11] and Krieger embedding like problems, for asymptotically expansive systems. In these previous works the symbolic extension is built as the intersection of decreasing subshifts whereas we build here the extension as the closure of increasing subshifts.

In Section 2 we introduce the main notions. We then state our main theorem. Section 4 is devoted to the proof of our embedding Theorem for zero-dimensional asymptotically expansive systems. In Section 5 we deal with the general case by reduction to the zero-dimensional one. Finally we will see that systems embeddable in our sense are asymptotically expansive. In this way we get a new characterization of asymptotical expansiveness.

2. Background

We will always consider topological dynamical systems (X,T), i.e. X is a compact metrizable space and $T : X \to X$ is an invertible continuous map. We also always assume that (X,T) has finite topological entropy. We denote by $\mathcal{M}(X,T)$ the set of T-invariant Borel probability measures and we endow this set with the weak $*$-topology.

2.1. **Small boundary property and essential partitions.** We recall here some facts about the small boundary property. A Borel subset of X is called a **null** (resp. **full**) set if it has null (resp. full) measure for any T-invariant ergodic Borel probability measure.

A subset of X is said to have a **small boundary** when its boundary is a null set. A partition of X is called **essential** when any of its element have a small boundary. For a zero-dimensional topological dynamical system (Y,S), a Borel map $\psi : X \to Y$ is said to be **essentially continuous** if there exists a basis of clopen sets $\mathcal{B}$ of Y such that for any $B \in \mathcal{B}$ the set $\psi^{-1}(B)$ has a small boundary. Observe that it easily implies that the map induced by ψ from $\mathcal{M}(X,T)$ to $\mathcal{M}(Y,S)$ is continuous. We also say finally that (X,T) has the **small boundary property** if X admits finite essential partitions with arbitrarily small diameter.

This property has been investigated in [Lin95], [Lin99], [Gut15] and used in the theory of symbolic extensions and entropy structures developed in [BD04],[Dow11]. In particular any dynamical system of finite topological entropy with an aperiodic minimal factor (Theorem 6.2 in [Lin99]) or any finite dimensional aperiodic system (Theorem 3.3 of [Lin95]) satisfies the small boundary property. In fact in [Lin95] it is proven that any finite dimensional system admits a basis of neighborhoods whose boundaries have zero measure for any aperiodic invariant measures. When $Per(X,T)$ is a zero-dimensional subset of X one may easily arrange the construction of [Lin95] to ensure that any element of the basis has a small boundary. Thus any finite dimensional system with a zero-dimensional set of periodic points has the small boundary property. This result may also follow from Lemma 3.7 of [Kuz95]. We refer to [Gut15] for some others dynamical properties implying the small boundary property.

2.2. **Asymptotic h-expansiveness.** Given two finite open covers $\mathcal{U}$ and $\mathcal{V}$ of X, we define the topological conditional entropy $h(\mathcal{V}|\mathcal{U})$ of $\mathcal{V}$ given $\mathcal{U}$ as

$$h(\mathcal{V}|\mathcal{U}) := \lim_n \frac{1}{n} \sup_{U^n \in \mathcal{U}^n} \log \min \sharp\{\mathcal{F}_n \subset \mathcal{V}^n,\ U^n \subset \bigcup_{V^n \in \mathcal{F}_n} V^n\}.$$

A map is said to be **asymptotically h-expansive** when for any decreasing sequence of open covers $(\mathcal{U}_k)_k$ whose diameter goes to zero we have

$$\lim_k \sup_{\mathcal{V}} h(\mathcal{V}|\mathcal{U}_k) = 0,$$

or equivalently when we have

$$\inf_{\mathcal{U}} \sup_{\mathcal{V}} h(\mathcal{V}|\mathcal{U}) = 0.$$

We refer to [Dow11] for basic properties of the topological conditional entropy and different characterizations of asymptotically h-expansiveness.

These notions were introduced by Misiurewicz in the seventies. One important consequence of the asymptotically h-expansiveness property is the upper semicontinuity of the measure theoretical entropy function and thus the existence of a measure of maximal entropy. A large class of dynamical systems satisfies this property, e.g. continuous piecewise monotone interval maps [Mis80], endomorphisms on compact groups, C^∞ maps on compact manifolds [Buz14],...

A topological extension $\pi : (Y, S) \to (X, T)$ is called **principal** when it preserves the entropy of invariant measures, i.e. $h_\nu(S) = h_\mu(T)$ for any T-invariant measure μ and for any S-invariant measure ν projecting on μ. Ledrappier [Led78] proved that if π is a principal extension then T is asymptotically h-expansive if and only if the same holds for S.

2.3. **Asymptotic per-expansiveness.** Similarly we introduce now the new notion of asymptotical per-expansiveness. With the previous notations we let

$$per(T|\mathcal{U}) := \limsup_n \frac{1}{n} \sup_{U^n \in \mathcal{U}^n} \log \sharp Per_n(X,T) \cap U^n$$

and we say (X, T) is **asymptotically per-expansive** when we have

$$\lim_k per(T|\mathcal{U}_k) = 0,$$

or equivalently when we have

$$\inf_{\mathcal{U}} per(T|\mathcal{U}) = 0.$$

We recall that a topological system (X, T) is expansive when there exists a finite open cover $\mathcal{U}$ with the following property: for any $x \neq y \in X$ there is $k \in \mathbb{Z}$ such that $T^k x$ and $T^k y$ do not lie in the same element of $\mathcal{U}$.

Obviously aperiodic systems and expansive systems are asymptotically per-expansive. We say that a topological dynamical system is **asymptotically expansive** when it is both asymptotically h- and per-expansive. Among asymptotically expansive systems which are neither expansive nor apeiodic we can cite [Bur16] toral quasi-hyperbolic automorphisms, piecewise expanding map of the interval or also generic piecewise affine surface homeomorphisms. The author recently proved [Bur16] that any C^∞ surface diffeomorphism, whose invariant measures have one negative and one positive Lyapunov exponents uniformly away from zero, is asymptotically per-expansive (observe that any periodic point is then hyperbolic, in particular the set of periodic points is zero-dimensional and the system has the small boundary property). In particular C^∞ Henon-like diffeomorphisms satisfy this property [1] (see also [Tak13]).

A topological extension is said to be **faithful** if the induced map between the sets of invariant probability measures is an homeomorphism. As any system has a principal faithful aperiodic extension (even zero-dimensional, see [DH12]), the factor of a asymptotically *per*-expansive map is not necessarily asymptotically *per*-expansive, even when the entropy of measures is preserved and the extension is faithful. Thus asymptotically *per*-expansiveness is not preserved under principal faithful extensions.

A topological extension $\pi : (Y, S) \to (X, T)$ is said to be **strongly faithful** if it is faithful and if it preserves periodic points, i.e. for any integer n we have $\pi(Per_n(Y, S)) = Per_n(X, T)$. In Subsection 6.3 we will show that a dynamical system with a principal strongly faithful asymptotically *per*-expansive extension is also asymptotically *per*-expansive.

2.4. **Symbolic extensions.** A **symbolic extension** of (X, T) is a topological extension by a subshift over a finite alphabet. The question of the existence of (principal, faithful) symbolic extension has led to a deep theory of entropy (we refer to [Dow11] for an introduction to the topic). One first positive result appeared in this area is the existence of principal symbolic extensions for asymptotically h-expansive systems [BFF02], [Dow01a]. More recently Serafin [Ser12] proved that such an extension could be chosen to be faithful when (X, T) is aperiodic[1]. Here we give a new proof of these results that we relate with our Krieger like embedding theorem (Main Theorem below) : the symbolic extension is just given by the inverse of the Krieger embedding.

3. Krieger embedding for asymptotically expansive systems

We state now our main result. For two topological dynamical systems (X, T) and (Y, S), a map $\phi : X \to Y$ is called **equivariant** when it semi-conjugates T with S, i.e. $\phi \circ T = S \circ \phi$. Moreover we say $\phi : X \to Y$ is **ϵ-injective** for some $\epsilon > 0$, if there exists $\delta > 0$, such that for any set $Z \subset Y$ with diameter less than δ the preimage $\phi^{-1}(Z)$ has diameter less than ϵ. Finally we will denote by $\phi^* : \mathcal{M}(X, T) \to \mathcal{M}(Y, S)$ the map induced by ϕ between the sets of probability invariant measures.

Main Theorem. *Let (X, T) be a topological dynamical system with the following properties :*

- *(X, T) has the small boundary property,*
- *$\frac{1}{n} \log \sharp Per_n(X, T) \leq \log K$ for any integer n,*
- *$h_{top}(T) < \log K$,*
- *(X, T) is asymptotically expansive.*

Then there exists an equivariant injective Borel map $\psi : X \to \{1, ..., K\}^{\mathbb{Z}}$ such that :

[1] This was extended by Downarowicz and Huczek to any asymptotically h-expansive systems [DH12]

- *ψ is a pointwise limit of a sequence of equivariant essentially continuous ϵ_k-injective maps ψ_k, where $(\epsilon_k)_k$ is going to zero,*
- *the induced map ψ^* is the uniform limit of $(\psi_k^*)_k$ on $\mathcal{M}(X,T)$, in particular ψ^* is a topological embedding,*
- *ψ^{-1} is uniformly continuous on $\psi(X)$ and the continuous extension π of ψ^{-1} on the closure $Y = \overline{\psi(X)}$ of $\psi(X)$ is a principal strongly faithful symbolic extension of (X,T) with $\psi^* = (\pi^*)^{-1}$.*

As previously discussed in Subsection 2.1 the above theorem applies to any asymptotically expansive finite dimensional system with finite topological entropy and finite exponential growth of periodic points, since in this case the set of periodic points is zero-dimensional. Observe also that the asymptotic *per*-expansiveness and the inequality $h_{top}(T) < \log K$ implies that the exponential growth of periodic point $per(T) := \limsup_n \frac{\log \sharp Per_n(X,T)}{n}$ also satisfies $per(T) \leq h_{top}(T) < \log K$.

The dynamical consequences of our statement characterize the asymptotic expansiveness. More precisely we have as a converse of our Main Theorem :

Proposition 3.1. *Let (X,T) be a topological dynamical system. Assume one of the two following properties :*

1. *there exists a Borel equivariant injective map $\psi : X \to \{1, ..., K\}^{\mathbb{Z}}$ such that ψ^* is a topological embedding,*
2. *there exists a strongly faithful principal extension $\pi : (Y,\sigma) \to (X,T)$ with Y a closed σ-invariant subset of $\{1, ..., K\}^{\mathbb{Z}}$.*

Then (X,T) satisfies the following properties :

- $\frac{1}{n} \log \sharp Per_n(X,T) \leq \log K$ *for any integer n,*
- $h_{top}(T) \leq \log K$,
- *(X,T) is asymptotically expansive.*

Together with the Main Theorem we obtain in particular that when $h_{top}(T) < \log K$ and X has the small boundary property then the assumptions (1) and (2) in Proposition 3.1 are equivalent.

In contrast with Theorem 1.1 and 1.2, for topological systems (X,T) with $h_{top}(T) = \log K$, the existence of an embedding on the set of invariant measures of (X,T) and the K-full shift does not seems to create other constraints on (X,T): in particular such an embedding may not be onto as we can see in the following example.

There exists by Jewett-Krieger ergodic theorem for any integer K a uniquely ergodic dynamical system with topological entropy equal to $\log K$ (such a system is in particular aperiodic). It easily follows from the tail variational principle [Bur09] that uniquely ergodic systems are asymptotically h-expansive. When the unique ergodic measure is Bernoulli then Theorem 1.1 gives a Borel equivariant map inducing a non surjective embedding in the set of invariant measures of the K-full shift.

However the case of equality in the conditions on periodic points creates some "rigidity". Indeed, if we assume $\sharp Per_n(X,T) = \sharp Per_n(\{1,...,K\}^{\mathbb{Z}},\sigma)$ for any integer n, then for any Borel equivariant map $\psi : X \to \{1,...,K\}^{\mathbb{Z}}$ such that ψ^* is a topological embedding, the map ψ^* is in fact an homeomorphism because periodic measures are dense in the set of invariant measures of the full shift.

As the previously mentioned embedding theorems our main result brings some new contributions in the classification of dynamical systems. In [Dow06] T. Downarowicz investigates the characterization of *assignments* arising from topological systems. An assignment is a function ψ defined on an abstract metrizable Choquet simplex K, whose values are measure-theoretic dynamical systems, i.e., for $p \in K$, $\psi(p)$ has the form $(X_p, \mathcal{B}_p, \mu_p, T_p)$. Two assignments, ψ on a simplex K and ψ' on a simplex K', are said to be equivalent if there exists an affine homeomorphism of Choquet simplexes $\pi : K \to K'$ such that for every $p \in K$ the systems $\psi(p)$ and $\psi'(p')$, where $p' = \pi(p)$, are isomorphic. A topological dynamical system (X,T) determines a natural assignment on the simplex $\mathcal{M}(X,T)$ by the rule $\mu \mapsto (X, \mathcal{B}, \mu, T)$ with $\mathcal{B}$ being the usual Borel σ-algebra of X. As a consequence of our Main Theorem, subshifts and asymptotically expansive systems have the same assignments, i.e. the natural assignment of an asymptotically expansive system is equivalent to the natural assignment of some subshift.

4. The case of zero-dimensional systems

We first consider zero-dimensional dynamical systems. Then in Section 5 we will deal with general systems with the small boundary property by reduction to the zero-dimensional case.

We will prove the following strong version (continuity replaces essential continuity) of our Main theorem for zero-dimensional systems :

Proposition 4.1. *Let (X,T) be a zero-dimensional topological dynamical system with the following properties :*

- $\sharp Per_n(X,T) \leq \sharp Per_n(\{1,...,K\}^{\mathbb{Z}},\sigma)$ *for any integer n,*
- $h_{top}(T) < \log K$,
- (X,T) *is asymptotically expansive.*

Then there exists an equivariant injective Borel map $\psi : X \to \{1,...,K\}^{\mathbb{Z}}$ such that :

- *ψ is a pointwise limit of a sequence of equivariant continuous ϵ_k-injective maps ψ_k, where $(\epsilon_k)_k$ is going to zero,*
- *the induced map ψ^* is the uniform limit of $(\psi_k^*)_k$ on $\mathcal{M}(X,T)$, in particular ψ^* is a topological embedding,*
- *ψ^{-1} is uniformly continuous on $\psi(X)$ and the continuous extension π of ψ^{-1} on the closure $Y = \overline{\psi(X)}$ of $\psi(X)$ is a principal strongly faithful symbolic extension of (X,T) with $\psi^* = (\pi^*)^{-1}$.*

In the next Subsections 4.1, 4.2 and 4.3 we develop some tools used later in the proof of Proposition 4.1.

4.1. **Asymptotic expansiveness for zero-dimensional systems.** Let (X,T) be a zero-dimensional system. In the definition of asymptotic h-expansiveness given in Subsection 2.2 we may choose the open covers $\mathcal{U}$ and $\mathcal{V}$ to be finite clopen partitions with $\mathcal{V} > \mathcal{U}$, i.e. any element V of $\mathcal{V}$ is contained in a element U of $\mathcal{U}$. The topological conditional entropy $h(\mathcal{V}|\mathcal{U})$ may be then rewritten as follows

$$h(\mathcal{V}|\mathcal{U}) := \lim_n \frac{1}{n} \sup_{U^n \in \mathcal{U}^n} \log \sharp\{V^n \in \mathcal{V}^n,\ V^n \subset U^n\}.$$

The following lemma used later in the proof of Proposition 4.1 follows directly from the definition of asymptotical expansiveness independently from the above expression of the topological conditional entropy.

Lemma 4.1. *Let (X,T) be an asymptotically expansive zero-dimensional system. For all $\alpha > 0$ there exists a decreasing sequence $(\mathcal{V}_k)_k$ of clopen partitions whose diameter goes to zero, such that for all $k \geq 1$:*

- $h(\mathcal{V}_{k+1}|\mathcal{V}_k) < \alpha/2^k$,
- $per(T|\mathcal{V}_k) < \alpha/2^k$.

4.2. **The marker property for zero-dimensional systems.** One key tool in our construction is the following "marker property". A similar approach is used to build symbolic extensions in [BD04] where the product with an odometer is used to mark the shift space in the same way.

Lemma 4.2. *(Lemma 7.5.4 in* [Dow11]*) Let (X,T) be a zero-dimensional dynamical system. Then for every positive integer n and for every ϵ there exists a clopen set U such that :*

- *T^iU are pairwise disjoint for $i = 0, ..., n-1$,*
- *$\bigcup_{|i|<n} T^iU = X \setminus Per_n^\epsilon$, where Per_n^ϵ denotes a clopen ϵ-neighborhood of the set $\bigcup_{m\leq n} Per_m(X,T)$ of periodic points with least period less than or equal to n.*

4.3. **Some tools on shift spaces.** We consider in this section a finite set Λ and a compact metrizable space X. We let P_0 be the zero coordinate partition of $\Lambda^{\mathbb{Z}}$.

4.3.1. *Decreasing metrics on $\Lambda^{\mathbb{Z}}$.* The product space $\Lambda^{\mathbb{Z}}$ will be endowed with different metrics. We first let d be a metric inducing the usual product topology (we consider the discrete topology on Λ), for example the Cantor metric defined as

$$\forall x, y \in \Lambda^{\mathbb{Z}},\ d(x,y) := 2^{-k} \text{ where } k = \min\{i \geq 0,\ x_{-i} \neq y_{-i} \text{ or } x_i \neq y_i\}.$$

Also for any integer N we consider the metric d_N defined as

$$\forall x, y \in \Lambda^{\mathbb{Z}},\ d_N(x,y) := \sup_{n \geq N} \left\{ \frac{1}{2n+1} \sharp\{i,\ |i| \leq n,\ x_i \neq y_i\} \right\}.$$

For any N the topology given by d_N is stronger than the product topology. Observe now that the sequence of metrics $(d_N)_N$ is nonincreasing, i.e. $(d_N(x,y))_N$ is nonincreasing in N for any $x, y \in \Lambda^{\mathbb{Z}}$. We let d_∞ be the (shift-invariant) pseudometric on $\Lambda^{\mathbb{Z}}$ given by the limit of $(d_N)_N$

$$d_\infty(x,y) := \lim_N d_N(x,y).$$

The pseudometric d_∞ is called the Besicovitch pseudometric on the shift space. Topological properties of the induced quotient metric space and dynamical properties of cellular automata on this space were studied in [BFP99]. In particular, although we do not used here directly, this quotient metric space is known to be complete.

4.3.2. *Convergence of functions taking values in* $\Lambda^{\mathbb{Z}}$. We consider now maps from X to $\Lambda^{\mathbb{Z}}$ and we define different kinds of convergence. A sequence of such maps $(\psi_k)_k$ is said to converge to ψ :

- **pointwisely with respect to** d, when for all $x \in X$,
$$d(\psi_k x, \psi x) \xrightarrow{k\to+\infty} 0,$$
- **uniformly with respect to** d_∞, when
$$\sup_{x\in X} d_\infty(\psi_k x, \psi x) \xrightarrow{k\to+\infty} 0,$$
- **uniformly with respect to** $(d_N)_N$, when
$$\sup_{x\in X} d_N(\psi_k x, \psi x) \xrightarrow{N,k\to+\infty} 0.$$

Observe that if $(\psi_k)_k$ is converging to ψ uniformly with respect to the decreasing sequence of pseudometrics $(d_N)_N$, then it is also converging to ψ uniformly with respect to its limit d_∞.

We let $\mathcal{M}(\Lambda^{\mathbb{Z}}, \sigma)$ be the set of probability Borel measures endowed with the following metric d_* inducing the weak $*$-topology : $\forall \mu, \nu \in \mathcal{M}(\Lambda^{\mathbb{Z}}, \sigma)$,
$$d_*(\mu, \nu) = \sum_n \frac{|\mu(A_n) - \nu(A_n)|}{2^n},$$
where $(A_n)_n$ is a given enumeration of $\bigcup_{N\in\mathbb{N}} P_0^N$. We also consider the space $\mathcal{K}(\Lambda^{\mathbb{Z}}, \sigma)$ of compact subsets of $\mathcal{M}(\Lambda^{\mathbb{Z}}, \sigma)$ endowed with the Hausdorff metric d_H associated to d_*. The set $\mathcal{M}(\Lambda^{\mathbb{Z}}, \sigma)$ being compact it is well known that $\mathcal{K}(\Lambda^{\mathbb{Z}}, \sigma)$ is also compact. For all $x \in \Lambda^{\mathbb{Z}}$ we let $\phi(x) \in \mathcal{K}(\Lambda^{\mathbb{Z}}, \sigma)$ be the set of limits of the empirical measures, i.e. the accumulation points of the sequence $(\frac{1}{n}\sum_{0\le k<n} \delta_{\sigma^k x})_n$ for the weak $*$-topology.

Lemma 4.3. *Let* $\psi : X \to \Lambda^{\mathbb{Z}}$ *be a uniform limit with respect to* d_∞ *of* $(\psi_k)_k$. *Then* $(\phi\circ\psi_k)_k$ *converge uniformly to* $\phi\circ\psi$ *with respect to* d_H.

When X is $\Lambda^{\mathbb{Z}}$ and x, y are two points in $\Lambda^{\mathbb{Z}}$ with $d_\infty(x,y) = 0$, then by taking ψ constant equal to x and ψ_k constant equal to y for any k, we get then $\phi(x) = \phi(y)$.

Proof. It is enough to prove that for any $A \in \bigcup_{N\in\mathbb{N}} P_0^N$ we have
$$\lim_k \sup_{x\in X} \limsup_n \frac{1}{n}\left|\sum_{0\le l<n} \left(\delta_{\sigma^l\circ\psi_k(x)}(A) - \delta_{\sigma^l\circ\psi(x)}(A)\right)\right| = 0.$$

Fix $N \in \mathbb{N}$, $A \in P_0^N$, $x \in X$ and $k \in \mathbb{N}$. Then
$$\begin{aligned}\limsup_n \frac{1}{n}\left|\sum_{0\le l<n} \left(\delta_{\sigma^l\circ\psi_k(x)}(A) - \delta_{\sigma^l\circ\psi(x)}(A)\right)\right| &\le \limsup_n \frac{1}{n}\sum_{0\le l<n} \left|\delta_{\sigma^l\circ\psi_k(x)}(A) - \delta_{\sigma^l\circ\psi(x)}(A)\right|, \\ &\le \limsup_n \frac{N}{n}\sharp\{0\le l\le n+N,\ (\psi_k(x))_l \neq (\psi(x))_l\}, \\ &\le N d_\infty(\psi_k(x), \psi(x)).\end{aligned}$$

By uniform convergence of $(\psi_k)_k$ to ψ with respect to d_∞, this last term goes to zero uniformly in x when k goes to infinity. This concludes the proof the lemma. □

For a subset Y of $\Lambda^{\mathbb{Z}}$ we let $\overline{Y}$ be the closure of Y for the product topology and we let Y^∞ be the d_∞-saturated set of Y, i.e. $Y^\infty = \{x \in \Lambda^{\mathbb{Z}},\ \exists y \in Y,\ d_\infty(x,y) = 0\}$.

Lemma 4.4. *Let $\psi : X \to \Lambda^{\mathbb{Z}}$ be both a pointwise limit with respect to d and a uniform limit with respect to $(d_N)_N$ of $(\psi_k)_k$. Assume moreover the maps $(\psi_k)_k$ are continuous (for the product topology on $\Lambda^{\mathbb{Z}}$), then*

$$\overline{\psi(X)} \subset \psi(X)^\infty.$$

Proof. Let $(y_n = \psi(x_n))_n$ be a sequence converging with respect to d in $\psi(X)$ to say y. We consider a sequence of continuous maps $(\psi_k)_k$ converging to ψ uniformly with respect to $(d_N)_N$. By definition for all $\epsilon > 0$ there exist integers K and M such that for all $k \geq K$ and for all $N \geq M$ we have for all integers n that $d_N(\psi(x_n), \psi_k(x_n)) \leq \epsilon$. Observe the function $(x, y) \in (X, d)^2 \mapsto d_N(x, y)$ is lower semicontinuous as a supremum of continuous functions. Up to extracting a subsequence we may assume by compacity of X that $(x_n)_n$ is converging to $x \in X$. Therefore by taking the limit in n in the previous inequality we obtain by continuity of ψ_k for all $k \geq K$ and for all $N \geq M$

$$d_N(y, \psi_k(x)) \quad \leq \quad \epsilon.$$

and then by pointwise convergence of $(\psi_k)_k$ to ψ in $(\Lambda^{\mathbb{Z}}, d)$ we have for all $N \geq M$

$$d_N(y, \psi(x)) \leq \epsilon.$$

Finally we let N go to infinity to get :

$$d_\infty(y, \psi(x)) \leq \epsilon.$$

This concludes the proof of the lemma as $\epsilon > 0$ may be chosen arbitrarily small. □

4.3.3. *Dynamical consequences.* We let now T be an invertible map acting continuously on X. For a Borel map $\xi : X \to \Lambda^{\mathbb{Z}}$ we let $\xi^* : \mathcal{M}(X,T) \to \mathcal{M}(\Lambda^{\mathbb{Z}}, \sigma)$ be the map induced by ξ on the set of invariant Borel probability measures. As done for the full shift in the previous subsection we denote for any $x \in X$ by $\phi(x) \subset \mathcal{M}(X,T)$ the set of limits of empirical measures at x, i.e. the set of accumulation points of the sequence $(\frac{1}{n}\sum_{0 \leq k < n} \delta_{T^k x})_n$ for the weak-* topology.

Lemma 4.5. *Let $\psi : X \to \Lambda^{\mathbb{Z}}$ be a uniform limit with respect to d_∞ of $(\psi_k)_k$. Assume moreover ψ and $(\psi_k)_k$ are equivariant Borel maps.*

Then the induced maps $(\psi_k^*)_k$ converge uniformly to ψ^* with respect to d_*.

Proof. By the ergodic decomposition it is enough to consider only ergodic T-invariant measures μ. By Birkhoff ergodic theorem we have for μ-almost every x and for all finite cylinders A in $\Lambda^{\mathbb{Z}}$:

$$\mu(\psi^{-1}A) = \lim_n \frac{1}{2n-1} \sum_{|l|<n} \delta_{T^l x}(\psi^{-1}A).$$

By equivariance we have

$$\begin{aligned}\mu(\psi^{-1}A) &= \lim_n \frac{1}{2n-1}\sum_{|l|<n}\delta_{\psi\circ T^l(x)}(A),\\ &= \lim_n \frac{1}{2n-1}\sum_{|l|<n}\delta_{\sigma^l\circ\psi(x)}(A).\end{aligned}$$

Since this holds for any A the sequence of empirical measures $(\frac{1}{2n-1}\sum_{|l|<n}\delta_{\sigma^l\circ\psi(x)})_n$ is converging to $\psi^*\mu$ in the weak-*topology, in others terms we have $\{\phi\circ\psi(x)=\psi^*\mu\}$ for x in a set E^μ of full μ-measure. Similarly for any integer k we have $\phi\circ\psi_k(x)=\{\psi_k^*\mu\}$ for x in a set E_k^μ of full μ-measure.

As by Lemma 4.3 the sequence $(\phi\circ\psi_k)_k$ converges uniformly to $\phi\circ\psi$ with respect to d_H we conclude that ψ_k^* converges to ψ^* uniformly with respect to d_*. Indeed for any μ we take $x_\mu\in E^\mu\cap\bigcap_k E_k^\mu$ and we conclude that:

$$\begin{aligned}\sup_\mu d_*(\psi_k^*\mu,\psi^*\mu) &= \sup_{x_\mu,\mu} d_*(\phi\circ\psi_k(x_\mu),\phi\circ\psi(x_\mu)),\\ &\leq \sup_x d_H(\phi\circ\psi_k(x),\phi\circ\psi(x)),\\ &\xrightarrow{k\to+\infty} 0.\end{aligned}$$

□

Remark 4.1. *The above lemma may also be proved by using the Ornstein's d-bar distance. Indeed it follows from Theorem I.9.10 in* [Sh96] *that for any $\mu\in\mathcal{M}(X,T)$ the d-bar distance of $\psi^*\mu$ and $\psi_k^*\mu$ is less than $\sup_{x\in X} d_\infty(\psi_k x,\psi(x))$. Moreover it is well known the d-bar distance is stronger than the weak-star topology.*

Assuming moreover continuity of the maps $(\psi_k)_k$ we prove the identity $\psi^*\phi(x)=\phi\circ\psi(x)$ for every $x\in X$.

Lemma 4.6. *Let $\psi: X\to\Lambda^{\mathbb{Z}}$ be a uniform limit with respect to d_∞ of $(\psi_k)_k$. Assume moreover ψ (resp. $(\psi_k)_k$) are equivariant Borel (resp. continuous) maps.*

Then for all $x\in X$,

$$\psi^*\phi(x)=\phi\circ\psi(x).$$

Proof. The maps ψ_k being continuous the associated maps ψ_k^* induced on the set of Borel probability measures on X endowed with the weak-* topology are also continuous and therefore we have the equality $\psi_k^*\phi=\phi\circ\psi_k$. By the above Lemma 4.5, the left member goes uniformly to $\psi^*\phi$ (with respect to d_H), whereas the right member goes uniformly to $\phi\circ\psi$ when k goes to infinity according to Lemma 4.3. □

Lemma 4.7. *Let $\psi: X\to\Lambda^{\mathbb{Z}}$ be both a pointwise limit with respect to d and a uniform limit with respect to $(d_N)_N$ of $(\psi_k)_k$. Assume moreover the maps $(\psi_k)_k$ are continuous equivariant.*

Then any σ-invariant measure μ on $\overline{\psi(X)}$ is supported on $\psi(X)$, i.e. $\mu(\psi(X))=1$.

Proof. Let μ be a σ-invariant ergodic measure on $\overline{\psi(X)}$. According to Birkhoff ergodic theorem there is $y\in\overline{\psi(X)}$ with $\phi(y)=\{\mu\}$. By Lemma 4.4 one can find $x\in X$ such that $d_\infty(y,\psi(x))=0$. As previously discussed after Lemma 4.3 it implies that $\phi\circ\psi(x)=\phi(y)=\mu$. Finally by Lemma 4.6 we have $\psi^*\phi(x)=\phi\circ\psi(x)=\mu$. This concludes the proof as $\psi^*\phi(x)$ is supported on $\psi(X)$. □

4.4. **Proof of Propostion 4.1.** We first deal with aperiodic systems. For general systems with periodic points the proof is a little more involved as we need to encode periodic points. In this last case we will adapt the construction of Krieger [Kri82] (Theorem 1.2) using the *per*-expansiveness property and the marker property given in Lemma 4.2.

Observe first that in any case one only needs to embed our system in a full shift with some finite alphabet. Indeed the topological entropy and the cardinality of periodic points with least period n for any n will be preserved by our Borel embedding so that by applying Krieger Theorem for topological expansive systems (Theorem 1.2) we get after a composition an embedding in the shift space with the desired number of letters (but also in any subshift of finite type with the suitable lower bound on the topological entropy and cardinality of periodic points).

4.4.1. *Construction of the embedding, the aperiodic case.* We consider an aperiodic asymptotically h-expansive zero-dimensional dynamical system (X,T). Let K be an integer with $\log K > h_{top}(T)$. We let $\alpha > 0$ be the difference $\alpha = \log K - h_{top}(T)$.

0. Dynamical Markers. We let $(\mathcal{V}_k)_k$ be a sequence of nested clopen partitions as in Lemma 4.1. We let n_1 be such that for $n > n_1$:

$$\sharp\mathcal{V}_1^n \leq e^{n(h_{top}(T)+\alpha/4)} < K^{n(1-\alpha/2)-1}$$

and for $k > 1$ we let n_k be such that for $n \geq n_k/2$:

$$\sup_{U^n \in \mathcal{V}_k^n} \sharp\{V^n,\ V^n \in \mathcal{V}_{k+1}^n \text{ with } V^n \subset U^n\} \quad < \quad K^{n\alpha/2^k - 1}.$$

We may also ensure that the sequence $(n_k)_k$ satisfies $\alpha n_k >> 2^k$ and $n_k > 2n_{k-1}$ for all integers k. We will build an injective map from X to $\{|, +, \|, \circ, 1, ..., K\}^{\mathbb{Z}}$ which conjugates T to the shift as in the ergodic generator Krieger theorem [Kri82] (Theorem 1.1 of the Introduction) by first encoding the dynamic with respect to the covers $\mathcal{V}_1, \mathcal{V}_2, ...\ ,\mathcal{V}_k,...$ on finite pieces of orbits of length of order $n_1, n_2, ..., n_k, ...$ These pieces of orbits are given by the topological Rokhlin towers $(U_k)_k$ given by Lemma 4.2 with respect to the sequences $(n_k)_k$. More precisely, we will consider the following nonincreasing sequence of $\mathbb{Z}$-subsets $(G_k(x))_k$ for any $x \in X$. We first let $G_1(x) := \{l \in \mathbb{Z},\ f^l x \in U_1\}$. Then we define for any integer $k > 1$ the subset $G_k(x)$ of $\mathbb{Z}$ by induction on k as the subset of integers $i \in G_{k-1}(x)$, such that if j denotes the successor of i in G_{k-1} then there is $l \in [i, j[$ with $f^l x \in U_k$ (see Figure 1). Observe that the distance between two consecutive integers in $G_k(x)$ has length larger than or equal to $n_k/2$ and less than $4n_k$. We let $\underline{n}$ be the sequence of integers $(n_k)_k$ and $G(x) := (G_k(x))_k$ be the nonincreasing sequence of the subsets $G_k(x)$.

1. Marker structure. We consider the sequence $G = G(x)$ for some fixed x. For simplicity we also denote by G_k the set $G_k(x)$. For any integer $k > 0$ an empty k-block associated to G will be a finite word u^k in $\{|, +, \|, *, \circ\}$ indexed with respect to G_k : it starts and finishes at consecutive integers $i_k < j_k \in G_k$, that is $u^k := (u_{i_k}, ..., u_{j_k - 1})$. The length $j_k - i_k$ of such a block u^k is denoted by $|u^k|$. The empty k-blocks and their k-marker, k-filling, k-free positions are then defined by induction as follows. The exact meaning of the symbols in the alphabet will be given later on, but in order to help the reader we roughly explain it now:

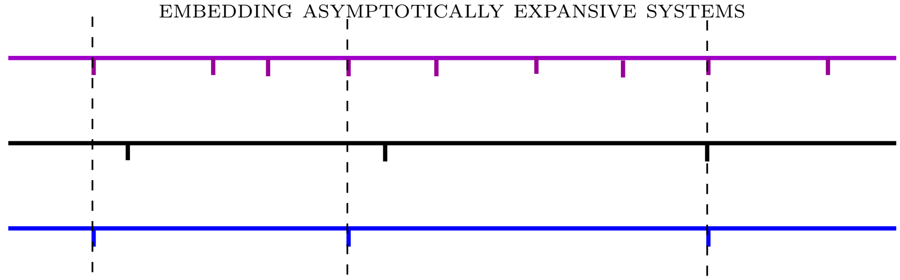

Figure 1: ***The set*** $G_k(x)$ ***from the return times in*** U_k***.*** *The first line and dashes in purple represent the element in* $G_k(x)$*. On the second line we draw the times in* U_{k+1}*. The last line gives the associated times in* $G_{k+1}(x)$*.*

- $|$ and $\|$ are marking the times in G_k,
- $+$ point out the lack of a marker $\|$,
- $*$ represent the filling positions, which are the places where will be written the code of finite orbits,
- $\circ$ are called the free positions. They correspond to the remaining positions : we still did not assign them any value at a given step of the construction.

Let us be more precise now. Firstly a empty 1-block is a word u^1 such that:

- the first coordinate of u^1, the 1-marker position, coincides with $|$,
- the next $[(1-\alpha/2)|u^1|]$ coordinates[2] of u^1, called the 1-filling positions, all take the value $*$,
- the remaining coordinates, called the 1-free positions, all take the value $\circ$. The first free position in u_1 is called the 2-marker position of u_1.

For $k > 1$ an empty k-block u^k is obtained from a concatenation of consecutive empty $(k-1)$-blocks, where we change $\circ$ to $\|$ at the k-marker position of the first $(k-1)$-block. This position in u^k defines the k-marker position. Moreover we change the k-marker positions of the others $(k-1)$-blocks in u^k to $+$. These positions allow to detect the first coordinate $i_k \in G_k$ of the k-block u^k. The $[\alpha|u^k|/2^k]$ next $(k-1)$-free positions are the k-filling positions of u^k and the remaining $(k-1)$-free positions of the concatenation define the k-free positions of u^k (see Figure 2 below). The $(k+1)$-marker position of u^k is the first k-free position in u^k.

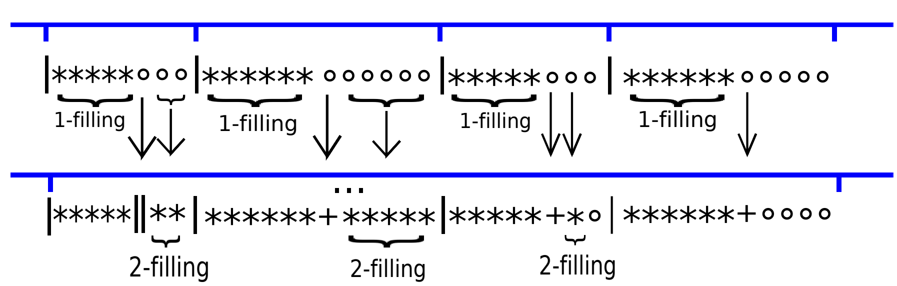

Figure 2: ***Empty*** **1-** ***and*** **2*-blocks.*** *For* $k = 1, 2$ *the empty* k*-blocks and their marker, filling and free positions are represented on the* k^{th} *blue line. Elements in* G_k *are given by the blue dashes.*

---

[2] $[x]$ denotes the integer part of a real number x

The empty k-code $\psi_k^\emptyset$ with respect to G is then just the infinite word in $\{|, +, \|, *, \circ\}^{\mathbb{Z}}$ obtained by concatenation of all empty k-blocks.

2. First scale encoding. let τ_1 be the first return time of y in U_1, that is $\tau_1(y) := \inf\{n > 0,\ T^n y \in U_1\}$. For any $n \geq n_1$ we consider an injective map from $\mathcal{V}_1^n$ to the set of words of length $[(1-\alpha/2)n]$ with letters in $\{1, ..., K\}$. Then to any $y \in U_1$ we associate such a word corresponding to $V_1^{\tau_1(y)}(y)$. The 1-code $\psi_1(x)$ of x is then obtained by replacing in the empty 1-blocks in $\psi_1^\emptyset(x)$ associated to $G(x)$ the symbols $*$'s in the 1-filling positions by these words for $y = T^l x$ with l being the return times in U_1, which corresponds to the first index of the empty 1-blocks.

3. Higher scales encoding. We will now encode the dynamics with respect to $\mathcal{V}_2$ conditionally to $\mathcal{V}_1$. For any $n \geq n_2/2$ and for any $V_1^n \in \mathcal{V}_1^n$ we consider an injective map from $\{V_2^n \in \mathcal{V}_2^n,\ V_2^n \subset V_1^n\}$ to the $[\alpha n/4]$-words of $\{1, ..., K\}^{\mathbb{Z}}$. Then in an empty 2-block of size n of $\psi_2^\emptyset(x)$ we replace the symbols $*$'s at the 2-filling positions by the $[\alpha n/4]$-words of $\{1, ..., K\}^{\mathbb{Z}}$ associated to $V_2^n(y)$ conditionally to $V_1^n(y)$ for $y = T^l x$ with l the first index of the empty 2-blocks. In a similar way we build ψ_k for any $k > 2$. The sequence $(\psi_k(x))_k$ is converging pointwisely for the product topology in $\{1, ..., K\}^{\mathbb{Z}}$ as for any i the i^{th} coordinate of $\psi_k(x)$ is constant after some rank. We let $\psi(x)$ be the pointwise limit of $\psi_k(x)$.

4. Decoding. The k-code $\psi_k(x)$ of x may be deduced from $\psi(x)$ by replacing the l-marker and l-filling positions in $\psi(x)$ for $l > k$ by k-free positions. Thus we get in this way a sequence of continuous maps $\pi_k : \psi(X) \to \{1, ..., K\}^{\mathbb{Z}}$ such that $\pi_k \circ \psi(x) = \psi_k(x)$. We can then identify $\mathcal{V}_1^n(x)$ for any integer n by reading the words in the 1-filling positions of $\psi_1(x)$ and finally $\mathcal{V}_l^n(x)$ inductively on l by reading in $\psi_l(x)$ the words in the l-filling positions. Consequently any fiber of ψ_k is contained in a unique element of $\mathcal{V}_k^\infty = \{\bigcap_{n \in \mathbb{Z}} T^{-n} V_k^n,\ V_k^n \in \mathcal{V}_k\}$. In particular ψ_k is ϵ_k-injective with ϵ_k being the diameter of $\mathcal{V}_k$. As the diameter of $\mathcal{V}_k$ goes to zero as k goes to infinity it follows that the limit ψ is injective.

5. Continuity and convergence of $(\psi_k)_k$. The maps $(\psi_k)_k$ are continuous because $(\mathcal{V}_k)_k$ are open covers and U_k are clopen sets. Moreover $\psi_k(x)$ and $\psi(x)$ differs on any k-block only at the k-free positions. Recall that these positions represent a proportion of at most $\alpha/2^k$ positions in k-blocks. Moreover for any $n > n_k^2$ the segment $[-n, n]$ is the union of at most $2(2n+1)/n_k$ k-blocks and 2 k-subblocks (whose length is less than or equal to $4n_k$). Then we have

$$\sharp\{i,\ |i| \leq n \text{ and } (\psi_k)^i(x) \neq (\psi)^i(x)\} \leq (2n+1)\alpha/2^k + 2(2n+1)/n_k + 8n_k.$$

Thus we conclude that $d_N\left(\{i,\ (\psi_l)^i(x) \neq (\psi)^i(x)\}\right) \lesssim \alpha/2^k$ for any $x \in X$, any $N > n_k^2$ and any $l \geq k$ because we took $\alpha n_k >> 2^k$ and $\{i,\ (\psi_l)^i(x) \neq (\psi)^i(x)\} \subset \{i,\ (\psi_k)^i(x) \neq (\psi)^i(x)\}$. Therefore ψ_k goes uniformly to ψ w.r.t. $(d_N)_N$ when k goes to infinity.

6. Principal strongly faithful symbolic extension as the inverse of Krieger embedding. The inverse of ψ is (uniformly) continuous on $\psi(X)$. Indeed if $\psi(x)$ and $\psi(y)$ coincide on their $[-4n_k, 4n_k]$ coordinates then they belong to the same l-block for any $l \leq k$ and in particular x and y belong to the same element of $\mathcal{V}_k$.

Lemma 4.7 applies to ψ, thus any T-invariant measure on $\overline{\psi(X)}$ is supported on $\psi(X)$. Finally as $\psi : (X,T) \to (\psi(X), \sigma)$ is a Borel isomorphism, the induced map on the set of invariant measures is bijective and preserve the measure theoretical entropy. This proves π is faithful and principal.

Remark 4.2. *Observe that $\psi(X)$ is in general not closed. Indeed let (X,T) be the odometer to base $(p_k)_k = (2^k)_k$. For any positive integer n we let x_n be the point in X given by $x_n = (...,0,...0,2^n,0,...)$ where 2^n is at the $(n+1)^{th}$ coordinate and U_n be the clopen set of points whose n first coordinates are zero. The sequence $(U_n)_n$ satisfies the properties of the Marker Lemma (Lemma 4.2). Then we consider the Borel embedding $\psi : X \to \{0,1\}^{\mathbb{Z}}$ given by the previous construction. Let y be a limit point of $(\psi(x_n))$. If it belongs to $\psi(X)$ the point y is necessarily $\psi(0)$ as $\pi_k(y) = \lim_n \pi_k \circ \psi(x_n) = \lim_n \psi_k(x_n) = \psi_k(0)$. Now the -1 position of $\psi(0)$ is a $n+1$-free position for all n whereas for any n the -1 position of $\psi(x_n)$ is a $n+1$-filling position (indeed the zeroth coordinate is in the middle of an $n+1$-block of $\psi(x_n)$ and at the beginning of an $n+1$ block of $\psi(0)$).*

4.4.2. *Construction of the embedding, the periodic case.* We consider now the general case, i.e. we consider asymptotically expansive zero-dimensional systems not necessarily aperiodic ones. Here we will distinguish among k-blocks regular ones and singular ones, which encode respectively the dynamics far from or close to the periodic points. Each singular block is the repetition of a brick, which is a finite word encoding the periodic point associated to the singular block (its length is in particular equal to the period). We let again $\alpha = \log K - h_{top}(T) > 0$.

0. Dynamical Markers. We consider a sequence of clopen partitions $(\mathcal{V}_k)_k$ given by Lemma 4.1 and we let $\underline{n} = (n_k)_k$ be a sequence of integers satisfying for any k and for all $n > n_k/2$

$$\sup_{U^n \in \mathcal{V}_k^n} \sharp\{V^n,\ V^n \in \mathcal{V}_{k+1}^n \text{ with } V^n \subset U^n\} \leq e^{n\alpha/2^k}$$

and moreover

$$(1) \qquad \sup_{V_k^n \in \mathcal{V}_k^n} \log\{\sharp Per_n(X,T) \cap V_k^n\} \quad < \quad \alpha/2^k.$$

We also choose n_1 large enough so that for $n \geq n_1$ we have $\sharp Per_n(X,T) < K^{n-1}$ and $\sharp \mathcal{V}_1^n < K^{n(1-\alpha/2)-1}$. Moreover we may ensure that $(n_k)_k$ satisfy $\alpha n_k >> 2^k$ and $n_k > 16 n_{k-1}$ for any $k \in \mathbb{N}$ (we let $n_0 = 0$). Far from the $n_1, n_2, ..., n_k, ...$-periodic points we will encode the dynamics with respect to the covers $\mathcal{V}_1, \mathcal{V}_2$, ... ,$\mathcal{V}_k$,... on finite pieces of orbits of length of order $n_1, n_2, ..., n_k, ...$ These pieces of orbits are given again by the topological Rokhlin towers over $(U_k)_k$ given in Lemma 4.2 with respect to the sequences $\underline{n} = (n_k)_k$ and $(\epsilon_k)_k$. This last sequence will be precised latter on. But we already assume ϵ_k so small that any consecutive return times in U_k with distance larger than or equal to $2n_k$ corresponds to the visit of a ϵ_k-neighborhood of a single periodic orbit. We may also assume that three consecutive return times in U_k, which are at least $2n_k$-far from each other, are associated to the same periodic orbit.

For any periodic orbit with least period $n \leq n_1$ we choose arbitrarily one point x in the orbit. For such a periodic point we let $G_1(f^l x) = F_1(f^l x) = l + n\mathbb{Z}$ and

$E_1(x) = \emptyset$ for any integer l. Let us now define $G_1(x), F_1(x), E_1(x)$ for aperiodic or n-periodic points x with $n > n_1$. Let l be a return time of x in U_1. We distinguish three cases:

- there are other return times in U_1, say k, m, with $l - 4n_1 < k < l < m < l + 4n_1$. We let in this case $l' = l$,
- there is a return time k in U_1 with $l - 4n_1 < k < l$ (resp. $l < k < l + 4n_1$), but no return times p with $l < p < l + 4n_1$ (resp. $l - 4n_1 < p < l$), in this case $T^{l+n_1}x, T^{l+n_1+1}x, ..., T^{l+2n_1}x$ (resp. $T^{l-2n_1}x, T^{l-2n_1+1}x, ..., T^{l-n_1}x$) belong to the ϵ_1-neighborhood of a single periodic orbit with period m less than or equal to n_1. Then we let l' be the least integer in $[l, l + n_1[$ (resp. largest integer in $]l-n_1, l]$) such that $T^{l'+cm}x$ is ϵ_1-close to the distinguished point in this periodic orbit where c is a positive (resp. negative) integer with $l + n_1 \leq l' + cm \leq l + 2n_1$ (resp. $l - 2n_1 \leq l' + cm \leq l - n_1$).
- there is no return time k in U_1 with $|k - l| < 4n_1$.

The set $E_1(x)$ is then defined as the set of integers l' as defined above for return times l in U_1 of the two first kinds. Observe two consecutive integers in $E_1(x)$ are at least n_1-far from each other. Now if (k', l') are two consecutive integers in $E_1(x)$ with $l' - k' > 4n_1$ then l' and k' are of the second kind and $l' - k' \in m\mathbb{Z}$ where m is the period of the single periodic orbit close to the piece of orbit $f^{k'}(x), ..., f^{l'}(x)$. Then we let $F_1(x)$ be the union of $[k', l'[\cap (k' + m\mathbb{Z})$ over all such pairs (k', l') (see Figure 3). Thus two consecutive integers in the union $G_1(x) = E_1(x) \cup F_1(x)$ are at most $4n_1$-far from each other. As the set U_1 is clopen the maps $x \mapsto E_1(x), F_1(x), G_1(x)$ are continous in the following sense : for any bounded interval of integers $I \subset \mathbb{Z}$ and for all $x \in X$ the sets $E_1(y) \cap I, F_1(y) \cap I$ and $G_1(y) \cap I$ are constant for y in a neighborhood of x.

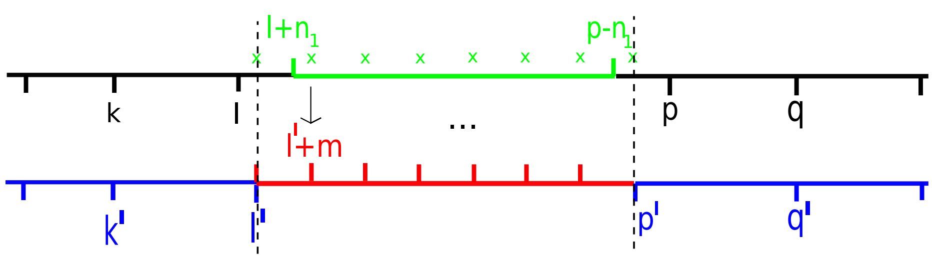

Figure 3: ***The set $G_1(x)$ from the return times in U_1.*** *On the first line the part in black corresponds to times in $\bigcup_{|l|<n_1} T^l U_1$ whereas the green part are times in the ϵ_1-neighborhood of Per_{n_1}. Black dashes are times in U_1 and the green crosses correspond to the visits of the ϵ_1-neighborhood of the distinguished point of the periodic orbit. On the second line the elements of $E_1(x)$ and $F_1(x)$ are respectively represented by blue and red dashes.*

The set $E_k(x), F_k(x)$ are then defined by induction on $k > 1$ in a similar way as follows. We assume the sets $E_{k-1}(x), F_{k-1}(x)$ already built and they satisfy :

- two consecutive integers in $G_{k-1}(x) = E_{k-1}(x) \cup F_{k-1}(x)$ are at most $5n_{k-1}$-far from each other,
- the maps $x \mapsto E_{k-1}(x), F_{k-1}(x)$ are continuous.

Let us define now $E_k(x)$ and $F_k(x)$. Firstly for a n-periodic point x with $n \leq n_k$ either $n \leq n_{k-1}$ and we let $G_k(x) = F_k(x) = F_{k-1}(x)$, $E_k(x) = \emptyset$ or $n_{k-1} < n \leq n_k$. In this last case we choose a point, say $f^l(x)$, in the orbit of x with $l \in G_{k-1}(x)$ and finally we let $G_k(x) = F_k(x) = l + n\mathbb{Z}$ and $E_k(x) = \emptyset$.

For other points we classify return times in U_k and define $E_k(x)$ and $F_k(x)$ in a similar way as for $k = 1$. Firstly observe that we can take ϵ_k so small that for any $x \in Per_{n_k}^{\epsilon_k}$ the set $G_{k-1}(x)$ coincides with the set $G_{k-1}(y) = E_{k_1}(y)$ on $[-n_k, n_k]$ where y is the periodic point which is ϵ_k-closed to x. We consider l be a return time of x in U_k. We distinguish three cases:

- there are other return times in U_k, say j, m, with $l - 4n_k < j < l < m < l + 4n_k$. We let in this case l' be the largest integer in $G_{k-1}(x)$ less than l (by the induction hypothesis we have $|l' - l| < 5n_{k-1}$),
- there is a return time j in U_k with $l - 4n_k < j < l$ (resp. $l < j < l + 4n_k$), but no return time p with $l < p < l + 4n_k$ (resp. $l - 4n_k < p < l$), in this case $T^{l+n_k}x, T^{l+n_k+1}x, ..., T^{l+2n_k}x$ (resp. $T^{l-2n_k}x, T^{l-2n_k+1}x, ..., T^{l-n_k}x$) belong to the ϵ_k-neighborhood of a single periodic orbit with period m less than or equal to n_k. Then we let l' be the least integer in $G_{k-1}(x)$ larger (resp. largest integer less) than l such that $T^{l'+cm}x$ is ϵ_k-close to the distinguished point in this periodic orbit where c is a positive (resp. negative) integer with $l + n_k \leq l' + cm \leq l + 2n_k$ (resp. $l - 2n_k \leq l' + cm \leq l - n_k$).
- there are no return times j in U_k with $|j - l| < 4n_k$.

The set $E_k(x)$ is then defined as the set of integers l' as defined above for return times l in U_k of the two first kinds. Observe two consecutive integers in $E_k(x)$ are at least $n_k/2$-far from each other. Now if (k', l') are two consecutive integers in $E_k(x)$ with $l' - k' > 4n_k$ then l' and k' are of the second kind and $l' - k' \in m\mathbb{Z}$ where m is the period of the single periodic orbit close to the piece of orbit $f^{k'}(x), ..., f^{l'}(x)$. Then we let $F_k(x)$ be the union of $[k', l'[\cap (k' + m\mathbb{Z})$ over all such pairs (k', l') (see Figure 4).

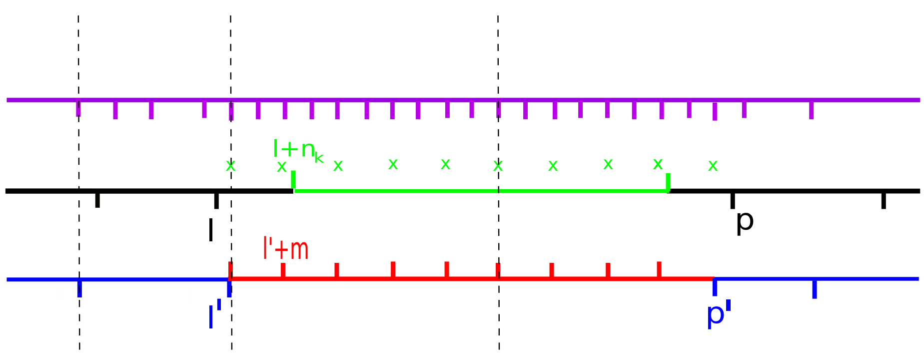

Figure 4: ***The set $G_k(x)$ from $G_{k-1}(x)$ and the return times in U_k.*** *On the first line in purple is represented the times in $G_{k-1}(x)$. The second line corresponds, as the first line in Figure 3, to the return times in U_k (in black) and in the ϵ_k-neighborhood of Per_{n_k} (in green). On the last line we draw the elements of $E_k(x)$ and $F_k(x)$ with respectively blue and red dashes.*

Thus two consecutive integers in the union $G_k(x) = E_k(x) \cup F_k(x)$ are at most $5n_k$-far from each other. The maps $x \mapsto E_k(x), F_k(x)$ are continuous. Moreover it follows from the construction that $G_k(x) \subset G_{k-1}(x)$ for all integers k and for all $x \in X$.

1. Marker structure. We consider the nonincreasing sequence $G = (G_k)_k$ of subsets of $\mathbb{Z}$ associated to a given point $x \in X$. Recall that for any k, the set G_k is the union of the two subsets E_k and F_k with the following properties. Moreover any two consecutive $p < q$ in $E_k \cup \{-\infty, +\infty\}$ satisfy :

- either $n_k/2 \leq q - p \leq 4n_k$, then there are no element of F_k in the interval $[p, q[$. Any word over the interval of integers $[p, q[$ will be called a regular k-block (see Figure 5 below).
- or $q - p > 4n_k$, then there exists an integer $0 < m \leq n_k$ such that $q - p \in m\mathbb{Z} \cup \{+\infty\}$ and $F_k \cap [p, q[= (p + m\mathbb{Z}) \cap [p, q[$ (resp. $F_k \cap [p, q[= (q + m\mathbb{Z}) \cap [p, q[$, resp. $F_k = n + m\mathbb{Z}$ for some integer n) when p is finite (resp. $p = -\infty$ and q is finite, resp. $p = -\infty$ and $q = +\infty$). Any word over the interval of integers $[p, q[$ will be called a singular k-block, whereas words over intervals of the form $[p + km, p + (k+1)m[$ with $k \in \mathbb{N}$ and $p + (k+1)m \leq q$ (resp. $[q + (k-1)m, q + km[$ with $-k \in \mathbb{N}$, resp. $[n + km, n + (k+1)m[$ with $k \in \mathbb{Z}$) are said to be l-bricks with $n_{l-1} < m \leq n_l$.

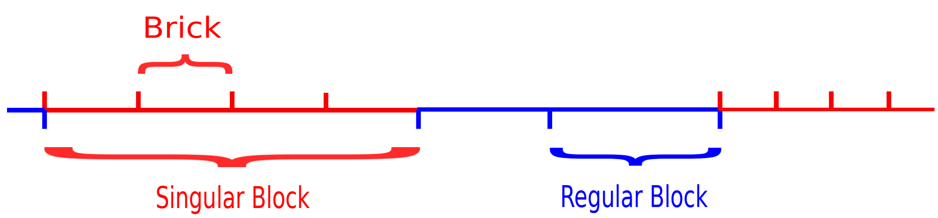

Figure 5: *k-blocks and l-bricks with $l \leq k$ for a given k.* *Elements in F_k are given by red dashes and elements in E_k by blue ones. Regular k-blocks (resp. bricks) are the blue intervals (resp. red) between two consecutive blues (resp. red) dashes. A singular k-block is a maximal red interval : it is the self concatenation of a single l-brick with $l \leq k$.*

The empty k-blocks and k-bricks are words in the alphabet $\{*, \times, \circ,], [, +, (,), |, \times, \times|, \times], \times[\}$ over k-blocks and k-bricks respectively. They are defined by induction on k, together with their marker, filling, free positions, as follows.

Empty 1-blocks. Firstly a empty regular 1-block is a regular 1-block u^1 such that

- the first and last letter of u^1, resp. called the left and right 1-marker positions, coincide respectively with (and),
- the next $[(1-\alpha)|u_1|]$ coordinates of u^1, called the 1-filling positions all take the value $*$,
- the remaining coordinates, called the 1-free positions, all take the value $\circ$.

The 2-marker position of a 1-regular block is the first 1-free position of this block. An empty 1-brick is a 1-brick whose first letter $\times$ represents the marker position and other ones $*$ are 1-filling positions.

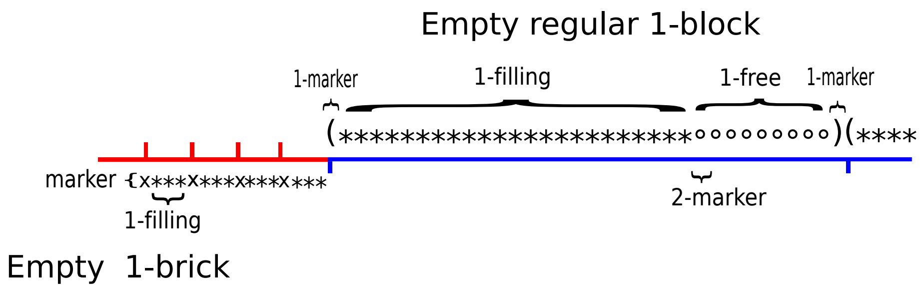

Figure 6: ***Empty* 1*-blocks.*** *The* 2*-marker,* 1*-marker,* 1*-filling and* 1*-free positions of an empty regular* 1*-block and the marker and* 1*-filling positions of* 1*-bricks are represented.*

Empty k- blocks from empty $k-1$-blocks. Empty k-block are obtained from a concatenation of empty $(k-1)$-blocks and l-bricks with $l \leq k-1$ after the following modifications which will ensure that all empty regular k-blocks and k-bricks (with $k \neq 1$) have a proportion $\alpha/2^{k+1}$ of k-free positions.

Let $u^k := (u_{i_k}, ..., u_{j_k - 1})$ be a concatenation of empty $(k-1)$-blocks and l-bricks with $l \leq k-1$ in such a way u^k defines a regular k-block. Either i_k belongs to E_{k-1}, which means u^k starts with a empty regular $(k-1)$-block, and then we change to [the value at the k-marker coordinate of this $(k-1)$-block or i_k is in F_{k-1}, which means u^k starts with a empty l-brick with $l \leq k-1$, and then we change to [(or $\times$[when $l = 1$) the value at the marker position of this brick. We also modify the k-marker position of the last $(k-1)$-regular block or l-brick of our k-block u^k by letting the symbol] (or $\times$] when $l = 1$). The marker positions of the remaining $(k-1)$-regular blocks and l-bricks with $l \leq k$ in u^k are changed to $+$.

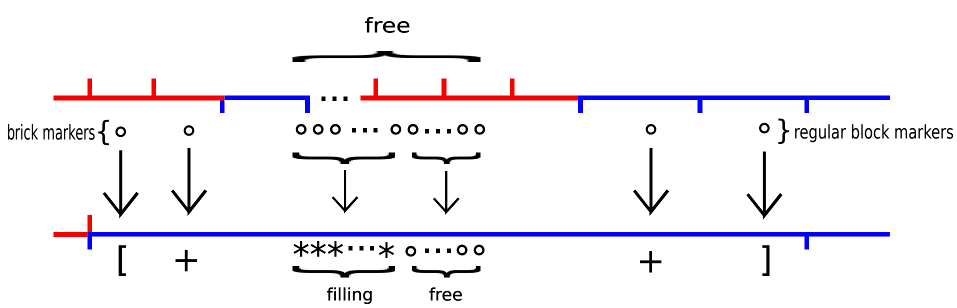

Figure 7: $(k+1)$***-regular block from*** k***-blocks.*** *A regular* $(k+1)$*-block is obtained from a concatenation of regular* k*-blocks and* l*-bricks with* $l \leq k$*.*

We finally modify the values in the empty maximal singular 1-subblocks v^k (i.e. maximal repetition of a 1-brick) in u^k as follows. In such a subblock v^k, we make k-free the l^{th} positions with $l > \max\big([|v^k|(1-\alpha/2^k)], m_1 + 1\big)$, where m_1 is the length of the repeated 1-bricks in v^k, by letting the value $\circ$ at these coordinates, with one exception : we only let the potential symbol] at the marker position of the last 1-brick of v_k (it appears when v^k is the end of u^k, see Figure 11.).

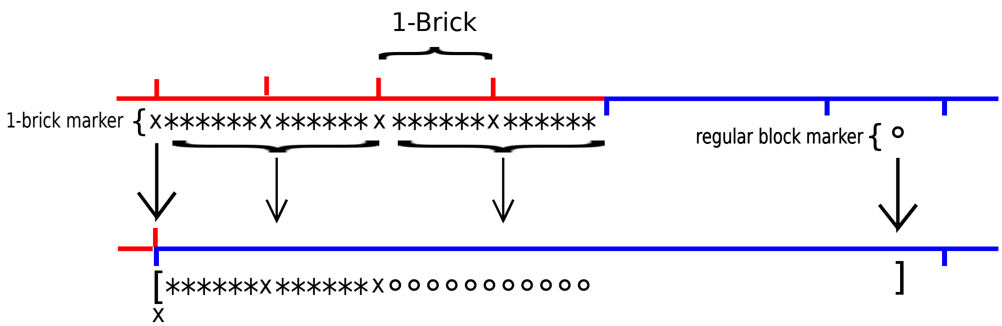

Figure 8: **1-*brick in a regular* $(k+1)$*-block (or* $(k+1)$*-brick).*** *We illustrate the particular case of* 1*-bricks in a* k*-singular block which take part in a* $(k+1)$*-regular block (or a* $(k+1)$*-brick). Some of the* 1*-filling positions of these* 1*-bricks are deleted. Such a change occurs only in this situation.*

Assume now that $u^k := (u_{i_k}, ..., u_{j_k-1})$ defines a l-brick with $l \leq k$. When $l \leq k-1$ we let the brick remain unchanged. Then an empty k-brick is a concatenation of $(k-1)$-regular blocks or l-bricks with $l < k$ (see Figure 9).

To mark the period of this k-brick we change to $|$ (or $\times|$ in the case of 1-bricks) the symbol at the k-marker position of the $(k-1)$-regular block or at the marker position

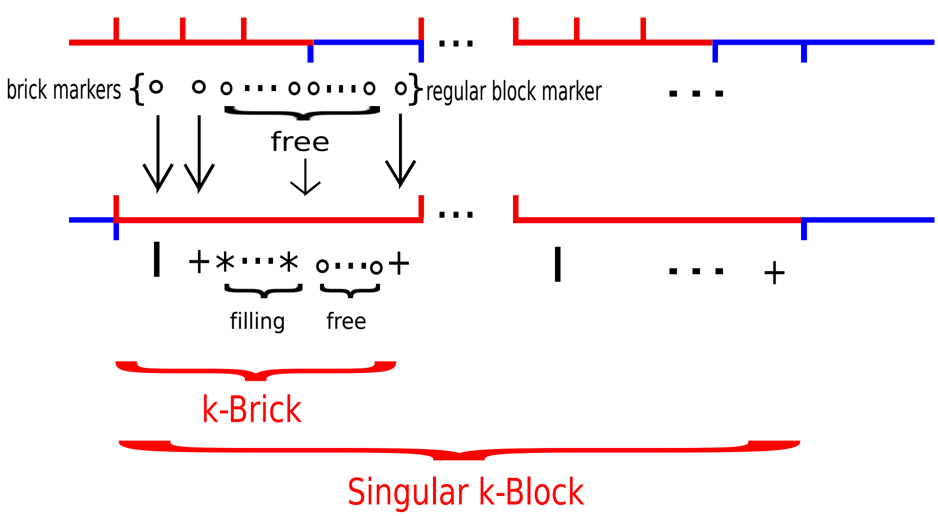

Figure 9: ***k-brick from*** $(k-1)$***-blocks.*** *In the first line, we can see the* $(k-1)$*-blocks. The second line represents the* k*-singular block given by the repetition of a* k*-brick, which is a concatenation of a* $(k-1)$*-regular block and* l*-bricks with* $l < k$*. We also visualize the changes to get the empty* k*-code from the empty* $(k-1)$*-code.*

of the l-brick by which the k-brick begins. We then make k-free the positions in the maximal singular 1-subblocks of u^k in the same way as above.

The k-filling positions (resp. k-free) of a k-regular block or a k-brick u^k are the $\alpha/2^k|u^k|$ first (resp. remaining) $(k-1)$-free positions of u_k. Finally the $(k+1)$-marker position of a regular k-block (resp. the marker position of a k-brick) is the first k-free position in this block (resp. in this brick). The empty k-code $\psi_k^\emptyset$ with respect to G is then just the infinite word obtained by concatenation of all empty k-blocks.

2. *First scale encoding.* Empty regular 1-blocks are encoded as for aperiodic systems. For any $n \leq n_1$ we consider an injection ψ_n from $Per_n(X,T)$ to $\{1,...,K\}^n$. Now we fill up an empty 1-brick by writing the image by ψ_n of the associated distinguished periodic point (without erasing the marker position $\times$).

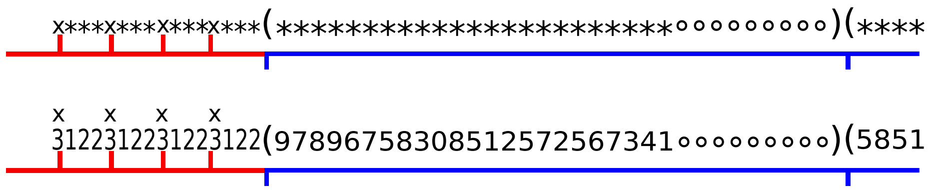

Figure 10: ***Filling up empty* 1*-blocks* ($K = 9$).** *We fill up empty* 1*-blocks. For regular* 1*-blocks we proceed as in the aperiodic case. For singular* 1*-block we repeat the code of the associated periodic point by preserving the marker* $\times$.

3. *Higher scale encoding.* Empty regular k-blocks are encoded by writing in the k-filling positions the code with respect to $\mathcal{V}_k$ conditionally to $\mathcal{V}_{k-1}$ as in the aperiodic case. We explain now the procedure for an l-brick with $l \leq k$. Such a brick is associated to a distinguished periodic point with least period $m \in]n_{l-1}, n_l]$. We distinguish then two cases : either $l \leq k-1$ and we let ψ_k be equal to ψ_{k-1} on this brick, or $l = k$ and in the first $\alpha n/2^k$ k-filling positions of the brick we write the code of $\mathcal{V}_k^n$ conditionally to $\mathcal{V}_{k-1}^n$ and in the next $\alpha n/2^k$ k-filling positions we specify the periodic point by asymptotic *per*-expansiveness according to Inequality (1) on p.15.

The sequence $(\psi_k(x))_k$ is converging pointwisely for the product topology in $\mathcal{A}^{\mathbb{Z}}$ (with $\mathcal{A}$ being the finite alphabet given by all symbols previously used) as for any i the i^{th} coordinate is changed at most two times. Indeed, k-free positions are changed at most one time. For $k \geq 2$ the k-filling positions are not changed but for $k = 1$ some of them become free and then filled again. However this last case may happen at any coordinate at most one time as we leave 1-bricks forever. We let $\psi(x)$ be the pointwise limit of $\psi_k(x)$.

4. *Decoding.* The k-code $\psi_k(x)$ may be deduced on k-regular blocks from $\psi(x)$ by identifying k-blocks by induction as follows. Firstly regular 1-block correspond to finite words in $\psi(x)$ delimited by two consecutive parentheses (and). In these blocks we may read the associated element of $\mathcal{V}_1^n$ by looking at the filling 1-positions.

Now in a 1-singular block a complete 1-bricks has not been distorted by higher scale encoding (see Figure 8) and we may read the associated periodic point between the symbols $\times$.

Then regular 2-blocks are identified as words between consecutive marked times by [and] with length in $[n_2/2, 4n_2]$, where these markers occur at the 2-marker position of a 1-regular block or at a marker position of a 1-brick. Note that we may find in a 2-singular block some consecutive parenthesis [and] coming from the k-encoding with $k > 2$, however they are at least $n_3/2$-far from each other.

The length of 2-bricks may be found thanks to the symbols | at the 2-marker position of 1-block or at a marker position of a 1-block and we may identify the associated periodic point at the 2-filling positions.

Similarly one may recover $\psi_k(x)$ and the k-blocks from $\psi(x)$ by induction on k and then read either the associated periodic point ϵ_k-close to the piece of orbit for singular k-block or the orbit with respect to $\mathcal{V}_k$ for regular ones. We can assume $2\epsilon_k$ is less than the Lebesgue number of the open cover $T^l\mathcal{V}_k$ for any k and any $|l| \leq n_k$. In particular this proves that any two points with the same image under ψ_k belong to the same element of $\mathcal{V}_k$. Thus ψ_k is ϵ'_k-injective with ϵ'_k being the diameter of $\mathcal{V}_k$ and ψ is injective.

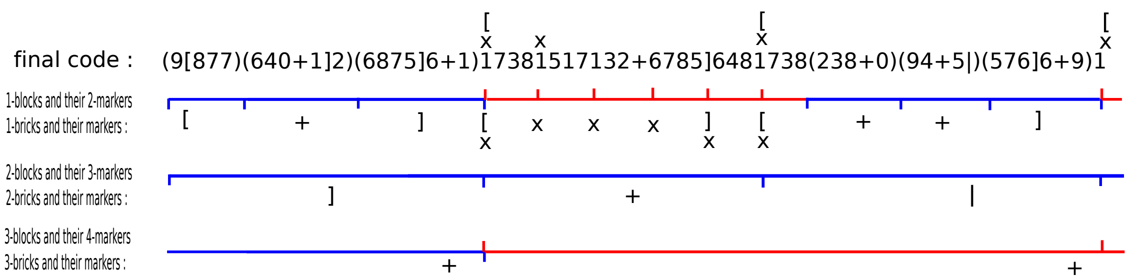

Figure 11: ***Example of decoding.*** *We recover from a part of a limit code $\psi(x)$ the associated structure of k-blocks (resp. bricks) and their $(k+1)-$markers (resp. markers) for $k = 1, 2, 3$.*

5. Continuity and convergence of $(\psi_k)_k$. One proves as in the aperiodic case the codes $(\psi_k)_k$ are continuous by the clopen property of $(\mathcal{V}_k)_k$, $(U_k)_k$ and $(Per^{\epsilon_k}_{n_k})_k$. As we change at each step a proportion $\alpha/2^k$ of coordinates on subblocks of size larger than $n_k/2$ and less than or equal to $4n_k$ the sequence $(\psi_k)_k$ is converging again uniformly with respect to $(d_N)_N$.

6. Principal strongly faithful symbolic extension as the inverse of Krieger embedding. The inverse of ψ is (uniformly) continuous on $\psi(X)$. Indeed if $\psi(x)$ and $\psi(y)$ coincide on their $[-8n_k, 8n_k]$ coordinates then the points x and y belong either to the same regular k-block or are ϵ_k-close to the same n_k-periodic point. In both cases x and y belong to the same element of $\mathcal{V}_k$. The other desired properties of the extension are shown as in the aperiodic case.

5. The general case

Now we will deduce the general case from the zero-dimensional one by using the small boundary property. More precisely this last property allows us to embed the system in a zero-dimensional one as follows :

5.1. Embedding systems with the small boundary property in zero-dimensional ones.

Proposition 5.1. *(p. 73 [Dow05]) Let (X,T) be an (resp. aperiodic, resp. asymptotically per-expansive, resp. asymptotically h-expansive) topological dynamical system with the small boundary property. Then there exists a zero dimension (resp. aperiodic , resp. asymptotically per-expansive, resp. asymptotically h-expansive) system (Y,S) and a Borel equivariant injective map $\psi : X \to Y$ such that*

- *ψ is continuous on a full set,*
- *the induced map $\psi^* : \mathcal{M}(X,T) \to \mathcal{M}(Y,S)$ is an homeomorphism,*
- *ψ^{-1} is uniformly continuous on $\psi(X)$ and extends to a continuous principal strongly faithful extension π on $Y = \overline{\psi(X)}$ of (X,T).*

For completeness we sketch the proof of Proposition 5.1. We consider a topological dynamical system with the small boundary property. Let $(P_k)_k$ be a nonincreasing sequence of essential partitions of X whose diameter goes to zero. We consider the shift σ over $\prod_k P_k^{\mathbb{Z}}$. Let $\psi : X \to \prod_k P_k^{\mathbb{Z}}$ be defined by $\psi(x) = (P_k(T^n x))$, it is injective and semi-conjugates T with the shift, $\sigma \circ \psi = \psi \circ T$. Clearly ψ is continuous on the full subset of points whose orbit does not intersect the boundary of the P_k's. Moreover the induced map ψ^* is a topological embedding. Indeed to prove the continuity of ψ^* it is enough to see that $\psi^*\nu(A)$ converges to $\psi^*\mu(A)$ when ν goes to μ for any finite cylinder A in $\prod_k P_k^{\mathbb{Z}}$. But for such A the closure and the interior of $\psi^{-1}A$ have the same measure for any invariant measure so that the previous property of continuity holds.

The inverse ψ^{-1} satisfies $\psi^{-1}\left((A_n^k)_{n,k}\right) = \bigcap_{n,k} \overline{T^{-n}A_{n,k}}$ and is thus clearly uniformly continuous. The continuous extension π of ψ^{-1} maps $\overline{\psi(X)} \setminus \psi(X)$ on the boundaries of the $(T^n P_k)_{n,k}$'s and thus any σ-invariant measure on $\overline{\psi(X)}$ is supported on $\psi(X)$. As $\psi(Per_n(X,T)) = Per_n(\psi(X),S)$ for any integer n we conclude that π is a strongly faithful principal extension of (X,T).

Principal extensions preserves the asymptotic h-expansiveness property and strongly faithful extensions preserves the aperiodicity. Finally if n-periodic points in $Y = \overline{\psi(X)}$ lie in the same P_k^n for some k their π-image are n-periodic points for T in the same $\overline{P_k}^n$. As these periodic points are in fact in $\psi(X)$ and π is one-to-one on $\psi(X)$, we conclude asymptotic *per*-expansiveness is also preserved by this extension.

5.2. Reduction to the zero-dimensional case. By applying the above Proposition 5.1, our Main Theorem follows from Proposition 4.1. Indeed if (X,T) is as in our Main Theorem, then the zero-dimensional extension (Y,S) of (X,T) obtained in the above proposition satisfies the assumption of Proposition 4.1. The desired embedding is then given by the composition of the embedding in the zero-dimensional system (Y,S) and the Krieger embedding of (Y,S) in the shift with the desired finite alphabet. The property of convergence and continuity of the embedding and its inverse follows then from basic properties of composition.

6. Proof of Propostion 3.1 and 6.1

6.1. Topological embedding in the set of invariant measures $\Rightarrow$ asymptotic h-expansiveness. Assume $\psi : X \to \Lambda^{\mathbb{Z}}$ is a Borel equivariant embedding such that ψ^* is a topological embedding. Recall that P_0 denotes the zero coordinate partition of $\Lambda^{\mathbb{Z}}$. Then for any $\mu \in \mathcal{M}(X,T)$ we have $h(\mu) = h(\psi^*\mu) = h(\psi^*\mu, P_0) =$

$h(\mu, \psi^{-1}P_0)$. In the present paper such a partition $\psi^{-1}(P_0)$ is said to be generating. Let $B \in P_0$ and $A = \psi^{-1}B \in \psi^{-1}P_0$ and let μ be T-invariant measure. As ψ^* is continuous and B is a clopen set we have

$$\begin{aligned}\limsup_{\nu\to\mu} \nu(A) &= \limsup_{\nu\to\mu} \psi^*\nu(B),\\ &\leq \lim_{\xi\to\psi^*\mu} \xi(B),\\ &\leq \psi^*\mu(B) = \mu(A).\end{aligned}$$

Similarly we prove $\liminf_{\nu\to\mu} \nu(A) \geq \mu(A)$. Therefore for any $A \in \psi^{-1}P_0$, the function $\mu \mapsto \mu(A)$ is continuous on $\mathcal{M}(X,T)$. In general it does not imply that A has a small boundary, for example when there is an attracting fixed point in the boundary of $B \in \psi^{-1}P_0$ (the Dirac measure at this point is then isolated in $\mathcal{M}(X,T)$). Finally asymptotic h-expansiveness follows from the following lemma.

Lemma 6.1. *Let (X,T) be a topological dynamical system admitting a finite generating partition P such that for any $A \in P$, the function $\mu \mapsto \mu(A)$ is continuous on $\mathcal{M}(X,T)$, then (X,T) is asymptotically h-expansive.*

Proof. Let $(h_k)_k$ be the Lebegue entropy structure (see Section 6.3 in [Dow05]). For any integer k the function h_k is defined as the entropy of $\mu \times \lambda$ with respect to an essential partition P_k of $(X \times \mathbb{S}^1, T \times R)$, where R is an irrational circle rotation and $(P_k)_k$ is a nonincreasing sequence of essential partitions of $X \times \mathbb{S}^1$ whose diameters go to zero. By the tail variational principle (Theorem 5.1.4 in [Dow05]) T is asymptotically h-expansive if and only if $(h_k)_k$ is converging uniformly to h. By assumption on P, the maps $\mu \mapsto h(\mu\times\lambda, P\times\mathbb{S}^1 \vee P_k|P_k)$ are upper semicontinuous on $\mathcal{M}(X,T)$. Moreover they are nonincreasing in k and converging to zero. It is well known that the convergence is in fact uniform for such sequences of functions. As $h(\mu) - h_k(\mu) \leq h(\mu \times \lambda, P \times \mathbb{S}^1 \vee P_k|P_k)$ for any T-invariant measure μ the sequence $h - h_k$ is also converging uniformly to zero and thus T is asymptotically h- expansive. □

6.2. **Topological embedding in the set of invariant measures ⇒ asymptotic *per*-expansiveness.** Let $\psi : (X,T) \to (\{1, ..., K\}^{\mathbb{Z}}, \sigma)$ be a Borel equivariant embedding such that ψ^* is a topological embedding and let $(\mathcal{U}_k)_k$ be a sequence of open covers such that the diameter of $\mathcal{U}_k$ goes to zero. We will show that the exponential growth in n of the set $U_k^n \cap Per_n(X,T)$ goes to zero uniformly in $U_k^n \in \mathcal{U}_k^n$ when k goes to infinity. Assume first (X,T) has the small boundary property. Let $(P_l)_l$ be a sequence of essential partitions with $diam(P_l) \xrightarrow{l\to 0} 0$. Let $\epsilon > 0$. For any l there exist k_l and n_l such that any U_k^n with $k > k_l$ and $n > n_l$ meets at most $e^{\epsilon n}$ elements of P_l^n (see Lemma 6 in [Bur09]). Thus it is enough to prove that the exponential growth in n of the set $A^n \cap Per_n(X,T)$ goes to zero uniformly in $A^n \in P_l^n$ when l goes to infinity. Take A^n so that the previous intersection has maximal cardinality. We consider the associated empirical measures $\mu_n := \frac{1}{\sharp A^n \cap Per_n(X,T)} \sum_{x\in A^n\cap Per_n(X,T)} \delta_x$ and $\nu_n := \frac{1}{n}\sum_{k=0}^{n-1} T^*\mu_n$ for any integer

n. Let $P = \psi^{-1}(P_0)$. Following [Bur09] p.368 l.6, we get for any $m < n$

$$\begin{aligned}\frac{H_{\nu_n}(P^m|P_l^m)}{m} &= \frac{H_{\psi^*\nu_n}(P_0^m|\psi(P_l)^m)}{m};\\ &\geq \frac{H_{\psi^*\mu_n}(P_0^n|\psi(P_l)^n) - 3m\log K}{n};\\ &\geq \frac{\log \sharp A^n \cap Per_n(X,T) - 3m\log K}{n}.\end{aligned}$$

The last inequality follows from the injectivity of ψ, the inclusion $\psi(Per_n(X,T)) \subset Per_n(\Lambda^{\mathbb{Z}}, S)$ and the inequalities $\sharp Per_n(\Lambda^{\mathbb{Z}}, S) \cap B^n \leq 1$ for any given $B^n \in P_0^n$. Observe also that ν_n is invariant: it is the barycenter of the periodic measures associated to the periodic points in $A^n \cap Per_n(X,T)$. Let ν be a weak limit of $(\nu_n)_n$. As seen above in Subsection 6.1 the sequence $(\nu_n(A))_n$ converges to $\nu(A)$ for any $A \in P$. By taking the limit in n and then m we get thus by upper semicontinuity:

$$h(\nu, P|P_l) \geq \limsup_n \frac{\log \sharp A^n \cap Per_n(X,T)}{n}.$$

However by asymptotic h-expansiveness, which follows from Subsection 6.1, the left member goes to zero uniformly in ν when l goes to infinity. This concludes the proof when (X,T) has the small boundary property. For general systems one may consider the product with an irrational circle rotation and apply the previous proof by replacing μ_n with is product with a Dirac measure of the circle, ψ by its product with the circle identity and P by its product with the circle.

6.3. **Principal strongly faithful asymptotic *per*-expansive extension $\Rightarrow$ asymptotic *per*-expansiveness.** We prove now the second item of Proposition 3.1. We only have to prove asymptotic *per*-expansiveness as principal extensions preserve asymptotical h-expansiveness as proved by Ledrappier [Led78]. Let $\pi : (Y,S) \to (X,T)$ be a principal extension. Equivalently the topological conditional entropy $h(S|T)$ of S w.r.t. T vanishes (see Section 6.3 of [Dow11] for precise definitions and basic properties). In particular, for any $\epsilon > 0$ there exists an open cover $\mathcal{U}$ of X such that for any open cover $\mathcal{V}$ and for any $U^n \in \mathcal{U}^n$ with large n, the set $\pi^{-1}U^n$ may be covered by $e^{\epsilon n/2}$ element of $\mathcal{V}^n$. If we assume moreover the extension to be strongly faithful it preserves periodic orbits. In particular the number of n-periodic points for T in U^n is equal to the number of n-periodic points for S in $\pi^{-1}U^n$. Therefore if (Y,S) is itself asymptotically *per*-expansive, one may choose $\mathcal{V}$ so that the number n-periodic points for S in any V^n is less than $e^{\epsilon n/2}$. One concludes that there at most $e^{\epsilon n}$ n-periodic points of (X,T) in U^n. This proves (X,T) is asymptotically *per*-expansive.

6.4. **ϵ-injective essentially continuous map $\Rightarrow$ small boundary property.** In Krieger Theorem for topological dynamical systems (X,T) recalled in Theorem 1.2, the existence of a topological embedding from X to a shift space with finite alphabet forces the space X to be zero-dimensional. Analogously we have now :

Proposition 6.1. *Let (X,T) be a topological dynamical system and let Y be a zero-dimensional topological space. Assume that for any $\epsilon > 0$ there exists an essentially continuous ϵ-injective map from X to Y. Then (X,T) has the small boundary property.*

Proof. Let (X,T) be a topological dynamical system and let Y be a zero-dimensional space. Assume that for any $\epsilon > 0$ there exists an ϵ-injective essentially continuous map ψ. Let $\delta > 0$ be such that for any set Y with diameter less than δ the set $\psi^{-1}Y$ has diameter less than ϵ. We let $\mathcal{B}$ be a basis of clopen sets such that $\psi^{-1}B$ has a small boundary for any $B \in \mathcal{B}$. Consider then a finite clopen partition P of Y with diameter less than δ such that any element of P is a finite intersection of elements of $\mathcal{B}$. In particular, for any $A \in P$, the set $\psi^{-1}A$ is a finite intersection of small boundary sets and thus it has itself small boundary. As the diameter of $\psi^{-1}A$ is less than ϵ, it concludes the proof of Proposition 6.1. □

Appendix A. Appendix : Equality in Krieger topological embedding problem

Proposition A.1. *Let (X,T) be a topological dynamical system with $h_{top}(T) = \log K$. Assume also there is a topological equivariant embedding $\psi : X \to \{1,...,K\}^{\mathbb{Z}}$. Then ψ is a topological conjugacy.*

Proof. It is enough to prove that ψ is onto. Let P be the clopen cover of X given by $P := \psi^{-1}P_0$. Then for $\delta > 0$ and $n \in \mathbb{N}$, any (δ, n) separated set E in $A^n \in P^n$ has cardinality bounded by a constant. Indeed there is $\delta' > 0$ depending only on δ and ψ such that $\psi(E)$ is (δ', n) separated in $\psi(A^n) \in P_0^n$. In particular $h_{top}(T) = h_{top}(T,P) := \inf_n \frac{\log \sharp P^n}{n}$. But as $\sharp P = K$ and $h_{top}(T) = \log K$ we have $\sharp P^n = K^n$ for all $n \in \mathbb{N}$. Fix $y \in \{1,...,K\}^{\mathbb{Z}}$ and let us show there exists $x \in X$ with $\psi(x) = y$. Let $n \in \mathbb{N}$. As $\sharp P^{2n+1} = K^{2n+1}$ there is $x_n \in X$ such that the k^{th} coordinates of $\psi(x_n)$ with $|k| \leq n$ coincide with those of y. Then if x is an accumulation point of $(x_n)_n$ we get $\psi(x) = y$ by continuity of ψ. □

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