PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 3, March 2014, Pages 957–968
S 0002-9939(2013)12067-8
Article electronically published on December 23, 2013

EXISTENCE OF MEASURES OF MAXIMAL ENTROPY FOR $\mathcal{C}^r$ INTERVAL MAPS

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(Communicated by Nimish Shah)

ABSTRACT. We show that a $\mathcal{C}^r$ $(r > 1)$ map of the interval $f : [0, 1] \to [0, 1]$ with topological entropy larger than $\frac{\log \|f'\|_\infty}{r}$ admits at least one measure of maximal entropy. Moreover the number of measures of maximal entropy is finite. It is a sharp improvement of the 2006 paper of Buzzi and Ruette in the case of $\mathcal{C}^r$ maps and solves a conjecture of J. Buzzi stated in his 1995 thesis. The proof uses a variation of a theorem of isomorphism due to J. Buzzi between the interval map and the Markovian shift associated to the Buzzi-Hofbauer diagram.

1. INTRODUCTION

Entropy is an important invariant of conjugacy which estimates the dynamical complexity of a system by counting the number of distinguishable orbits. This can be done at a topological or a measure theoretical level which are related by a variational principle: the topological entropy is the supremum of the entropy of invariant probability measures. Measures which realize the supremum are remarkable, as they reflect all the complexity of the system from the point of view of entropy. In this paper we discuss the existence of such measures for $\mathcal{C}^r$ interval maps with $r > 1$.

We first recall the definition of entropy for compact topological dynamical systems. Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map.

Let $S \subset X$, $\delta > 0$ and $n \in \mathbb{N}$. A subset E of S is an (n, δ) separated set of S if for all $x, y \in E$ with $x \neq y$ there exists $0 \leq k < n$ with $d(f^k x, f^k y) > \delta$. We denote by $s(n, \delta, S)$ the maximal cardinality of an (n, δ) separated set of S. A subset F of X is an (n, δ) covering set of S if for all $x \in S$ there exists $y \in F$ with $d(f^k x, f^k y) < \delta$ for all $0 \leq k < n$. We denote by $r(n, \delta, S)$ the minimal cardinality of an (n, δ) covering set of S. The topological entropy of f denoted by $h_{top}(f)$ is defined as follows:

$$h_{top}(f) := \lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log s(n, \delta, X).$$

If μ is an invariant ergodic probability measure we define the measure theoretical entropy of μ by counting the orbits in the sets of μ-measure larger than $\lambda \in]0, 1[$:

$$h_\lambda(f, \mu, \delta) := \limsup_{n \to +\infty} \frac{1}{n} \log \inf_{\mu(Y) > \lambda} s(n, \delta, Y)$$

Received by the editors April 24, 2012.
2010 *Mathematics Subject Classification.* Primary 37E05, 37A35.

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and then for any $\lambda \in]0,1[$ (the limit below does not depend on λ by ergodicity):

$$h(f,\mu) := \lim_{\delta \to 0} h_\lambda(f,\mu,\delta).$$

For general invariant measures ν one can define the entropy $h(f,\nu)$ of ν as $h(f,\nu) := \int h(f,\mu) dM_\nu(\mu)$ where $\nu = \int \mu dM_\nu(\mu)$ is the ergodic decomposition of ν. This definition is due to Katok [11]. Observe that one can use covering sets instead of separated sets in the above definitions, as we always have $r(n,\delta,S) \leq s(n,\delta,S) \leq r(n,\delta/2,S)$ [18].

The topological entropy and the measure theoretical entropy are related by the well known variational principle [12], [13]:

$$h_{top}(f) = \sup_{\mu \in M(X,f)} h(f,\mu) = \sup_{\mu \in M_e(X,f)} h(f,\mu)$$

where $M(X,f)$ is the set of f-invariant probability measures and $M_e(X,f) \subset M(X,f)$ the subset of ergodic mesures.

A measure $\mu \in M(X,f)$ is said to be maximal if $h(\mu) = h_{top}(f)$. Such measures do not always exist and the number of ergodic maximal measures may be infinite. Under assumptions of expansiveness we can ensure the existence of maximal measures. By using Y. Yomdin's theory, S. Newhouse showed that if f is a $\mathcal{C}^\infty$ map defined on a smooth compact manifold X, then the entropy function h is upper semicontinuous on the compact set $M(X,f)$ endowed with the weak-star topology and therefore f admits a maximal measure [16]. Moreover there exist counter-examples to the existence of maximal measures and to their finitude in the case of $\mathcal{C}^r$ maps for any finite $r > 0$ (see [6] for interval maps and [15] for diffeomorphisms on a compact manifold of dimension larger than or equal to 4).

Main Theorem. *Let $r > 1$ and $f : [0,1] \to [0,1]$ be a $\mathcal{C}^r$ map, such that $h_{top}(f) > \frac{\log \|f'\|_\infty}{r}$. Then f admits a maximal measure. Moreover the number of ergodic maximal measures is finite.*

This result is sharp according to the examples of [6], where the topological entropy is precisely equal to $\frac{\log \|f'\|_\infty}{r}$.

In [7], J. Buzzi and S. Ruette have shown the same result under the stronger assumption $h_{top}(f) > \frac{2\log \|f'\|_\infty}{r}$. We essentially combine the strategy of their proof with an estimate of the number of monotone branches with big derivatives used in [10]. Following the isomorphism theorems of [5] and [8] we prove that the Markov shift given by the Buzzi-Hofbauer diagram preserves the entropy of invariant measures with positive entropy. In particular f admits a maximal measure if and only if the Markov shift does. The works of B. M. Gurevič and S. Savchenko then give some properties of Markov shifts that do not have maximal measures. These properties allow us to bound the topological entropy of f from above by $\frac{\log \|f'\|_\infty}{r}$ by shadowing large pieces of typical orbits with critical points.

To prove the existence of maximal measures one uses in most known examples an argument of upper semicontinuity of the entropy function. Here it is not the case as the entropy of a $\mathcal{C}^r$ map may not be upper semicontinuous, even at large entropy measures. However we show that a large defect of upper semicontinuity only occurs at small entropy measures.

The definitions of the Buzzi-Hofbauer diagram and the associated Markov shift together with known isomorphism theorems are recalled in Section 2. In Section 3

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we introduce a property of shadowing for invariant measures that we use in the proof of the Main Theorem presented in Section 4. The last section deals with the defect of upper semicontinuity of the entropy function.

2. Markov shift

We recall in this section the construction of the Buzzi-Hofbauer Markov diagram for $\mathcal{C}^1$ interval maps. We first recall the definitions and some properties of the Markov shift given by an oriented graph and the symbolic dynamic associated to the natural partition in monotone branches.

2.1. **Markov shift on oriented graphs.** Let $\mathcal{G}$ be a countable oriented graph. Let $u, v \in \mathcal{G}$. We use the notation $u \to v$ when there is an oriented arrow from u to v. Consider $\Sigma(\mathcal{G}) := \{(v_n)_n \in \mathcal{G}^{\mathbb{Z}},\ \forall n \in \mathbb{Z}, v_n \to v_{n+1}\}$. The Markov shift on $\mathcal{G}$ is the shift $\sigma((v_n)_n) = (v_{n+1})_n$ on $\Sigma(\mathcal{G})$. Remark that $\Sigma(\mathcal{G})$ is a priori not compact. We define the entropy $h(\mathcal{G})$ as the supremum of $h(\sigma, \xi)$ over all σ-invariant probability measures ξ (we refer to [18] for entropy in the general context of measure preserving systems). A σ-invariant measure ξ is said to be maximal if $h(\sigma, \xi) = h(\mathcal{G})$. If F is a finite set of vertices we write $[F] := \{(v_n)_n \in \Sigma(\mathcal{G}),\ v_0 \in F\}$.

The proposition below follows from results of Gurevich and Savchenko [14]:

Proposition 1 ([7])**.** *Let $\mathcal{G}$ be a countable oriented graph with entropy $0 < h(\mathcal{G}) < +\infty$. Assume that $\Sigma(\mathcal{G})$ does not admit maximal measure or that $\Sigma(\mathcal{G})$ admits an infinite number of maximal ergodic measures.*

Then there exists a sequence of ergodic σ-invariant measures $(\xi_m)_m$ such that:

- $\lim_m h(\sigma, \xi_m) = h(\mathcal{G})$;
- *for all finite set of vertices F,* $\lim_m \xi_m([F]) = 0$.

2.2. **Symbolic dynamics associated to the natural partition.** We call a monotone branch (with respect to f) any subinterval J of $[0,1]$ such that the interval map f is monotone on J. Let C be the critical set of f, i.e. the set of points which do not belong to the interior of any monotone branch of f. It is a compact subset of the set of vanishing points of the derivative f'. Let us denote by P the set of connected components of $[0,1] \setminus C$. The two-sided **symbolic dynamic** $\Sigma(f, P)$ associated to f is defined as the shift on the closure in $P^{\mathbb{Z}}$ (for the product topology) of the two-sided sequences $A = (A_n)$ such that for all $n \in \mathbb{Z}$ and $l \in \mathbb{N}$ the word $A_n \dots A_{n+l}$ is **admissible**, i.e. $\bigcap_{l=0}^{k} f^{-l} A_{n+l} \neq \emptyset$. The **follower set** of a finite P-word $B_n \dots B_{n+l}$ is $fol(B_n \dots B_{n+l}) := \{A_{n+l} A_{n+l+1} \dots \in P^{\mathbb{N}}$, s.t. $\exists (A_n) \in \Sigma(f, P)$ with $A_n \dots A_{n+l} = B_n \dots B_{n+l}\}$. Let $N := \{A \in \Sigma(f, P)\, \forall n_0\ \exists n \geq n_0,\ fol(A_{-n} \dots A_0) \neq fol(A_{-n_0} \dots A_0)\}$. Then the **Markovian subshift** $\Sigma_M(f, P)$ of $\Sigma(f, P)$ is defined as follows:

$$\Sigma(f, P) \setminus \Sigma_M(f, P) := \bigcup_{p \in \mathbb{Z}} \sigma^p N.$$

By an argument of ergodicity [6] one can see that any ergodic invariant probability measure ν on $\Sigma(f, P)$ with $\nu(\Sigma_M(f, P)) \neq 1$ satisfies $\nu(N) = 1$.

The symbolic dynamics extends the dynamic on the interval as follows. For any $A = (A_n) \in \Sigma(f, P)$ we let $\pi(A) := \bigcap_{k \in \mathbb{N}} \overline{\bigcap_{l=0}^{k} f^{-l} A_l}$. Since f is monotone on each element of P the set $\bigcap_{l=0}^{k} f^{-l} A_l$ is an interval for all $k \in \mathbb{N}$. In particular $\pi(A)$ is a point or a compact nontrivial interval, but this last possibility occurs only for

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a countable set of elements of $\Sigma(f,P)$. Therefore the application π extends to a Borelian map from $\Sigma(f,P)$ to the interval $[0,1]$. Now we recall Lemma 5.3 of [6]:

Proposition 2 ([6], [8])**.** *The projection $\pi : \Sigma(f,P) \to [0,1]$ induces a bijection π^* preserving entropy between ergodic invariant probability measures of $(\Sigma(f,P),\sigma)$ and $([0,1],f)$ with positive entropy.*

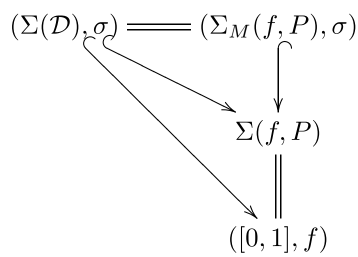
2.3. **Markov diagram and isomorphism theorem.** Let $\mathcal{P}$ be the set of admissible P-words. We endow $\mathcal{P}$ with the following equivalence relation. We say $A_{-n}\dots A_0 \sim B_{-m}\dots B_0$ if and only if there exists $0 \le k \le \min(m,n)$ such that:

- $A_{-k}\dots A_0 = B_{-k}\dots B_0$;
- $\bigcap_{i=0}^{n} f^i A_{-i} = \bigcap_{i=0}^{k} f^i A_{-i}$, i.e. $fol(A_{-n}\dots A_0) = fol(A_{-k}\dots A_0)$;
- $\bigcap_{i=0}^{n} f^i B_{-i} = \bigcap_{i=0}^{k} f^i B_{-i}$, i.e. $fol(B_{-m}\dots A_0) = fol(B_{-k}\dots A_0)$.

We endow $\mathcal{D} := \mathcal{P}/\sim$ with a structure of oriented graph in the following way [6]: there exists an oriented arrow $\alpha \to \beta$ between two elements α,β of $\mathcal{D}$ if and only if there exists an integer n and $A_{-n}\dots A_0A_1 \in \mathcal{P}$ such that $\alpha \sim A_{-n}\dots A_0$ and $\beta \sim A_{-n}\dots A_0A_1$. This graph is known as the Buzzi-Hofbauer diagram. We recall now the corresponding isomorphism theorem (Theorem 5.7 of [6]):

Theorem 1 ([6], [8])**.** *The map $\pi : \Sigma(\mathcal{D}) \to \Sigma(f,P)$ defined by $\pi((\alpha_n)_n) = B_n$, where B_n is the last letter of the word α_n, realizes a measurable conjugacy between $\Sigma(\mathcal{D})$ and $\Sigma_M(f,P)$. In particular any invariant measure ν on $\Sigma(f,P)$ with $\nu(\Sigma_M(f,P)) = 1$ lifts to a unique measure in $\Sigma(\mathcal{D})$ with the same entropy.*

To summarize we have the following diagram. All the semiconjugacies in the diagram preserve the entropy of measures. The sign $=$ means that the semiconjugacy is a bijection between ergodic invariant measures with positive entropy. In the whole paper all the semi-conjugacies will be denoted by π (without any confusion).



We let $\mathcal{D}_N$ be the subset of $\mathcal{D}$ generated by elements of $\bigcup_{k=1}^{N} P^k$, i.e. $\alpha \in \mathcal{D}_N$ if and only if there exists $0 \le k < N$ and $A_{-k}\dots A_0 \in \mathcal{P}$ such that $\alpha \sim A_{-k}\dots A_0$. We call **the significative part** of $\alpha \in \mathcal{D}$ the word $A_{-n_\alpha}\dots A_0$ where n_α is the smallest integer N such that $\alpha \in \mathcal{D}_N$ (in particular $fol(A_{-n_\alpha}\dots A_0) \neq fol(A_{-n_\alpha+1}\dots A_0)$ when $n_\alpha > 0$). We will also make use of the following lemma proved in [7]:

Lemma 1 ([7])**.** *Let $(\xi_m)_m$ be a sequence of σ-invariant ergodic measures on $\Sigma(\mathcal{D})$ such that:*

- *$h(\sigma,\xi_m) > 0$ for all m;*
- *for any finite set of vertices F of $\mathcal{D}$, $\lim_m \xi_m([F]) = 0$.*

Then for all $N \in \mathbb{N}$ we have $\lim_m \xi_m([\mathcal{D}_N]) = 0$.

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3. The shadowing property

We introduce in this section a notion of shadowing for invariant measures, which is convenient in the proof of the Main Theorem. For any $A \in \Sigma(f,P)$ and any $N \in \mathbb{N}$ we first define $r_N(A)$ as the least $n \in \mathbb{N} \cup \{+\infty\}$ with $n > N$ such that $fol(A_{-n-1}\dots A_0) \neq fol(A_{-n}\dots A_0)$. Observe that this last inequality implies that there exists a point y in ∂A_{-n-1} such that $f(y) \in \bigcap_{l=0}^n f^{-l}A_{-n+l}$: *the point y shadows the piece of orbit $A_{-n-1}\dots A_0$.*

An ergodic invariant measure of $\Sigma(f,P)$, respectively a sequence $(\nu_m)_m$ of ergodic invariant measures of $\Sigma(f,P)$, is said to satisfy the **shadowing property** when for all integers N,

$$\nu(r_N < +\infty) = 1,$$

$$\text{respectively } \lim_m \nu_m(r_N < +\infty) = 1.$$

The next two lemmas obviously follow from the definitions of $\mathcal{D}_N$ and $\Sigma_M(f,P)$. The proofs are left to the reader.

Lemma 2. *Let ν be an ergodic σ-invariant measure on $\Sigma(f,P)$ such that $\nu(\Sigma_M(f,P)) = 0$. Then ν satisfies the shadowing property.*

Lemma 3. *Let $(\xi_m)_m$ be a sequence of ergodic σ-invariant measures of $\Sigma(\mathcal{D})$ such that $\lim_m \xi_m([\mathcal{D}_N]) = 0$ for all integers N. Then the induced sequence $(\nu_m)_m = (\pi^*\xi_m)_m$ on $\Sigma(f,P)$ satisfies the shadowing property.*

Let $\epsilon > 0$ and $N \in \mathbb{N}$ and let ν be an ergodic invariant probability measure on $\Sigma(f,P)$ with $\nu(\{r_N < +\infty\}) > 1-\epsilon$ and $h(\sigma,\nu) > 0$. For any such triple (ν, N, ϵ) we will associate a set of large ν-measure for which we manage to estimate the number of monotone branches intersecting its π-image in $[0,1]$. These estimates will then be used in the next section to prove the Main Theorem.

We choose N' large enough (depending on ν) such that $\nu(\{r_N < N'\}) > 1-\epsilon$, and by ergodicity we may consider a Borel set B of $\Sigma(f,P)$ with $\nu(B) > 1-\epsilon$ such that $\frac{1}{n}\sharp\left\{0 \le k \le n,\ \sigma^k A \in \{r_N < N'\}\right\} > 1-\epsilon$ for all A in B and for all large n. To any $A \in B$ and for any large n we associate disjoint intervals of integers $[a_i, b_i]$ for $i \in I := \{1,\dots,j\}$ such that:

- $[a_i, b_i] \subset [0, n[$ for all $i \in I$;
- $n_i := b_i - a_i > N$ for all $i \in I$;
- $k_n := \sharp[0,n[- \bigcup_{i\in I}[a_i,b_i] < N' + \epsilon n$;
- $fol(A_{a_i-1}A_{a_i}\dots A_{b_i}) \neq fol(A_{a_i}A_{a_i}\dots A_{b_i})$.

This is done as follows. We put $a_0 := n$ and we define by induction for $i \ge 1$:

$$b_i := \max\left\{k \in [0, a_{i-1}],\ \sigma^k A \in \{r_N < N'\}\right\},$$

$$a_i := b_i - r_N(\sigma^{b_i}A).$$

We stop the process at the last integer j satisfying $b_j - r_N(\sigma^{b_i}A) \ge 0$. Finally we put $b_{j+1} = 0$.

Let $\mu = \pi^*\nu \in M([0,1],f)$. We remark that $h(f,\mu) = h(\sigma,\nu) > 0$ by Proposition 2 and that $\mu(\pi(B)) > 1-\epsilon$. We will denote by $\lambda_\mu := \int \log|f'|(x)d\mu(x)$ the Lyapunov exponent of μ. We consider a (δ, n) separated set $F \subset \pi(B)$ with maximal cardinality. For any $x \in F$ we choose $A \in B$ with $x = \pi(A)$ and we let $a_i(x) = a_i(A), b_i(x) = b_i(A)$ and $k_n(x) = k_n(A)$.

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Claim. Up to dividing the cardinality of F by $e^{\phi(\epsilon,N)}$ for some function ϕ satisfying $\phi(\epsilon,N)\xrightarrow[]{\substack{\epsilon\to 0,\\ N\to+\infty}} 0$, one can assume $a_i(x)$, $b_i(x)$ independent of $x\in F$ and thus $k_n(x)$, but also $l_k(x) := [-\log^-|f'(f^kx)|]+1$ for $k\notin\bigcup_{i\in I}]a_i,b_i]$ (the symbol $[.]$ denotes the usual integer part of a real number and $\log^- = \min(\log,0)$ is the negative part of the logarithm).

Proof of the Claim. For points in a set of arbitrarily large μ-measure and large n we have by ergodicity

$$\frac{1}{n}\sum_{k\notin\bigcup_{i\in I}]a_i,b_i]}[-\log^-|f'(f^kx)]+1\leq S := -\int\log^-|f'|(x)d\mu(x)+2.$$

This last integral is finite, more precisely less than $\log^+\|f'\|_\infty$, because we have $\lambda_\mu=\int\log|f'|(x)d\mu(x)\geq h(f,\mu)>0$ by Ruelle inequality. Recall also that $k_n := \sharp[0,n[-\bigcup_{i\in I}[a_i,b_i]<N'+\epsilon n$. Therefore the number of possible sequences $(l_k(x))_k$ with $k\in[0,n[-\bigcup_{i\in I}[a_i,b_i]$ and for $x\in F$ is less than $\binom{nS}{k_n}$ and thus by Stirling's formula grows exponentially as $nSH(\frac{k_n}{nS})$ with $H(t)=-t\log t-(1-t)\log(1-t)$. Our Claim then follows from $\lim_{t\to 0}H(t)=0$ and $\limsup_n\frac{k_n}{nS}\leq\epsilon$. □

Now by Lemma 4.1 of [10] (see also Lemma 3 of [2] for an alternative proof) the number of elements of P, that is, the number of monotone branches where the absolute value of f' at some point exceeds e^{-l}, is less than $ce^{\frac{l}{r-1}}$ for any $l\in\mathbb{N}$, where c is some constant depending only on f. For any $k\notin\bigcup_{i\in I}]a_i,b_i]$, the set f^kF meets at most $2ce^{\frac{l_k}{r-1}}$ monotone branches for f^{L_k} with $L_k=n_i+1$ if $k=a_i$ for some $i\in I$ and $L_k=1$ if not and with l_k the common value of the function l_k in F, i.e. $l_k=l_k(x)$ for any $x\in F$. Indeed for any $x=\pi(A)\in F$ we have $fol(A_{a_i-1}A_{a_i}\dots A_{b_i})\neq fol(A_{a_i}A_{a_i}\dots A_{b_i})$, and thus there exists y in the boundary of A_{a_i} such that $f^{t-a_i}y\in A_t$ for $t=a_i,\dots,b_i$. Therefore $f^{a_i}x$ belongs to the monotone branch for f^{n_i+1} given by $A_{a_i},\dots,A_{b_i}$ if and only if so does one of the two boundary points of A_{a_i}. Finally recall that the number of possible intervals $A_{a_i}\in P$ is at most $ce^{\frac{l_{a_i}}{r-1}}$.

4. Proof of the Main Theorem

We will first prove that an ergodic measure with the shadowing property has zero entropy.

Proposition 3. *Let $f:[0,1]\to\mathbb{R}$ be a $\mathcal{C}^r$ interval map with $r>1$ and let $\Sigma(f,P)$ be the symbolic dynamics associated to the natural partition as defined in Section 2.2. For any ergodic σ-invariant probability measure ν on $\Sigma(f,P)$ satisfying the shadowing property we have*

$$h(f,\pi^*\nu)=0.$$

This last proposition improves, in the case of $\mathcal{C}^r$ maps with $r>1$, Theorem 6.1 of [5], which holds for $\mathcal{C}^1$ maps but asserts only that $h(f,\mu)\leq h_{top}(C)$. Together with Proposition 2 and Lemma 2 we get:

Corollary 1. *The projection $\pi:\Sigma_M(f,P)\to[0,1]$ induces a bijection between ergodic invariant measures of $(\Sigma_M(f,P),\sigma)$ and $([0,1],f)$ with positive entropy, which is entropy preserving.*

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Finally we asymptotically bound from above the entropy of a sequence of ergodic σ-invariant measures satisfying the shadowing property.

Proposition 4. *Let $(\nu_m)_m$ be a sequence of ergodic σ-invariant measures on $\Sigma(f,P)$ satisfying the shadowing property. Then we have for any weak limit $\mu := \lim_k \mu_{m_k}$ of $(\mu_m)_m$ with $\mu_m = \pi^*\nu_m$:*

$$\limsup_k h(f,\mu_{m_k}) \leq \frac{\int \log^+ |f'| d\mu}{r}.$$

Together with Theorem 1, Lemma 1, Lemma 3 and Corollary 1 we easily get as a consequence:

Corollary 2. *The projection $\pi : \Sigma(\mathcal{D}) \to [0,1]$ induces a bijection π^* preserving entropy between ergodic invariant measures of $(\Sigma(\mathcal{D}),\sigma)$ and $([0,1],f)$ with entropy larger than $\frac{\log \|f'\|_\infty}{r}$.*

We will prove Proposition 3 and Proposition 4 later. We first deduce the Main Theorem from these propositions.

Proof of the Main Theorem. We consider a $\mathcal{C}^r$ $(r > 1)$ interval map, $f : [0,1] \to [0,1]$, such that $h_{top}(f) > \frac{\log \|f'\|_\infty}{r}$. We argue by contradiction: assume f does not admit a maximal measure or admits an infinite number of ergodic maximal measures; then it also does for the Markov shift $(\Sigma(\mathcal{D}),\sigma)$ according to Corollary 2. Moreover $h_{top}(f) = h(G)$. Therefore by Proposition 1, there exists a sequence $(\xi_m)_m$ of ergodic σ-invariant measures on $\Sigma(\mathcal{D})$ such that $\lim_m h(\sigma,\xi_m) = h_{top}(f)$ and such that for any finite set of vertices F we have $\lim_m \xi_m([F]) = 0$. It follows from Proposition 1 that $\lim_m \xi_m([\mathcal{D}_N]) \to 0$ for all $N \in \mathbb{N}$, and the sequence $(\nu_m)_m = (\pi^*\xi_m)_m$ of induced measures on $\Sigma(f,P)$ therefore satisfies the shadowing property according to Lemma 3. Finally we conclude by Proposition 4 by writing $(\pi^*\nu_m)_m = (\mu_m)_m$ such that $\limsup_m h(f,\mu_m) = \limsup_m h(\sigma,\xi_m) \leq \frac{\log \|f'\|_\infty}{r}$, which contradicts the assumption $h_{top}(f) > \frac{\log \|f'\|_\infty}{r}$. □

We now proceed to the proofs of Proposition 3 and Proposition 4.

Proof of Proposition 3. We consider an ergodic σ-invariant measure ν on $\Sigma(f,P)$ with the shadowing property and $h(\sigma,\nu) > 0$. Let $\mu := \pi^*\nu$. Fix $\delta > 0$ and let us prove $h_{1/2}(f,\mu,\delta) < \delta$. As $\lambda_\mu \geq h(\mu) > 0$ the function $\log|f'|$ is integrable with respect to μ, and therefore one can choose a large integer K such that

$$-\frac{1}{r-1}\int_{|f'|<e^{-K}} (\log|f'| - 1)\, d\mu < \delta/4.$$

Let $P_K := \{A \in P,\ \sup_{x\in A} |f'(x)| \geq e^{-K}\}$. We now choose $\frac{1}{2} > \epsilon > 0$ and $N \in \mathbb{N}$ such that $\phi(\epsilon, N) < \delta/4$, $\left(\delta^{-1}c\sharp P_K\right)^\epsilon < e^{\delta/4}$ and $-x\log x < \delta/4$ for all $0 < x < 2\epsilon$. As ν satisfies the shadowing property we have $\nu(\{r_N < +\infty\}) = 1$. Let F, B, (a_i),... be the data associated to the triple (ν, N, ϵ) as in the end of Section 3. Let $\mathcal{K}_n = \{k \notin \bigcup_{i\in I}]a_i, b_i], l_k > K\}$. By taking a smaller set B one can assume by ergodicity that $\frac{1}{n}\sum_{k\in\mathcal{K}_n} \frac{l_k}{r-1} < \delta/4$ for large enough n. Let $E_\delta = 2\delta\mathbb{N} \cap [0,1]$. For any $k \notin \bigcup_{i\in I}]a_i, b_i]$ the set $F_k := \bigcup_{\substack{0\leq l<L_k \\ (A_k,\dots,A_{k+L_k-1})}} f|_{A_{k+l}}^{-l} E_\delta$, where the union is over all monotone branches $\bigcap_{0\leq l<L_k} f^{-l}A_{k+l}$ for f^{L_k} intersecting f^kF, is a (δ, L_k)

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covering set of f^kF by a standard argument. Therefore F may be $(2\delta, n)$ covered by at most $\prod_k \sharp F_k$ points, and then we have for some constant C:

$$\begin{aligned} r(n,3\delta,\pi(B)) &\leq r(n,2\delta,F)\\ &\leq e^{\phi(\epsilon,N)n}\prod_k \sharp F_k\\ &\leq e^{\phi(\epsilon,N)n}\delta^{-k_n}\prod_k L_k \prod_{k\in\mathcal{K}_n} ce^{-\frac{l_k}{r-1}}\prod_{k\notin\mathcal{K}_n}\sharp P_K\\ &\leq Ce^{\phi(\epsilon,N)n}(\delta^{-1}c\sharp P_K)^{\epsilon n}e^{-\frac{\sum_{k\in\mathcal{K}_n} l_k}{r-1}}\prod_k L_k. \end{aligned}$$

By assumption the terms $e^{\phi(\epsilon,N)n}$, $(\delta^{-1}c\sharp P_K)^{\epsilon n}$ and $e^{-\frac{\sum_{k\in\mathcal{K}_n} l_k}{r-1}}$ are all less than $e^{\delta n/4}$. Now by the arithmetic mean-geometric mean inequality we have $\left(\prod_k L_k\right)^{1/k_n} \leq \frac{\sum_k L_k}{k_n} \leq \frac{n}{k_n}$, and therefore for large n we get by the choice of ϵ:

$$\frac{1}{n}\log\prod_k L_k \leq -\frac{k_n}{n}\log\left(\frac{k_n}{n}\right) < \delta.$$

By using Katok's definition of entropy recalled in the introduction we finally conclude that $h_{1/2}(f,\mu,\delta) < \delta$. □

Proof of Proposition 4. Let $\epsilon > 0$ and $N \in \mathbb{N}$. For m large enough we have $\nu_m(\{r_N < +\infty\}) > 1-\epsilon$. We can also clearly assume $h(\sigma,\nu_m) > 0$. Let F, B, (a_i),... be the data associated as above to the triple (ν_m, N, ϵ). The set F meets at most $c^{2(\epsilon n+N')}\prod_k e^{-\frac{l_k}{r-1}}$ monotone branches for f^n. As at most n/δ (n,δ)-separated points may lie in the same monotone branch for f^n for any $\delta > 0$, we have

$$\begin{aligned} s(n,\delta,\pi(B)) &= \sharp F\\ &\leq \frac{n}{\delta}e^{\phi(\epsilon,n)n}c^{2(\epsilon n+N')}\prod_k e^{-\frac{l_k}{r-1}}\\ &\leq \frac{n}{\delta}e^{\phi(\epsilon,n)n}c^{2(\epsilon n+N')}e^{\frac{\int -\log|f'|^-(x)d\mu_m(x)}{r-1}}. \end{aligned}$$

Then by using Katok's entropy formula we get up to changing $\phi(\epsilon,N)$:

$$\begin{aligned} h(f,\mu_m) &\leq \frac{\int -\log|f'|^-(x)d\mu_m(x)}{r-1}+\phi(\epsilon,N)\\ &\leq \frac{\int \log^+|f'|(x)d\mu_m(x)-\lambda_{\mu_m}}{r-1}+\phi(\epsilon,N). \end{aligned}$$

But we also have by Ruelle inequality $h(f,\mu_m) \leq \lambda_{\mu_m}$. By combining these two inequalities we get

$$\begin{aligned} h(f,\mu_m) &\leq \frac{r-1}{r}h(f,\mu_m)+\frac{1}{r}\lambda_{\mu_m}\\ &\leq \frac{\int \log^+|f'|(x)d\mu_m(x)}{r}+\phi(\epsilon,N). \end{aligned}$$

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Then by taking the lim sup in m and the limits in ϵ and N we get by upper semicontinuity of $\xi \mapsto \int \log^+ |f'|(x)d\xi(x)$:

$$\limsup_m h(f,\mu_m) \quad \leq \quad \frac{\int \log^+ |f'|(x)d\mu(x)}{r}.$$

□

5. Nonupper semicontinuity of the entropy function

This section deals with the properties of upper semicontinuity of the entropy function for $\mathcal{C}^r$ interval maps $f:[0,1]\to[0,1]$. To simplify the notation we let $h(.)$ for $h(f,.)$. We consider the defect $\dddot{h}$ of upper semicontinuity defined as follows:

$$\forall \mu \in M(X,f),\ \dddot{h}(\mu) = \limsup_{\nu\to\mu} h(\nu) - h(\mu).$$

Following Y. Yomdin [19] it was proved by J. Buzzi [6] that for $\mathcal{C}^r$ $(r \geq 1)$ maps f on a compact manifold X of dimension d:

$$\dddot{h} \quad \leq \quad \frac{d \log \|f'\|_\infty}{r}. \tag{1}$$

The author has refined this inequality in [1] at the measure theoretical level as follows: $\dddot{h} \leq \frac{d\,\lambda^+}{r}$, where λ is the maximal Lyapunov exponent. For one dimensional maps the works of T. Downarowicz and A. Maass [10] on symbolic extensions give another proof of this inequality (without referring to Yomdin's theory and semi-algebraic tools) but also allow us to estimate the entropy of measures with a large defect. We state this remark as a corollary of the Antarctic theorem of T. Downarowicz and A. Maass [10].

Corollary 3. *Let f be a $\mathcal{C}^r$ interval map with $r \geq 1$. For any $\mu \in M([0,1],f)$ we let $a_\mu \in [0,1]$ be such that $\dddot{h}(\mu) = a_\mu \frac{\lambda_\mu}{r}$. Then*

$$h(\mu) \quad \leq \quad (1-a_\mu)\lambda_\mu^+. \tag{2}$$

In particular a measure μ at which the defect is "maximal", $a_\mu = 1$, has zero entropy.

Let us explain in more detail how the above corollary follows from the Antarctic theorem. In [10] the authors study the convergence of entropy structures (h_k). Such a sequence is a nondecreasing sequence of functions converging pointwise to the entropy function in some specific way (we refer to the book [9] for further details). One may choose the functions h_k to be upper semicontinuous. The Antarctic theorem then states that

$$\lim_k \left(h - h_k + \frac{\lambda^+}{r-1}\right)^{\cdots} = 0. \tag{3}$$

We now explain how Corollary 3 follows from the above estimate.

Proof of Corollary 3. Together with the triangular inequality, $(f+g)^{\cdots} \leq \dddot{f} + \dddot{g}$, we get by upper semicontinuity of h_k:

$$\left(h + \frac{\lambda^+}{r-1}\right)^{\cdots} = 0.$$

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Now consider a sequence $(\nu_n)_n$ of invariant measures such that $\lim_n h(\nu_n) = \dddot{h} + h(\mu) = a_\mu \frac{\lambda_\mu^+}{r} + h(\mu)$. Then we conclude with the Ruelle inequality that

$$\begin{aligned} h(\mu) + \frac{\lambda_\mu^+}{r-1} &\geq \limsup_n \left(h(\nu_n) + \frac{\lambda_{\nu_n}^+}{r-1} \right) \\ &\geq \left(1 + \frac{1}{r-1} \right) \lim_n h(\nu_n), \\ (r-1)h(\mu) + \lambda_\mu^+ &\geq a_\mu \lambda_\mu^+ + r h(\mu), \end{aligned}$$

that is,

$$(1 - a_\mu)\lambda_\mu^+ \geq h(\mu).$$

□

Remark 1. The estimate (3) holds true for $\mathcal{C}^r$ $(r > 1)$ surface diffeomorphisms [4], and therefore so does Corollary 3.

We review now some situations where upper semicontinuity of the entropy may fail. A classical criteria to ensure the existence of maximal measures is the upper semicontinuity of the entropy function. But this does not hold for general $\mathcal{C}^r$ maps, as shown by the example produced in [6]. In this example the Dirac measure at the fixed point 0 is a limit of ergodic measures $(\mu_n)_n$ with entropy $\frac{\log \|f'\|_\infty}{r} > 0$. One can easily modify this example to get a map with the same property but also admitting an invariant measure with $h(\mu) \simeq \log \|f'\|_\infty$ (see for example [3]). Then for any $a \in [0,1]$ we have with $\nu_n = a\delta_0 + (1-a)\mu_n$:

$$\begin{aligned} \dddot{h}(a\delta_0 + (1-a)\mu) &\geq \limsup_n h(\nu_n) - h(a\delta_0 + (1-a)\mu) \\ &\geq a \frac{\log \|f'\|_\infty}{r}. \end{aligned}$$

This proves that inequality (2) obtained in the above corollary is sharp because $h(a\delta_0 + (1-a)\mu) \simeq (1-a)\log \|f'\|_\infty$. In particular entropy may not be upper semicontinuous even at large entropy measures. In the previous example the sequence $(\nu_n)_n$ and its limit are not ergodic.

We present now a similar example where the sequence $(\nu_n)_n$ is ergodic (but not its limit). We just outline the construction. To simplify the exposition the parameter a_μ will be equal to $\frac{1}{2}$; i.e. for any $r > 1$ we will exhibit an example with a sequence (ν_n) of ergodic invariant measures converging to some invariant measure μ such that $h(\mu) = \frac{\log \|f'\|_\infty}{2}$ and $\limsup_n h(\nu_n) = \frac{\log \|f'\|_\infty}{2}\left(1 + \frac{1}{r}\right)$.

We consider a $\mathcal{C}^r$ interval map f as follows. The point 0 is a repulsing fixed point, $f'(0) = \|f'\|_\infty = 2$ and f is linear close to zero. Moreover there exist disjoint intervals of monotonicity $(J_0, J_1, I_n^1, \dots, I_n^{N_n},\ n \in \mathbb{N})$ such that:

- $f(J_0) = f(J_1)$ covers the union U of all these intervals;
- each interval I_n^k for $k \leq N_n$ is n times successively expanded near 0 and $f^n I_n^k$ also covers U;
- $\lim_n \frac{\log N_n}{n} = \frac{\log f'(0)}{r}$.

Such a map can be built by adapting the construction in [17] and [3]. We skip the technical details and we refer these papers to the interested reader.

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The collection of intervals $\bigcap_{l=0}^{n-1} f^{-l}J_i \bigcap_{l=n}^{2n-1} f^{-l}I_n^j$ for $i = 1, 2$ and $j = 1, \ldots, N_n$ defines a $2n$-horseshoe; i.e. the corresponding compact invariant set for f^{2n} is conjugated to a full shift with $2^n N_n$ symbols. Let ν_n be the maximal entropy invariant measure carried by this horseshoe. The sequence $(\nu_n)_n$ is converging to $\frac{1}{2}(\delta_0 + \mu)$ where μ is supported by the horseshoe defined by J_0 and J_1, in particular $h(\mu) \leq \log 2$. We therefore have

$$\lim_n h(\nu_n) = \frac{\log \|f'\|_\infty}{2}\left(1 + \frac{1}{r}\right) \text{ and } h(\lim_n \nu_n) \leq \frac{\log \|f'\|_\infty}{2}.$$

As already said the limit however is not ergodic. It leads to the following open question:

Question. Is the entropy function of a $\mathcal{C}^r$ interval map with $r > 1$ upper semicontinous at ergodic measures with positive entropy?

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