Entropy of physical measures for $\mathcal{C}^\infty$ systems

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Paris, November 26, 2019

X a Borel space and $f : X \circlearrowleft$ measurable,
$\pi : V \to X$ a measurable vector fiber bundle over X equipped with a measurable Riemannian metric $\| \cdot \|_x$ on each fiber $V_x := \pi^{-1}x$,
$F : V \circlearrowleft$ a measurable vector bundle morphism with $\pi \circ F = f \circ \pi$.

Definition (Pointwise Lyapunov exponent)

For $x \in X$ and $v \in V_x$ we let

$$\chi(x, v) := \limsup_{n \to +\infty} \frac{1}{n} \log \|F^n(v)\|_{f^n x}.$$

For a fixed x, the set $\{v,\ \chi(x, v) \leq \lambda\}$ is a vector subspace nondecreasing in $\lambda \in \mathbb{R} \cup \{\pm\infty\}$.

Definition (Lyapunov flag)

There exist

- $r = r(x) \in \mathbb{N} \setminus \{0\}$,
- *vector spaces* $V_x = V_1(x) \supsetneq \cdots \supsetneq V_r(x) \neq 0$,
- $+\infty \geq \beta_1(x) > \cdots \beta_r(x) \geq -\infty$

$$s.t.\ \forall v \in V_i(x) \setminus V_{i+1}(x),\ \chi(x,v) = \beta_i(x).$$

The maps r, β_i, V_i *are measurable. Moreover* r *and* β_i *are* f*-invariant.*

Let us denote by $\chi_j(x) \in \{\beta_i(x),\ i = 1, \cdots, r\}$ for $j = 1, \cdots, \dim(V)$ the Lyapunov exponents at x counted nonincreasingly with multiplicity, i.e.

$$\sharp\{j,\ \chi_j = \beta_i\} = \dim(V_i) - \dim(V_{i+1}).$$

χ^k the top Lyapunov exponent of $\Lambda^k F : \Lambda^k V \circlearrowleft$,
$\chi_\Lambda = \max_k \chi^k$ of $\Lambda F : \Lambda V = \oplus_k \Lambda^k V \circlearrowleft$.

Definition

x is said **Lyapunov regular** *when* $\chi^k(x) = \sum_{j=1}^k \chi_j(x)$ *for all* k*.*

By Oseledets theorem the set of Lyapunov regular points has full measure for any f-invariant probability measure when $\int \log^+ |||F_x||| d\mu(x) < +\infty$.

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Remark : $\chi_\Lambda^+(x) = \sum_j \chi_j^+(x)$ for x Lyapunov regular.

Empirical exponent

(X, d) compact metric, $f : X \circlearrowleft$ continuous,
$(\mathcal{M}, \mathfrak{d})$ compact set of f-invariant probas,
$\mathcal{M}_e$ the subset of ergodic measures.

Definition

For $x \in X$, we let $p\omega(x)$ be the compact subset of $\mathcal{M}$ consisting of accumulation points of the sequence of empirical measures

$$(\mu_n^x)_n := \left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x}\right)_{n\in\mathbb{N}}.$$

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$(\mathcal{KM}, \mathfrak{d}^{Hau})$ compact set of nonempty closed subsets of $\mathcal{M}$.

<u>Remark :</u> $p\omega : x \mapsto p\omega(x)$ from X to $\mathcal{KM}$ is Borel measurable.

$F : V \circlearrowleft$ a **continuous** Riemannian bundle morphism over $f : X \circlearrowleft$.

$$\forall x \in X \, \forall p \in \mathbb{N}, \quad \lambda_p(x) := \limsup_n \frac{1}{n} \sum_{l=0}^{n-1} \log^+ \|F^p\|_{f^l x}.$$

The sequence $(\lambda_p(x))_p$ being subadditive, we may define the *empirical exponent* $\lambda(x)$ *at* x *as*

$$\lambda(x) := \lim_p \frac{\lambda_p(x)}{p} \geq \chi_1^+(x).$$

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Lemma

$$\lambda(x) = \max_{\mu \in p\omega(x)} \int \chi_1^+ \, d\mu.$$

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Definition

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*x is said $p\omega$-**regular** when $\lambda_\Lambda(x) = \chi^+_\Lambda(x)$.*

The set of $p\omega$-regular points have full μ-measure for any $\mu \in \mathcal{M}_e$.

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Any point in the bassin of a hyperbolic attractor is $p\omega$-regular w.r.t. the derivative cocycle.

Statements

M compact manifold, $f : M \circlearrowleft$ a $\mathcal{C}^1$ map,
Exponent w.r.t. derivative cocycle df on TM.

Theorem (Ruelle)

f $\mathcal{C}^1$,

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For a $\mathcal{C}^\infty$ system the converse inequality holds *physically* :

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<u>Remark :</u> Ruelle's inequality may be restated for f $\mathcal{C}^1$ as

$$\forall x \in M,\ \sup_{\mu \in p\omega(x)} h(\mu) \leq \lambda_\Lambda(x).$$

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However we have for f $\mathcal{C}^r$ with $r > 1$,

$$\text{for } \mathrm{Leb} \text{ a.e. } x, \quad \sup_{\mu \in p\omega(x)} h(\mu) \geq \chi_\Lambda^+(x) + \frac{\lambda_\Lambda(x) - \chi_\lambda^+(x)}{r-1} + \frac{d}{r}\lambda(x).$$

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We conjecture

$$\text{for Leb a.e. } x, \quad \sup_{\mu \in p\omega(x)} h(\mu) \geq \chi_\Lambda^+(x) + \frac{\lambda_\Lambda(x) - \chi_\lambda^+(x)}{r-1}.$$

$B_\mu := \{x,\ p\omega(x) = \{\mu\}\}$ basin of μ,
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Corollary

Let μ be a physical measure of a $\mathcal{C}^\infty$ system. Then

$$h(\mu) \geq \overline{\chi_\Lambda|_{B_\mu}},$$

with $\overline{\chi_\Lambda|_{B_\mu}}$ the essential supremum of χ_Λ on B_μ.

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The Corollary does not hold true anymore for f $\mathcal{C}^r$ with $r < +\infty$.

For the $\mathcal{C}^\infty$ Bowen eight's attractor, we get $\chi_1(x) = 0$ for Leb-a.e. x in the eyes.

$f : M \circlearrowleft$ a $\mathcal{C}^1$ map,
μ is SRB when $h(\mu) = \int \chi_\Lambda^+(x)\, d\mu(x) > 0$.

For $\mu \in \mathcal{M}_e$, we let
$$B'_\mu := \left\{ x \in B_\mu,\ \forall j\ \chi_j(x) = \int \chi_j \, d\mu \right\}.$$
Remark : any point in B'_μ is Lyapunov regular and $p\omega$-regular.

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Theorem (Tsujii)

Let $f : M \circlearrowleft$ be a $\mathcal{C}^{1+}$ diffeo. Assume the union of B'_μ over all ergodic hyp. measures of saddle type μ has positive Lebesgue measure. Then there exists an ergodic hyp. SRB measure.

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Corollary

Let $f : M \circlearrowleft$ be a $\mathcal{C}^\infty$ map. Assume the set of points x $p\omega$-regular w.r.t. Λdf with $\chi_\Lambda(x) > 0$ has positive Lebesgue measure. Then there exists a (ergodic) SRB measure.

For surface diffeos the converse also holds true. The Corollary for f $\mathcal{C}^{1+}$ would follow from the above Conjecture.

Bounded distortion

$f : M \circlearrowleft \mathcal{C}^1$,

Definition

For $n \in \mathbb{N}$ a $\mathcal{C}^1$ disc D_n is said to have n-bounded distorsion when

$$\forall x, y \in D_n, \; \frac{|\operatorname{Jac}(d_x f^n|_{T_x D_n})|}{|\operatorname{Jac}(d_y f^n|_{T_y D_n})|} < 2.$$

For $x \in M$, $n \in \mathbb{N}$, $\epsilon > 0$ and a partition P of M, let $B_n(x, \epsilon)$ the dynamical ball

$$B_n(x, \epsilon) := \{y \in M, \; d(f^k x, f^k y) < \epsilon \; \forall 0 \leq k < n\}$$

and let P_x^n the element of $P^n = \bigvee_{k=0}^{n-1} f^{-k} P$ containing $x \in M$. When $\operatorname{diam}(P) < \epsilon$ we have $P_x^n \subset B_n(x, \epsilon)$ for all x.

Theorem (Folklore)

Let f be C^1 uniformly hyperbolic and D a local unstable disc. There exists a scale $\epsilon > 0$ such that for all n and for all $x \in X$ there is an unstable subdisc of D_n^x of D satisfying :

- $B_n(x, \epsilon) \cap D \subset D_n^x$,
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Proof of bounded distortion property :

- there is $0 < \lambda < 1$ with $\mathrm{diam}(D_n^x) < \epsilon\lambda^n$,
- $y \mapsto \mathrm{Jac}(d_y f|_{E_u})$ is Hölder.

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Geometrical method for SRB measures in UH systems

Building SRB measures for a $\mathcal{C}^{1+}$ uniformly hyperbolic system f :

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2. Bounded distortion : P be a partition with $\mathrm{diam}(P) < \epsilon$, then for all x we have

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\begin{aligned}
m(P_x^n) &\leq \frac{\mathrm{vol}(D_n^x)}{\mathrm{vol}(D)}, \\
&\leq \frac{2\,\mathrm{vol}(f^n D_n^x)}{\mathrm{vol}(D) \cdot \mathrm{Jac}(d_x f^n|_{E_u})}, \\
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3. Entropy computation : for $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f^k m$ and $\mu = \lim_k \mu_{n_k}$ with $\mu(\partial P) = 0$, we have

$$h(\mu, P) \geq \limsup_k -\frac{1}{n_k}\int \log m(P_x^{n_k})\, dm(x).$$

We get

$$
\begin{array}{rcl}
-\dfrac{1}{n_k}\displaystyle\int \log m(P_x^{n_k})\, dm(x) & \geq & \dfrac{1}{n_k}\displaystyle\int \log \mathrm{Jac}(df^{n_k}|_{E_u})\, dm, \\
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then by letting $k \to +\infty$

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h(\mu) & \geq & \displaystyle\int \chi_\Lambda^+\, d\mu.
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Reparametrization Lemma (Yomdin,)

Let $f : M \circlearrowleft$ be a $\mathcal{C}^\infty$ map and D a $\mathcal{C}^\infty$ k-disc.
For all $\gamma > 0$, there exists a scale $\epsilon > 0$ such that for all n large enough and for all $x \in X$ there is a finite family $\mathcal{F}_n^x$ of $\mathcal{C}^\infty$ k-subdiscs of D with $\sharp\mathcal{F}_n^x \leq e^{\gamma n}$ satisfying :

- $B_n(x,\epsilon) \cap D \subset \bigcup_{D_n \in \mathcal{F}_n} D_n$,
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- D_n *has n-bounded distorsion for any* $D_n \in \mathcal{F}_n^x$.

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Theorem (Misurewicz, Downarowicz-Newhouse,Buzzi)

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$$\forall \mu \in \mathcal{M}, \ \limsup_{\nu \to \mu} h(\nu) \leq h(\mu) + \frac{\dim(M)}{r} \int \chi_1^+ \, d\mu.$$

Essential domain and image

X, Y metric spaces, Y separable,
$\phi : X \to Y$ Borel measurable,
m Borel measure on X.

Definition (Essential image/domain)

$$\overline{\mathrm{Im}}_\phi(m) := \{y \in Y,\ \forall U \in \mathcal{V}(y)\ m(\phi^{-1}U) > 0\}$$

$$\overline{\mathrm{Dom}}_\phi(m) := \left\{x \in X,\ \phi(x) \in \overline{\mathrm{Im}}_\phi(m)\right\}.$$

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The essential image is the smallest closed subset K of Y for which $\phi(x) \in K$ for m-a.e. x. For $Y = \mathbb{R}$ the essential supremum is $\overline{\phi} = \sup\left(\overline{\mathrm{Im}}_\phi(m)\right)$. The essential domain has full m-measure.

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In the following, for $f : X \circlearrowleft$ continuous we consider $\phi = p\omega : X \to \mathcal{KM}$. When $X = M$ is a Riemannian manifold and m the Lebesgue measure then $\overline{\mathrm{Im}}_{p\omega}(m)$ is the set of physical-like measures as defined by Enrich and Catsigeras.

Key Proposition

Fix $a < \overline{\chi^k}$ and let $\mathrm{Leb}_a = \mathrm{Leb}|_{\{\chi^k > a\}}$.

Proposition

For $x \in \overline{\mathrm{Dom}}_{p\omega}(\mathrm{Leb}_a)$, there exists $\mu \in p\omega(x)$ s.t.

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Proof of Main Theorem : For $(a_l)_{l\in\mathbb{N}}$ dense in $\mathbb{R}^+$ we let

$$F_l = \{\chi^k \leq a_l\} \cup \overline{\mathrm{Dom}}_{p\omega}(\mathrm{Leb}_{a_l}).$$

Then $\mathrm{Leb}(\bigcap_l F_l) = 1$. Moreover for $x \in \bigcap_l F_l$ and for a_l with $\chi^k(x) > a_l$, there is $\mu_l \in p\omega(x)$ with $h(\mu_l) \geq a_l$.

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By u.s.c. of the entropy, any accumulation point $\mu \in p\omega(x)$ of $(\mu_{l_n})_n$ with $a_{l_n} \xrightarrow{\nearrow n} \chi^k(x)$ satisfies

$$h(\mu) \geq \limsup_n h(\mu_{l_n}) \geq \lim_n a_{l_n} = \chi^k(x).$$

Main lines of the proof

$x_* \in \overline{\mathrm{Dom}}_{p\omega}(\mathrm{Leb}_a)$ fixed,

1. NU Expanding disc : D a $\mathcal{C}^\infty$ embedded k-disc and $E \subset D$ with $\mathrm{Leb}_D(E) > 0$ s.t. for any $y \in E$
   - $\chi^k(y, \iota(T_y D)) > a$ with ι the Plücker embedding,
   - $\mathfrak{d}^{Hau}(p\omega(y), p\omega(x_*)) \ll 1$.

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2. Borel-Cantelli argument : for $n \in J \subset \mathbb{N}$ with $\sharp J = \infty$, subset E_n of E with $\frac{|\log \mathrm{Leb}_D(E_n)|}{n} \ll 1$ s.t. for any $y \in E_n$
   - $|\mathrm{Jac}(d_y f^n|_{T_y D})| \geq e^{na}$,
   - $\mathfrak{d}(\mu_n^y, p\omega(x_*)) \ll 1$,

   m_n proba induced on E_n by Leb_D,

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   - $\mathfrak{d}^{Hau}(p\omega(y), p\omega(x_*)) \ll 1$.
2. Borel-Cantelli argument : for $n \in J \subset \mathbb{N}$ with $\sharp J = \infty$, subset E_n of E with $\frac{|\log \mathrm{Leb}_D(E_n)|}{n} \ll 1$ s.t. for any $y \in E_n$
   - $|\,\mathrm{Jac}(d_y f^n|_{T_y D})| \geq e^{na}$,
   - $\mathfrak{d}(\mu_n^y, p\omega(x_*)) \ll 1$,

   m_n proba induced on E_n by Leb_D,
3. Bounded distortion for $\mathrm{Jac}(df|_{TD})$: P partition with $\mathrm{diam}(P) < \epsilon$ for $\gamma \ll 1$ as in Reparametrization Lemma,

Main lines of the proof

$x_* \in \overline{\mathrm{Dom}}_{p\omega}(\mathrm{Leb}_a)$ fixed,

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4. Entropy computation : for $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f^k m_n$ and $\mu = \lim_k \mu_{n_k}$ with $\mu(\partial P) = 0$, we have
$$\mathfrak{d}\left(\mu, p\omega(x_*)\right) \ll 1 \text{ and}$$
$$h(\mu, P) \gtrsim a.$$

We conclude by u.s.c. of the metric entropy on $\mathcal{M}$.

Step 1. Choice of D and E

$x_* \in \overline{\mathrm{Dom}}_{p\omega}(\mathrm{Leb}_a)$, i.e. $p\omega(x_*) \in \overline{\mathrm{Im}}_{p\omega}(\mathrm{Leb}_a)$, thus $\forall \eta > 0$,

$$E' = \{y,\ \chi^k(y) > a \text{ and } \mathfrak{d}^{Hau}(p\omega(y), p\omega(x_*)) < \eta\}$$

satisfies $\mathrm{Leb}(E') > 0$. By reducing E' the Lyapunov space V_2 w.r.t. $\Lambda^k df$ is continuous on E'.

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Since $\mathrm{Leb}(U \cap E') > 0$ one may take by Fubini $D \subset H_y$ with $\mathrm{Leb}_D(E') > 0$ for some $y \in U \cap E'$. Put $E = E' \cap D$.

Step 2 : Borel-Cantelli argument

Take

$$E_n := \{y \in E,\ |\mathrm{Jac}(d_y f^n|_{T_y D})| \geq e^{na} \text{ and } \mathfrak{d}(\mu_n^y, p\omega(x_*)) \leq \eta\}.$$

For $\gamma > 0$ small error term, let us show that

$$\exists J \subset \mathbb{N} \text{ with } |J| = +\infty \text{ s.t.}$$

$$\mathrm{Leb}_D(E_n) > e^{-n\gamma}.$$

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Argue by contradiction :

$$[\forall n \text{ large } \ \mathrm{Leb}_D(E_n) < e^{-n\gamma}]$$
$$\Rightarrow [\mathrm{Leb}_D(\limsup_n E_n) = 0.]$$

But $\limsup_n E_n \supset E$ and $\mathrm{Leb}_D(E) > 0$...

$\epsilon > 0$ given by Reparametrization Lemma w.r.t. small error term γ,

P be a finite partition with $\mathrm{diam}(P) < \epsilon$.

For all n large and all $x \in X$, let $\mathcal{F}_n^x$ be a family of $\mathcal{C}^\infty$ k-discs with $\sharp\mathcal{F}_n^x \leq e^{\gamma n}$ s.t.

- $P_x^n \subset B_n(x,\epsilon) \cap D \subset \bigcup_{D_n \in \mathcal{F}_n^x} D_n$,
- $\mathrm{vol}(f^n D_n) < 1$ for any $D_n \in \mathcal{F}_n^x$,
- D_n has n-bounded distorsion for any $D_n \in \mathcal{F}_n^x$.

m_n proba induced on E_n by Leb_D for $n \in J$,

P as above, $\mathcal{G}_n^x = \{D_n \in \mathcal{F}_n^x,\ D_n \cap E_n \neq \emptyset\}$ for $x \in M$.

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$$\begin{aligned}
m_n(P_x^n) &\leq \frac{1}{\mathrm{Leb}_D(E_n)} \sum_{D_n \in \mathcal{G}_n^x} \frac{\mathrm{vol}(D_n)}{\mathrm{vol}(D)}, \\
&\leq \frac{1}{\mathrm{Leb}_D(E_n)} \sum_{D_n \in \mathcal{G}_n^x} \frac{2e^{-na}\mathrm{vol}(f^n D_n)}{\mathrm{vol}(D)}, \\
m_n(P_x^n) &\leq \frac{e^{-na}\sharp\mathcal{G}_n^x}{\mathrm{Leb}_D(E_n)} \leq e^{-n(a-2\gamma)}.
\end{aligned}$$

Step 4 : Entropy computation

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P as above, $\mathcal{G}_n^x = \{D_n \in \mathcal{F}_n^x,\ D_n \cap E_n \neq \emptyset\}$ for $x \in M$.

$$
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m_n(P_x^n) &\leq \frac{e^{-na}\sharp\mathcal{G}_n^x}{\mathrm{Leb}_D(E_n)} \leq e^{-n(a-2\gamma)}.
\end{aligned}
$$

Therefore for $\mu = \lim_k \mu_{n_k}$ with $\mu(\partial P) = 0$:

$$
h(\mu, P) \geq \liminf_{n \in J} -\frac{1}{n} \int \log m_n(P_x^n)\, dm_n(x) \gtrsim a.
$$

Counterexample in finite smoothness

Theorem (Downarowicz-Newhouse)

Let f be a $\mathcal{C}^r$ surface diffeomorphism for $1 \leq r < +\infty$ with homoclinic tangency at a saddle hyp. fixed point p.

Then there exists g arbitrarily close to f admitting a sequence of horseshoes $(H_n)_n$ with $\lim_n h_{top}(H_n) = \frac{\min_i |\chi_i(p_g)|}{r}$ s.t.

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- $\sup_{x \in H_n} \left| \chi_1(x) - \frac{\chi_1(p_g)}{r} \right| \xrightarrow{n} 0$,
- $p\omega(H_n) \xrightarrow{n} \{\delta_{p_g}\}$,
- $HD^u(H_n) \xrightarrow{n} 1$.

Theorem (under preparation)

Let f be a $\mathcal{C}^r$ surface diffeomorphism for $1 \leq r < +\infty$ with a homoclinic tangency at a dissipative saddle hyp. fixed point p.

Then there exists g arbitrarily $\mathcal{C}^r$-close to f with $E \subset M$ s.t.

- $\forall x \in E, \chi_1(x) = \frac{\chi_1(p_g)}{r}$,
- $p\omega(E) = \{\delta_{p_g}\}$,
- $\mathrm{Leb}(E) > 0$.

Thank you for your attention !