SYMBOLIC EXTENSIONS FOR 3-DIMENSIONAL DIFFEOMORPHISMS

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Abstract. We prove that every $\mathcal{C}^r$ diffeomorphism with $r > 1$ on a three-dimensional manifold admits symbolic extensions, i.e. topological extensions which are subshifts over a finite alphabet. This answers positively a conjecture of Downarowicz and Newhouse in dimension three.

1. Introduction

A symbolic extension of a topological dynamical system is a topological extension given by a subshift over a finite alphabet. Existence and entropy of symbolic extensions have been intensively investigated in the last decades. M. Boyle and T. Downarowicz [3] characterized the problem of existence in terms of new entropic invariants related to weak expansiveness properties of the system. In particular asymptotically h-expansive systems always admit *principal* symbolic extensions, i.e. extensions that preserve the entropy of invariant measures [4].

For smooth systems on compact manifolds this theory appears to be of high interest. It is well known that Markov partitions allow to encode uniformly hyperbolic systems by finite-to-one symbolic extensions of finite type. Beyond uniform hyperbolicity, partially hyperbolic diffeomorphisms with one dimensional center satisfy the h-expansiveness property, hence admit principal symbolic extensions [12, 18]. More recently the second author with M. Viana and J. Yang showed that smooth systems with no principal symbolic extension are $\mathcal{C}^1$-close to diffeomorphisms with homoclinic tangencies [17].

Moreover the existence of symbolic extensions depends on the order of smoothness. While $\mathcal{C}^\infty$ systems are asymptotically h-expansive [8, 24] and thus admit principal symbolic extensions, there is a $\mathcal{C}^1$ open set of 3-dimensional diffeomorphisms [1] (resp. Lebesgue preserving diffeomorphisms [9, 15]) in which generic ones have no symbolic extension. In intermediate smoothness, i.e. for $\mathcal{C}^r$ systems with $1 < r < +\infty$, the existence was conjectured by T. Downarowicz and S. Newhouse in [15] and in general this problem is still open. It has been first proved for circle maps by T. Downarowicz and A. Maass [14] and then by the first author for surface diffeomorphisms [5, 6]. In this paper we continue the work of [6] to show existence of symbolic extensions for diffeomorphisms in dimension 3. We refer to the next section for the definitions and notations used in our main Theorem below.

Main Theorem. *Let f be a $\mathcal{C}^r$ diffeomorphism with $r > 1$ on a compact 3-dimensional manifold M. Then f admits a symbolic extension $\pi : (Y, S) \to (M, f)$ satisfying for all $\mu \in \mathcal{M}_{inv}(f)$:*

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$$\max_{\xi \in \mathcal{M}_{inv}(S),\ \pi\xi=\mu} h(S,\xi) = h(f,\mu) + \frac{\lambda_1^+(f,\mu)+\lambda_2^+(f,\mu)}{r-1},$$

where $\lambda_1^+(f,\mu) \geq \lambda_2^+(f,\mu)$ *denote the positive parts of the two largest Lyapunov exponents of* μ.

The ingredient of the present advance is mainly a new inequality relating the Newhouse local entropy of an ergodic measure and the local volume growth of smooth discs of unstable dimension (which is the number of positive Lyapunov exponents of the measure). Section 3 is devoted to the proof of this key estimate. Then for a 3-dimensional diffeomorphism, we may bound from above the Newhouse local entropy with respect to either f or f^{-1} by the local volume growth of curves, which implies the existence of symbolic extensions by combining with the *Reparametrization Lemma* developed in [6]. This is proved together with the Main Theorem in the last section.

2. Preliminaries

2.1. **Newhouse entropy structure and the Symbolic Extension Theorem.** Consider a topological system (M,f), i.e. a continuous map $f : M \to M$ on a compact metric space (M,d). For $x \in M, \varepsilon > 0, n \in \mathbb{N}$, we denote the n-step dynamical ball at x with radius ε by

$$B_n(x,\varepsilon,f) = \{y \in M : d(f^i(x), f^i(y)) < \varepsilon,\ i = 0,\cdots,n-1\}.$$

A subset N of M is said (n,δ)-separated when any pair $y \neq z$ in N satisfies $d(f^i(y), f^i(z)) > \delta$ for some $i \in [0, n-1]$. For any subset Λ of M and $\delta > 0$, denote by $s(n,\delta,\Lambda)$ the maximal cardinality of the (n,δ)-separated sets contained in Λ. For any $\Lambda \subset M$, $\varepsilon > 0$, define

$$h^*(f,\Lambda,\varepsilon) = \lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in\Lambda} s\left(n,\delta,B_n(x,\varepsilon,f)\cap\Lambda\right).$$

Denote by $\mathcal{M}_{inv}(f)$ (resp. $\mathcal{M}_{erg}(f)$) the set of all f-invariant (resp. ergodic f-invariant) Borel probability measures endowed with the usual metrizable weak-$*$ topology. Given $\mu \in \mathcal{M}_{erg}(f)$, for any $\varepsilon > 0$, Newhouse [20] defined the tail entropy of μ at the scale ε by letting

$$h^*(f,\mu,\varepsilon) = \lim_{\eta\to 1,\, 0<\eta<1}\ \inf_{\mu(\Lambda)>\eta} h^*(f,\Lambda,\varepsilon).$$

For $\mu \in \mathcal{M}_{inv}(f)$, assuming $\mu = \int_{\mathcal{M}_{erg}(f)} \nu\, dL_\mu(\nu)$ is the ergodic decomposition of μ, let

$$h^*(f,\mu,\varepsilon) = \int_{\mathcal{M}_{erg}(f)} h^*(f,\nu,\varepsilon)\, dL_\mu(\nu).$$

Entropy structures are *particular* non-increasing sequences of nonnegative functions defined on $\mathcal{M}_{inv}(f)$ which are converging pointwisely to the Kolmogorov-Sinai entropy function $h : \mathcal{M}_{inv}(f) \to \mathbb{R}^+$ (see [13] for a precise definition). They satisfy the following criterion for the existence of symbolic extensions.

Theorem 1 (Symbolic Extension Theorem [3, 14])**.** *Let* (M,f) *be a topological system. Assume* E *is a nonnegative affine upper semicontinuous function such that for all* $\mu \in \mathcal{M}_{inv}(f)$ *there is an entropy structure* $(h_k)_k$ *satisfying*

$$\lim_k \limsup_{\mathcal{M}_{erg}(f)\ni\nu\to\mu} (E + h - h_k)(\nu) \leq E(\mu). \tag{1}$$

Then there exists a symbolic extension $\pi : (Y, S) \to (M, f)$ *such that*

$$\max_{\xi \in \mathcal{M}_{inv}(S),\ \pi\xi = \mu} h(S, \xi) = (E + h)(\mu).$$

Letting $\varepsilon_k \to 0$, then the sequence $(h_k^{New})_k$ defined by $h_k^{New}(f, \mu) := h(f, \mu) - h^*(f, \mu, \varepsilon_k)$ for all $k \in \mathbb{N}$ and for all $\mu \in \mathcal{M}_{inv}(f)$ is an entropy structure [13]. As a matter of fact, for any $m \in \mathbb{Z} \setminus \{0\}$, $h_{m,k}^{New}(f, \mu) := h(f, \mu) - \frac{1}{|m|} h^*(f^m, \mu, \varepsilon_k)$ for all $k \in \mathbb{N}$ and for all $\mu \in \mathcal{M}_{inv}(f)$ is also an entropy structure (see Lemma 1 in [6]).

2.2. **Lyapunov exponents.** Let $f : M \to M$ be a differentiable map on a compact Riemannian manifold $(M, \|\cdot\|)$ of dimension d. Given $x \in M$, the Lyapunov exponent relative to a direction $v \in T_x M$ is the exponential growth rate given by the limit

$$\lim_{n \to \infty} \frac{1}{n} \log \|D_x f^n v\|, \tag{2}$$

which exists for almost every point x with respect to every f-invariant Borel probability measure μ by Oseledets theorem [21] (it does not depend on the Riemannian structure on M). Moreover, for μ-almost every point x, there exist values $\lambda_1(f, x) \geq \cdots \geq \lambda_{\mathsf{d}}(f, x)$ of the limit (2) and measurable flags of the tangent spaces $\{0\} = G_x^{\mathsf{d}+1} \subset G_x^{\mathsf{d}} \subset \cdots \subset G_x^1 = T_x M$ satisfying:

$$\lim_{n \to \infty} \frac{1}{n} \log \|D_x f^n v\| = \lambda_i(f, x), \quad \forall v \in G_x^i \setminus G_x^{i+1},\ 1 \leq i \leq \mathsf{d}.$$

For any $\mu \in \mathcal{M}_{inv}(f)$, $1 \leq i \leq \mathsf{d}$, we denote

$$\lambda_i(f, \mu) = \int \lambda_i(f, x)\, d\mu(x),$$

$$\sum_{j=1}^{i} \lambda_j^+(f, \mu) = \int \sum_{j=1}^{i} \lambda_j^+(f, x)\, d\mu(x).$$

For $\nu \in \mathcal{M}_{erg}(f)$, we have $\lambda_i(f, \nu) = \lambda_i(f, x)$ for all i and for ν-almost every x. By standard arguments the function $\mu \mapsto \sum_{j=1}^{i} \lambda_j^+(f, \mu)$ defines an affine upper semicontinuous function on $\mathcal{M}_{inv}(f)$ (see Lemma 3 in [6]). For a $\mathcal{C}^r$ diffeomorphism with $r > 1$ on a compact 3-dimensional Riemannian manifold, we will prove that $E = \frac{\sum_{j=1}^{2} \lambda_j^+(f, \cdot)}{r-1}$ satisfies Inequality (1), which together with Theorem 1 implies the Main Theorem.

2.3. **Nonuniformly hyperbolic estimates.** Assume now f is a diffeomorphism. In this case, Oseledets theorem provides for any $\mu \in \mathcal{M}_{inv}(f)$, for μ-a.e., $x \in M$, a decomposition on the tangent space $T_x M = E_x^{cs} \oplus E_x^u$ and $\rho_{cs}(x) \leq 0 < \rho_u(x)$ satisfying

- $\lim_{|n| \to \infty} \frac{1}{n} \log \|D_x f^n(v)\| \leq \rho_{cs}(x), \quad \forall\, 0 \neq v \in E_x^{cs}$;
- $\lim_{|n| \to \infty} \frac{1}{n} \log \|D_x f^n(w)\| \geq \rho_u(x), \quad \forall\, 0 \neq w \in E_x^u$;
- $\lim_{|n| \to \infty} \frac{1}{n} \log \sin \angle(E_{f^n(x)}^{cs}, E_{f^n(x)}^u) = 0.$

For any $\lambda_u > 0$, $0 < \gamma \ll \lambda_u$ and $k \in \mathbb{N}$, we consider the sets $\Lambda_k(\lambda_u, \gamma)$ consisting of points x in M with the following properties:

- $\|Df^n|E^{cs}_{f^i(x)}\| \le e^{k\gamma}e^{|i|\gamma}e^{n\gamma}, \quad \forall\, i \in \mathbb{Z},\, n \ge 1;$

- $\|Df^{-n}|E^{u}_{f^i(x)}\| \le e^{k\gamma}e^{|i|\gamma}e^{n(-\lambda_u+\gamma)}, \quad \forall\, i \in \mathbb{Z},\, n \ge 1;$

- $\sin\angle(E^{cs}_{f^i(x)}, E^{u}_{f^i(x)}) \ge e^{-k\gamma}e^{-|i|\gamma}, \quad \forall\, i \in \mathbb{Z}.$

From the definition, it holds that [2, 22]

- $T_xM = E^{cs}_x \oplus E^u_x$ is a continuous splitting on each $\Lambda_k(\lambda_u,\gamma)$;

- $f(\Lambda_k(\lambda_u,\gamma)) \subset \Lambda_{k+1}(\lambda_u,\gamma)$ and $f^{-1}(\Lambda_k(\lambda_u,\gamma)) \subset \Lambda_{k+1}(\lambda_u,\gamma)$ for any $k \in \mathbb{N}$;

- $x \in \bigcup_{k\in\mathbb{N}} \Lambda_k(\lambda_u,\gamma)$ provided $\lambda_u \le \rho_u(x)$;

- $\lim_{\lambda_u\to 0} \mu\left(\bigcup_{k\in\mathbb{N}} \Lambda_k(\lambda_u,\gamma)\right) = 1$ for any $\mu \in \mathcal{M}_{inv}(f)$.

Now fix λ_u and γ. For the sake of statements, we let $\Lambda_k = \Lambda_k(\lambda_u,\gamma)$ for any $k \in \mathbb{N}$ and $\Lambda^* = \bigcup_{k\in\mathbb{N}} \Lambda_k(\lambda_u,\gamma)$. Denote $\lambda'_u = \lambda_u - 2\gamma$. Given $x \in \Lambda^*$, define for all $v = v_{cs} + v_u$ and $w = w_{cs} + w_u$ with $v_{cs}, w_{cs} \in E^{cs}_x$ and $v_u, w_u \in E^u_x$,

$$\begin{aligned}
< v_{cs}, w_{cs} >' &= \sum_{n=0}^{+\infty} e^{-4n\gamma} < D_xf^n(v_{cs}), D_xf^n(w_{cs}) >,\\
< v_u, w_u >' &= \sum_{n=0}^{+\infty} e^{2n\lambda'_u} < D_xf^{-n}(v_u), D_xf^{-n}(w_u) >,\\
< v, w >' &= < v_u, w_u >' + < v_{cs}, w_{cs} >' .
\end{aligned}$$

There exists $a_1 = a_1(\gamma) > 1$ such that

$$\|v\| \le \|v\|' \le a_1 e^{k\gamma}\|v\|, \quad \forall\ v \in T_{\Lambda_k}M. \tag{3}$$

The norm $\|\cdot\|'$ is called a Lyapunov metric, with which f behaves uniformly on Λ^*:

$$\frac{1}{\|Df^{-1}\|}\|v_{cs}\|'_x \le \|D_xf(v_{cs})\|'_{f(x)} \le e^{2\gamma}\|v_{cs}\|'_x,$$

$$\frac{1}{\|Df\|}\|v_u\|'_x \le \|D_xf^{-1}(v_u)\|'_{f^{-1}(x)} \le e^{-\lambda'_u}\|v_u\|'_x.$$

In this manner, the splitting $T_{\Lambda^*}M = E^{cs} \oplus E^u$ is dominated with respect to $\|\,\|'$, i.e.

$$\frac{\|D_xf(v_{cs})\|'}{\|D_xf(v_u)\|'} \le e^{2\gamma-\lambda'_u}\frac{\|v_{cs}\|'}{\|v_u\|'}, \quad \forall\, 0 \ne v_{cs} \in E^{cs}_x,\, 0 \ne v_u \in E^u_x,\ x \in \Lambda^*,$$

$$\text{with } 2\gamma - \lambda'_u < 0.$$

We consider a $\mathcal{C}^r$ diffeomorphism f on a C^r smooth Riemanian manifold $(M, \|\cdot\|))$ with $r > 1$. Let $\alpha = \min\{r-1, 1\}$. We are going to state that the dominated behavior on each Λ_k can be extended to a $e^{-k\mathsf{d}\gamma'}$-neighborhood for $\gamma' = \alpha^{-1}\gamma$. Moreover, for attaining a preassigned local proximity of dominated splitting, we may choose a positive number b independently of k such that this proximity holds in a $be^{-k\mathsf{d}\gamma'}$-neighborhood of Λ_k.

Let d be the Riemannian distance on M and r be the radius of injectivity of $(M, \|\cdot\|)$. The ball at $x \in M$ of radius $R \in \mathbb{R}^+$ with respect to d is denoted by $B(x,R)$. Then for $y \in B(x,\mathsf{r})$ we use the identification

$$\begin{array}{rcl} T_{B(x,\mathsf{r})}M & \simeq & B(x,\mathsf{r}) \times T_xM, \\ (y,v) & \mapsto & \big(y, D_y(\exp_x^{-1})(v)\big) \end{array}$$

to "translate" the vector $v \in T_yM$ to the vector $\hat{v}_x := D_y(\exp_x^{-1})(v) \in T_xM$. Recall that the exponential map $(x,v) \mapsto \exp_x(v)$ defines a C^r map (thus $C^{1+\alpha}$) from TM to M with $D_x(\exp_x) = \mathrm{Id}_{T_xM}$. Since the diffeomorphism f is also $C^{1+\alpha}$ on the compact manifold M, there exist $K > 1, a_2 > 0$ such that

$$\forall x \in M,\ \forall (y,v) \in T_{B(x,a_2)}M, \qquad \tfrac{\|v\|}{2} \leq \|\hat{v}_x\| \leq 2\|v\|$$
$$\text{and } \|D_xf^{\pm}(\hat{v}_x) - \widehat{D_yf^{\pm}v}_{f(x)}\| \leq K\|v\|d(x,y)^{\alpha}.$$

For $x \in \Lambda^*$ and $(y,v) \in T_{B(x,a_2)}M$, we define $\|v\|''_x = \|\hat{v}_x\|'$ and we also let $<,>''_x$ be the associated scalar product on T_yM. It follows then from (3) that

$$\forall x \in \Lambda_k,\ \forall (y,v) \in T_{B(x,a_2)}M, \quad 2a_1e^{k\gamma}\|v\| \quad \geq \|v\|''_x = \|\hat{v}_x\|' \quad \geq \frac{\|v\|}{2}. \tag{4}$$

We write $\hat{v}_x$ as v whenever there is no confusion and we also denote by $T_yM = E_x^{cs} \oplus E_x^u$ the splitting of T_yM which translates to the splitting $T_xM = E_x^{cs} \oplus E_x^u$. Let $\lambda''_u = \lambda'_u - \gamma$ and let $a'_2 > 0$ such that $f^i(B(x,a'_2)) \subset B(f^ix, a_2)$ for all $x \in M$ and $i = 0, 1, -1$. Then define

$$\gamma_k = \min\left\{1, a'_2, \left(\frac{e^{-\lambda''_u} - e^{-\lambda'_u}}{4a_1e^{(k+1)\gamma}K}\right)^{\frac{1}{\alpha}}\right\}.$$

Then we have for all $x \in \Lambda_k$, for all $y \in B(x,\gamma_k)$ and for all $v_{cs/u} \in E_x^{cs/u} \subset T_yM$ (see [22] p.72 for further details) :

$$\Big(\frac{1}{\|Df^{-1}\|} - (e^{\gamma} - 1)\Big)\|v_{cs}\|''_x \leq \|D_yf(v_{cs})\|''_{f(x)} \leq e^{3\gamma}\|v_{cs}\|''_x, \tag{5}$$

$$\Big(\frac{1}{\|Df\|} - (e^{\gamma} - 1)\Big)\|v_u\|''_x \leq \|D_yf^{-1}(v_u)\|''_{f^{-1}(x)} \leq e^{-\lambda''_u}\|v_u\|''_x. \tag{6}$$

Define $\kappa(x) = \min\{k \in \mathbb{N} : x \in \Lambda_k\}$ for $x \in \Lambda^*$. Then the inequalities (5) and (6) hold for any $y \in B(x, \gamma_{\kappa(x)})$. Such sets $B(x,\gamma_{\kappa(x)})$ are called Lyapunov neighborhoods. Letting $\gamma' = \alpha^{-1}\gamma$, we have $\gamma_k = a_3e^{-k\gamma'} < 1$ for k large enough and some constant a_3 independent of k. We use d''_x to denote the distance induced by $\|\cdot\|''_x$ on $B(x,a_2)$ and $B''_x(y,r)$ to denote the ball centered at y with radius r in d''_x.

For the purpose of our use in the computation of tail entropy and local volume growth, we need to estimate the proximity of the dominated splitting in Lyapunov neighborhoods along orbits. For a splitting $F = F_1 \oplus F_2$ of an Euclidean space F with norm $\|\,\|$, and $\xi > 0$, we denote by $Q_{\|\,\|}(F_1,\xi)$ the cone of width ξ of F_1 in $\|\,\|$, i.e. the set $\{v = v_1 + v_2 \in F : v_1 \in F_1, v_2 \in F_2, \quad \|v_2\| \leq \xi\|v_1\|\}$. For any vector subspace G of F we let $\iota(G)$ be the Plücker embedding of G in the projective space $\mathbb{P}\bigwedge F$ of the Euclidean power exterior algebra $\bigwedge F$. When $A : F \to F'$ is a linear map between two finite dimensional Euclidean spaces F and F', we let $\bigwedge^l A$ be the induced map on the l-exterior power $\bigwedge^l F$ with l less than or equal to the dimension of F. With the above notations the map $x \mapsto \bigwedge^l D_x f$ is α-Hölder and one may assume its Hölder norm is less than K by taking K larger in advance. Observe that $\bigwedge^l F \ni u \mapsto \| \bigwedge^l Au\|$ induces a map on $\mathbb{P} \bigwedge^l F$ by letting $\| \bigwedge^l A(\mathbb{P}u)\| = \frac{\|\bigwedge^l Au\|}{\|u\|}$. Also we let $l_u(f,z)$ be the dimension of $E^u(z)$. When $\mu \in \mathcal{M}_{erg}(f)$, $l_u(f,z)$ is a constant for μ-a.e. z, which we denote by $l_u(f,\mu)$.

Lemma 1. *For any $\xi > 0$ small enough there exists $a_\xi > 0$ such that for any $x \in \Lambda^*$ and for any $y \in B\left(x, a_\xi \gamma^l_{\kappa(x)}\right)$ with $l = l_u(f,x)$ we have :*

(i) $\|D_y f(v)\|''_{f(x)} \geq e^{\lambda''_u - \gamma}\|v\|''_x$ *for all* $v \in Q_{\|\|''_x}(E^u_x, \xi)$ *and* $\|D_y f(v)\|''_{f(x)} \leq e^{4\gamma}\|v\|''_x$ *for all* $v \in Q_{\|\|''_x}(E^{cs}_x, \xi)$,

(ii) $D_y f(Q_{\|\|''_x}(E^u_x, \xi)) \subset Q_{\|\|''_{f(x)}}(E^u_{f(x)}, \xi)$ *and* $D_y f^{-1}(Q_{\|\|''_x}(E^{cs}_x, \xi)) \subset Q_{\|\|''_{f^{-1}(x)}}(E^{cs}_{f^{-1}(x)}, \xi)$,

(iii) $e^{-\gamma} \leq \frac{\|\bigwedge^l D_y f(\iota(G))\|''_{f(x)}}{\|\bigwedge^l D_x f(\iota(E^u_x))\|''_{f(x)}} \leq e^{\gamma}$ *for all l-plane* $G \subset Q_{\|\|''_x}(E^u_x, \xi)$.

Proof. Let $\xi > 0$ and $x \in \Lambda^*$.

(i) By the domination property $E^{cs}_x \oplus E^u_x$ at y with respect to $\|\cdot\|''_x$ given by the inequalities (5) and (6), the first item holds for small ξ independent of $\kappa(x)$.

(ii) Using the invariance of E^u and the domination property at x there exists $\varsigma \in (0,1)$ independent of x satisfying $D_x f(Q_{\|\|''_x}(E^u_x, \xi)) \subset Q_{\|\|''_{f(x)}}(E^u_{f(x)}, \varsigma\xi)$. Then for any $y \in B(x, a_\xi \gamma_{\kappa(x)})$, we get by the Inequalities (4)

$$\begin{aligned} \|D_x f - D_y f\|''_x &:= \max_{\|v\|''_x = 1} \|D_x f(v) - D_y f(v)\|''_{f(x)} \\ &\leq 4a_1 e^{\kappa(f(x))\gamma}\|D_x f - D_y f\| \\ &\leq 4Ka_1 e^{(\kappa(x)+1)\gamma}(a_\xi \gamma_{\kappa(x)})^\alpha \\ &\leq a_\xi^\alpha. \end{aligned}$$

For small γ, by (5) and (6) one has also

$$\frac{1}{2\|Df^{-1}\|} \leq \min_{\|v\|''_x=1} \|D_y f(v)\|''_{f(x)} \leq \max_{\|v\|''_x=1} \|D_y f(v)\|''_{f(x)} \leq 2\|Df\|.$$

It follows that for $\|v\|''_x = 1$, the angle $\angle''(D_y f(v), D_x f(v))$ with respect to $\|\cdot\|''_{f(x)}$ is less than $\arctan(\xi) - \arctan(\varsigma\xi)$ for a_ξ small enough. We conclude that $D_y f\left(Q_{\|\|''_x}(E^u_x), \xi\right) \subset Q_{\|\|''_{f(x)}}\left(E^u_{f(x)}, \xi\right)$ for any $y \in B(x, a_\xi \gamma_k)$. We prove similarly the cone invariance property for the center stable direction.

(iii) To prove the last item observe first that using again the domination property at x we get $\left|\frac{\|\bigwedge^l D_x f(\iota(G))\|''_{f(x)}}{\|\bigwedge^l D_x f(\iota(E^u_x))\|''_{f(x)}} - 1\right| \leq 1 - e^{-\gamma/2}$ for all l-planes $G \subset Q_{\|\|''_x}(E^u_x, \xi)$ for $\xi > 0$ small enough. As f is $e^{\lambda_u''}$-expanding in the unstable direction with respect to $\|\cdot\|''$ we have

$$\| \curlywedge^l D_x f(\iota(E_x^u))\|''_{f(x)} \geq e^{l\lambda_u''}.$$

Then arguing as above for $y \in B(x, a_\xi \gamma^l_{\kappa(x)})$, we have by Lemma 2 in the Appendix and the Inequalities (4) :

$$\| \curlywedge^l D_x f - \curlywedge^l D_y f\|''_{f(x)} \leq (4a_1 e^{\kappa(f(x))\gamma})^l \| \curlywedge^l D_x f - \curlywedge^l D_y f\| \leq a_\xi^\alpha.$$

Therefore we get for a_ξ small enough :

$$\begin{aligned}
\left| \frac{\| \curlywedge^l D_y f(\iota(G))\|''_{f(x)}}{\| \curlywedge^l D_x f(\iota(E_x^u))\|''_{f(x)}} - 1 \right| &\leq \frac{\left| \| \curlywedge^l D_y f(\iota(G))\| - \| \curlywedge^l D_x f(\iota(G))\|''_{f(x)} \right|}{\| \curlywedge^l D_x f(\iota(E_x^u))\|''_{f(x)}} \\
&+ \left| \frac{\| \curlywedge^l D_x f(\iota(G))\|''_{f(x)}}{\| \curlywedge^l D_x f(\iota(E_x^u))\|''_{f(x)}} - 1 \right| \\
&\leq \frac{a_\xi^\alpha}{e^{l\lambda_u''}} + 1 - e^{-\gamma/2} \\
&\leq 1 - e^{-\gamma}.
\end{aligned}$$

□

From the domination structure $E^{cs} \oplus E^u$ in the norm $\|\cdot\|''_x$, one may build a family of *fake* center-stable manifolds as follows.

Proposition 1. *With the notations of Lemma 1, for any $\xi > 0$ small enough, there exist $b_\xi \in (0, a_\xi)$ and families $\{\mathcal{W}_x^{cs} : x \in \Lambda^*\}$ of C^1 manifolds satisfying*

- *(i) uniform size: for $x \in \Lambda_k$, $k \in \mathbb{N}$, there is a C^1 map $\phi_x : E_x^{cs} \to E_x^u$ such that $\mathcal{W}_x^{cs}$ is locally given by the graph $\Gamma\phi_x := \{(z, \phi_x(z)),\ z \in E_x^{cs}\}$ of ϕ_x, i.e.*
$$\mathcal{W}_x^{cs} = \exp_x(\Gamma\phi_x) \cap B(x, a_\xi\gamma_k);$$
- *(ii) almost tangency: $T_y\mathcal{W}_x^{cs}$ lies in a cone of width ξ of E_x^{cs} in $\|\cdot\|''_x$ for any $y \in \mathcal{W}_x^{cs}$;*
- *(iii) local invariance: $f^{\pm}\mathcal{W}_x^{cs}(b_\xi\gamma_{\kappa(x)}) \subset \mathcal{W}^{cs}_{f^{\pm}(x)}$ with $\mathcal{W}_x^{cs}(\zeta)$ being the ball of radius ζ centered at x inside $\mathcal{W}_x^{cs}$ with respect to the distance induced by $\|\cdot\|''_x$ on $\mathcal{W}_x^{cs}$.*

Proof. By taking the exponential map at x we can assume without loss of generality that we are working in $\mathbb{R}^{\mathsf{d}}$. Let ξ and a_ξ be as in Lemma 1. For any $x \in \Lambda^*$, we can extend $f|_{B(x, a_\xi\gamma_{\kappa(x)})}$ to a diffeomorphism $\tilde{f}_x : \mathbb{R}^{\mathsf{d}} \to \mathbb{R}^{\mathsf{d}}$ such that

- $\tilde{f}_x(y) = f(y) \quad$ for $y \in B(x, a_\xi\gamma_{\kappa(x)})$;
- $\|D_y\tilde{f}_x - D_x f\|''_x \leq 2a_\xi^\alpha \quad$ for $y \in \mathbb{R}^{\mathsf{d}}$.

By taking a_ξ smaller, the properties of Lemma 1 hold with respect to $\tilde{f}_x$ for all $y \in \mathbb{R}^{\mathsf{d}}$. Let Ξ be the disjoint union given by $\Xi = \coprod_{x\in\Lambda^*}\{x\} \times \mathbb{R}^{\mathsf{d}}$ where Λ^* is endowed with the discret topology. Then $\tilde{f} = (\tilde{f}_x)_{x\in\Lambda^*}$ can be viewed as a map from Ξ to itself by letting $\tilde{f}(x, v) = \Big(f(x), \tilde{f}_x(v)\Big)$. Note that the global splitting $\coprod_{x\in\Lambda^*}\{x\} \times \mathbb{R}^{\mathsf{d}} = \coprod_{x\in\Lambda^*}\{x\} \times (E_x^{cs} \oplus E_x^u)$ is dominated with respect to $\tilde{f}$. By [16] §5, we can obtain a family $\{\mathcal{Y}_x^{cs} : x \in \Lambda^*\}$ of global C^1 submanifolds in $\mathbb{R}^{\mathsf{d}}$ which are C^1 graphs defined on E_x^{cs} such that we have for all $x \in \Lambda^*$:

$$\{x\} \times \mathcal{Y}_x^{cs} = \bigcap_{n=0}^{+\infty} \tilde{f}^{-n} \left(\{f^n(x)\} \times Q_{\|\|''_x}(E^{cs}_{f^n(x)}, \xi) \right),$$

$$\forall y \in \mathbb{R}^{\mathsf{d}},\ T_y \mathcal{Y}_x^{cs} \subset Q_{\|\|''_x}(E_x^{cs}, \xi).$$

In particular we get $\tilde{f}^{\pm}(\{x\} \times \mathcal{Y}_x^{cs}) \subset \{f^{\pm}(x)\} \times \mathcal{Y}^{cs}_{f^{\pm}(x)}$. Since we have $\tilde{f} \mid_{\{x\}\times B(x, a_\xi \gamma_{\kappa(x)})} = f \mid_{B(x, b_\xi \gamma_{\kappa(x)})}$, one concludes the proof by considering $\mathcal{W}_x^{cs} = \mathcal{Y}_x \cap B(x, a_\xi \gamma_{\kappa(x)})$ and taking much smaller b_ξ than a_ξ.

□

3. Tail entropy and local volume growth

Let $f : M \to M$ be a $\mathcal{C}^r$ diffeomorphism with $r > 1$ on a compact Riemannian manifold $(M, \|\|)$. In this section, we relate the Newhouse local entropy of an ergodic measure with the local volume growth of smooth *unstable* discs. We begin with some definitions. A $\mathcal{C}^r$ map σ, from the unit square $[0,1]^k$ of $\mathbb{R}^k$ to M, which is a diffeomorphism onto its image, is called a $\mathcal{C}^r$ k-disc. The $\mathcal{C}^r$ size of σ is defined as

$$\|\sigma\|_r = \sup\{\|D^q\sigma\| :\ q \le r,\ q \in \mathbb{R}^+\},$$

where $\|D^q\sigma\|$ denotes the $(q - [q])$-Hölder norm of $D^{[q]}\sigma$ for $q \notin \mathbb{N}$ and the usual supremum norm of the derivative $D^q\sigma$ of order q for $q \in \mathbb{N}$.

For any $\mathcal{C}^1$ smooth k-disc σ and for any $\chi > 0$, $1 \gg \gamma > 0$, $C > 1$ and $n \in \mathbb{N}$, we consider the set $\mathcal{H}_f^n(\sigma, \chi, \gamma, C)$ of points of $[0,1]^k$ whose exponential growth of the induced map on the k-exterior tangent bundle is almost equal to χ:

$$\mathcal{H}_f^n(\sigma,\chi,\gamma,C) := \left\{ t \in [0,1]^k \ :\ \forall 1 \le j \le n-1,\ C^{-1} e^{(\chi-\gamma)j} \le \| \textstyle\bigwedge^k D_t \left(f^j \circ \sigma\right) \| \le C e^{(\chi+\gamma)j} \right\}.$$

For $\Gamma \subset [0,1]^k$, we also denote by $|\sigma_{|\Gamma}|$ the k-volume of σ on Γ, i.e. $|\sigma_{|\Gamma}| = \int_\Gamma \|\bigwedge^k D_t\sigma\|\, d\lambda(t)$, where $d\lambda$ is the Lebesgue measure on $[0,1]^k$. Then given $\chi > 0$, $1 \gg \gamma > 0$, $C > 1$, $x \in M$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we define the local volume growth of σ at x with respect to these parameters as follows :

$$V_x^{n,\varepsilon}(\sigma|\chi,\gamma,C) := \left|(f^{n-1}\circ\sigma)_{|\Delta_n}\right|$$

$$\text{with } \Delta_n := \mathcal{H}_f^n(\sigma,\chi,\gamma,C) \cap \sigma^{-1} B_n(x,\varepsilon,f).$$

Proposition 2. *Let $\nu \in \mathcal{M}_{erg}(f)$ with $l = l_u(f,\nu) \ge 1$. Then for any $\varepsilon > 0$, $1 > \eta > 0$ and $\gamma > 0$, there exist a Borel set F_η with $\nu(F_\eta) > \eta$ and a constant $C > 1$, such that for all $\delta > 0$, all n large enough and all $x \in F_\eta$:*

$$s(n,\delta,B_n(x,\varepsilon,f)\cap F_\eta) \le e^{\gamma n} \sup_{\substack{\sigma\ l\text{-disk} \\ \text{with } \|\sigma\|_r \le 1}} V_x^{n,2\varepsilon}\left(\sigma \,\middle|\, \sum_i \lambda_i^+(\nu), \gamma, C\right).$$

In fact the l_u-discs can be chosen to be affine through the exponential map (see the proof of Proposotion 2 below). Let $v_k^*(f,\varepsilon)$ denote the local volume growth of k-disks :

$$v_k^*(f,\varepsilon)=\limsup_n \frac{1}{n}\sup_{x\in M}\sup_{\substack{\sigma\ k\text{-disk}\\ \text{with } \|\sigma\|_r\leq 1}}\log\left|(f^{n-1}\circ\sigma)_{|\sigma^{-1}B_n(x,\varepsilon,f)}\right|.$$

S. Newhouse [19, 20] proved that the Newhouse local entropy $h^*(f,\nu,\varepsilon)$ of an ergodic measure is less than or equal to the local volume growth of center-unstable dimension. As a direct consequence of Proposition 2, we improve this estimate by considering the local volume growth of unstable dimension.

Corollary 1. *With the above notations,*

$$\forall\varepsilon>0\ \forall\nu\in\mathcal{M}_{erg}(f),\ \ h^*(f,\nu,\varepsilon)\leq v^*_{l_u(f,\nu)}(f,2\varepsilon).$$

Such an inequality was established by K. Cogswell in [11] between the Kolmogorov-Sinai entropy and the global volume growth of unstable discs (in particular Cogswell's main result implies Corollary 1 for ε larger than the diameter of M).

Remark 1. *For any $\nu\in\mathcal{M}_{erg}(f)$, let us denote by $l_{cu}(f,\nu)$ the number of nonnegative Lyapunov exponents of ν. The following estimate is shown in [20] :*

$$\forall\varepsilon>0\ \forall\nu\in\mathcal{M}_{erg}(f),\ \ h^*(f,\nu,\varepsilon)\leq \sup_{\substack{\sigma\ l_{cu}(f,\nu)\text{-disk}\\ \text{with } \|\sigma\|_r\leq 1}}\limsup_n\frac{1}{n}\sup_{x\in M}\log\left|(f^{n-1}\circ\sigma)_{|\sigma^{-1}B_n(x,2\varepsilon,f)}\right|.$$

Observe the right-hand side term differs from the local volume growth $v^*_{l_{cu}(f,\nu)}(f,2\varepsilon)$ as we invert the supremum in σ with the limsup in n. We do not know if the above inequality still holds true for l_u in place of l_{cu}.

We prove now Proposition 2 which is the key new tool to prove the existence of symbolic extensions in dimension 3 combined with the approach developed in [6].

Proof of Proposition 2. Consider $\nu\in\mathcal{M}_{erg}(f)$ with $l=l_u(f,\nu)\geq 1$. Let $0<\gamma\ll\lambda_u:=\lambda_l(f,\nu)$ in the nonuniformly hyperbolic estimates of Section 2. Fix $\eta\in(0,1)$ and $k\in\mathbb{N}$ with $\nu(\Lambda_k(\lambda_u,\gamma))>\eta$. There is a subset F_η of $\Lambda_k=\Lambda_k(\lambda_u,\gamma)$ with $\nu(F_\eta)>\eta$ such that $\frac{1}{n}\|\bigwedge^l D_y(f^n|_{E^u_y})\|$ is converging uniformly in $y\in F_\eta$ to $\sum_i\lambda_i^+(f,y)=\sum_i\lambda_i^+(f,\nu)$ when n goes to $+\infty$. Let $\varepsilon\in(0,1)$ and $\varepsilon_k<\varepsilon$ to be precised. For any given $\hat{x}\in F_\eta$, $0<\delta<\varepsilon$, let E_n be a maximal (n,δ)-separated set in d for f in $B_n(\hat{x},\varepsilon,f)\cap F_\eta$. There exists $x\in E_n$ such that $E'_n=E_n\cap B(x,\varepsilon_k)$ satisfies $\sharp E'_n\geq A_1\left(\frac{\varepsilon_k}{\varepsilon}\right)^{\mathsf{d}}\sharp E_n$ for some universal constant A_1.

(i) Distance estimates in local charts

Since we only deal with the local dynamics around the orbit of x, we can assume without loss of generality that we are working in $\mathbb{R}^{\mathsf{d}}$ by taking the exponential map at x. Take $0<\varepsilon_k<(a_1e^{k\gamma})^{-1}$ so small that $B(x,\varepsilon_k)\subset B''_x(x,2a_1e^{k\gamma}\varepsilon_k)\subset B(x,\varepsilon)$ and consider

$$\hat{\mathcal{W}}^{cs}_x=(x+E^{cs}_x)\cap B''_x(x,2a_1e^{k\gamma}\varepsilon_k).$$

For $\theta_n=\beta_k e^{-n(4\gamma+l\gamma')}$ with $\beta_k=\beta_k(\delta)$ to be precised we let $\mathcal{A}^{cs}$ be a θ_n-net of $\hat{\mathcal{W}}^{cs}_x$ for d''_x satisfying $\sharp\mathcal{A}^{cs}\leq A_2\theta_n^{-\dim E^{cs}}=A_2\theta_n^{-(\mathsf{d}-l)}$ for some universal constant A_2. This means that any point of $\hat{\mathcal{W}}^{cs}_x$ is within a distance θ_n of $\mathcal{A}^{cs}$ for d''_x. For any $z\in\mathcal{A}^{cs}$, denote

$$I_z=\{z+v:\|v\|''_x\leq 4a_1e^{k\gamma}\varepsilon_k,\ v\in E^u_x\}.$$

For $y \in B''_x(x, 2a_1 e^{k\gamma}\varepsilon_k)$ we let $y = y_{cs} + y_u$ with $y_{cs} \in x + E^{cs}_x$ and $y_u \in E^u_x$. Observe that E^{cs}_x and E^u_x are orthogonal in $<,>''_x$, thus y_{cs} lies in $\hat{\mathcal{W}}^{cs}_x$ and there exists $z_y \in \mathcal{A}^{cs}$ with $\|y_{cs} - z_y\|''_x < \theta_n$. Therefore, when y also lies in Λ_k we get :

$$\begin{aligned}\|y_{cs} - z_y\|''_y &\leq 2a_1 e^{k\gamma}\|y_{cs} - z_y\| \\ &\leq 4a_1 e^{k\gamma}\|y_{cs} - z_y\|''_x \\ &\leq 4a_1 e^{k\gamma}\theta_n.\end{aligned}$$

For small $\xi \in (0, \frac{1}{4})$, let $b_\xi > 0$ be as in Lemma 1 and Proposition 1. Since the distributions E^{cs} and E^u are continuous on Λ_k, we may choose ε_k and β_k so small that for any $y \in E'_n$:

- the set $([y_{cs}, z_y] + E^u_x) \cap \mathcal{W}^{cs}_y$ defines a graph Γ_{ϕ_y} of a C^1 function $\phi_y : [y_{cs}, z_y] \subset E^{cs}_x \to E^u_x$,
- $E^{cs/u}_x \subset Q_{\|\|}\left(E^{cs/u}_y, \frac{\xi}{4a_1 e^{k\gamma}}\right) \subset Q_{\|\|''_y}\left(E^{cs/u}_y, \xi\right)$, these cones being defined with respect to the splitting $E^{cs}_y \oplus E^u_y$.

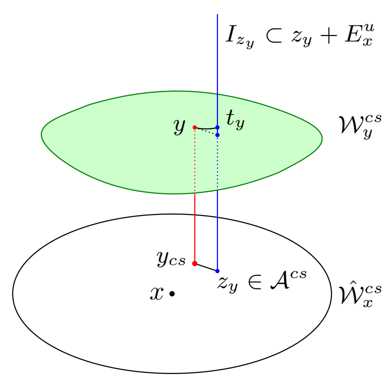

FIGURE 1. The transverse intersection at t_y of I_{z_y} and $\mathcal{W}^{cs}_y$ for $y \in E'_n$.

Let $\theta_y : [0,1] \to E^{cs}_x + E^u_x$ be the reparametrization of the graph of ϕ_y given by

$$\forall t \in [0,1],\ \theta_y(t) = y_{cs} + t(z_y - y_{cs}) + \phi(y_{cs} + t(z_y - y_{cs})).$$

Note that $\theta_y(0) = y$ and $\theta_y(1)$ is the intersection point of I_{z_y} and $\mathcal{W}^{cs}_y$. To simplify the notations we let $t_y := \theta_y(1)$. It follows from the almost tangency property of center-stable fake manifolds stated in Proposition 1 (ii) that

$$\theta'(t) \in Q_{\|\|''_y}(E^{cs}_y, \xi). \tag{7}$$

Moreover we have

$$z_y - y_{cs} \quad \in E^{cs}_x \subset Q_{\|\|''_y}(E^{cs}_y, \xi), \tag{8}$$

$$D_{y_{cs}+t(z_y-y_{cs})}\phi(z_y - y_{cs}) \quad \in E^u_x \subset Q_{\|\|''_y}(E^u_y, \xi). \tag{9}$$

From the above properties (7), (8), (9) and $\xi < \frac{1}{4}$, one deduces after an easy computation that $\|\theta'(t)\|''_y \leq 3\|z_y - y_{cs}\|''_y$ for all $t \in [0,1]$. For $w \in \Lambda^*$ let $d''_{\mathcal{W}^{cs}_w}$ be the distance induced respectively by $\|\|''_w$ on $\mathcal{W}^{cs}_w$. We have

$$\begin{aligned} d''_{\mathcal{W}^{cs}_y}(y,t_y) &\leq \int_{[0,1]} \|\theta'(t)\|''_y \, dt \\ &\leq 3\|y_{cs}-z_y\|''_y \\ &\leq 12a_1e^{k\gamma}\theta_n. \end{aligned}$$

(ii) Dynamical estimates along $\mathcal{W}^{cs}$

By the local invariance of center-stable manifolds stated in Proposition 1 (iii) we get for all $0<j\leq n$:

$$d''_{\mathcal{W}^{cs}_{f^j(y)}}\left(f^j(y),f^j(t_y)\right) \quad \leq \quad \|Df\,|_{T\mathcal{W}^{cs}_{f^{j-1}(y)}}\|''_{f^{j-1}(y)} d''_{\mathcal{W}^{cs}_{f^{j-1}(y)}}\left(f^{j-1}(y),f^{j-1}(t_y)\right),$$

and then by Lemma 1 (i),

$$d''_{\mathcal{W}^{cs}_{f^j(y)}}\left(f^j(y),f^j(t_y)\right) \quad \leq \quad e^{4\gamma} d''_{\mathcal{W}^{cs}_{f^{j-1}(y)}}\left(f^{j-1}(y),f^{j-1}(t_y)\right).$$

After an immediate induction we obtain for all $0\leq j\leq n$:

$$d''_{\mathcal{W}^{cs}_{f^j(y)}}\left(f^j(y),f^j(t_y)\right) \quad \leq \quad e^{4j\gamma} d''_{\mathcal{W}^{cs}_y}(y,t_y),$$

and therefore

$$\begin{aligned} d''_{\mathcal{W}^{cs}_{f^j(y)}}\left(f^j(y),f^j(t_y)\right) &\leq 12a_1e^{k\gamma}e^{4n\gamma}\theta_n \\ &\leq 12a_1e^{k\gamma}\beta_k e^{-nl\gamma'} \quad (\gamma'=\gamma/\alpha), \\ d''_{\mathcal{W}^{cs}_{f^j(y)}}\left(f^j(y),f^j(t_y)\right) &\leq 12a_1e^{k\gamma}\frac{\beta_k}{\gamma_{\kappa(y)}}\gamma_{\kappa(f^j(y))}, \end{aligned}$$

where $\kappa(p)$ and $\gamma_{\kappa(p)}$ for $p\in\cup_{k\in\mathbb{N}}\Lambda_k$ are defined as in Page 5. As y belongs to $E'_n\subset\Lambda_k$ we have $\kappa(y)\leq k$. Therefore we get for $\beta_k\leq\frac{b_\xi}{48a_1e^{k\gamma}}$:

$$\begin{aligned} \forall\, 0\leq j\leq n,\ d(f^j(t_y),f^j(y)) &\leq 2d''_{\mathcal{W}^{cs}_{f^j(y)}}\left(f^j(y),f^j(t_y)\right) \\ &\leq \frac{b_\xi}{2}\gamma_{\kappa(f^j(y))}, \end{aligned}$$

and we have similarly for $\beta_k<\frac{\delta}{48a_1e^{k\gamma}}$:

$$\forall\, 0\leq j\leq n,\ d(f^j(t_y),f^j(y))\leq\delta/4,$$

$$\text{i.e. } t_y\in B_n(y,\delta/4,f).$$

For $y\in E'_n$ we let now

$$\begin{aligned} W_n(t_y) &:= \bigcap_{j=0}^{n-1} f^{-j}\left(B''_{f^j(y)}(f^j(t_y),\frac{\delta}{8}e^{-lj\gamma'})\right)\bigcap I_z \\ &\subset B_n(t_y,\delta/4,f) \end{aligned}$$

$$\subset \quad B_n(y, \delta/2, f).$$

As E'_n is (n,δ)-separated, the sets $(W_n(t_y))_{y\in E'_n}$ are pairwise disjoint.

(iii) Dynamical estimates along E^u

For δ small enough (depending only on k), for any $j = 0, \cdots, n-1$, the ball $B''_{f^j(y)}\left(f^j(t_y), \frac{\delta}{8}e^{-lj\gamma'}\right)$ is contained in $B\left(f^j(y), b_\xi\gamma^l_{\kappa(f^j(y))}\right)$, since $d(f^j(t_y), f^j(y)) \leq \frac{b_\xi}{2}\gamma^l_{\kappa(f^j(y))}$. Let (e^i_x) be an orthonormal basis of E^u_x with respect to $\|\cdot\|''_x$. We consider the affine reparametrization of I_z, $z \in \mathcal{A}^{cs}$, given by $\sigma_z : [0,1]^{l_\nu} \to M$, $(t_i)_i \mapsto z + \sum_i (t_i - 1/2)4a_1 e^{k\gamma}\varepsilon_k e^i_x$. Noting that $E^u_x \in Q_{\|\|''_y}(E^u_y, \xi)$, by Lemma 1 (ii), for any $\tau \in \sigma_z^{-1}W_n(t_y)$ and for any $0 \leq j \leq n$, the vector space $D_{\sigma_z(\tau)}f^j(E^u_x)$ lies in $Q_{\|\|''_{f^j(y)}}\left(E^u_{f^j(y)}, \xi\right)$. Then by Lemma 1 (iii) we get

$$\begin{aligned}
\limsup_n \frac{1}{n}\log\|\curlywedge^l D_{\sigma_z(\tau)}f^n|_{E^u_x}\|''_y &= \limsup_n \frac{1}{n}\sum_{j=0}^{n-1}\log\|\curlywedge^l D_{f^j\circ\sigma_z(\tau)}f|_{D_{\sigma_z(\tau)}f^j(E^u_x)}\|''_{f^j(y)} \\
&\leq \limsup_n \frac{1}{n}\sum_{j=0}^{n-1}\log\|\curlywedge^l D_{f^j(y)}f|_{E^u_{f^j(y)}}\|''_{f^j(y)} + \gamma \\
&= \limsup_n \frac{1}{n}\log\|\curlywedge^l D_y f^n|_{E^u_y}\|''_y + \gamma.
\end{aligned}$$

Noting that $f^n(y) \in \Lambda_{k+n}$, we have by the Inequalities (4)

$$\forall v \in T_{f^n(\sigma_z(\tau))}M, \ \frac{\|v\|}{2} \leq \|v\|''_{f^n(y)} \leq 2a_1e^{(k+n)\gamma}\|v\|.$$

Then it follows from Lemma 2 in the Appendix that

$$\begin{aligned}
\limsup_n \frac{1}{n}\log\|\curlywedge^l D_\tau(f^n\circ\sigma_z)\| &= \limsup_n \frac{1}{n}\log\|\curlywedge^l D_{\sigma_z(\tau)}f^n|_{E^u_x}\| \\
&\leq \limsup_n \frac{1}{n}\log\|\curlywedge^l D_{\sigma_z(\tau)}f^n|_{E^u_x}\|''_y + l\gamma \\
&\leq \limsup_n \frac{1}{n}\log\|\curlywedge^l (D_y f^n|_{E^u_y})\|''_y + (l+1)\gamma \\
&\leq \limsup_n \frac{1}{n}\log\|\curlywedge^l (D_y f^n|_{E^u_y})\| + (2l+1)\gamma \\
&= \sum_i \lambda^+_i(f,\nu) + (2l+1)\gamma.
\end{aligned}$$

Similarly we also get :

$$\liminf_n \frac{1}{n}\log\|\curlywedge^l D_\tau(f^n\circ\sigma_z)\| \geq \sum_i \lambda^+_i(f,\nu) - (2l+1)\gamma.$$

Moreover the above limsup and liminf are uniform in $y \in E'_n$ and $\tau \in \sigma_z^{-1} W_n(t_y)$. Therefore for some $C > 1$ we have for n large enough,

$$\sigma_z^{-1} W_n(t_y) \subset \mathcal{H}_f^n\left(\sigma_z, \sum_i \lambda_i^+(f,\nu), (2l+2)\gamma, C\right).$$

By using Lemma 1 and classical arguments of graph transform, the set $f^j(W_n(t_y))$ for $0 \leq j \leq n-1$ defines a graph of a function from $B''_{f^j(y)}\left(f^j(t_y), \frac{\delta}{8} e^{-lj\gamma'}\right) \cap E^u_{f^j(y)}$ to $E^{cs}_{f^j(y)}$. Therefore the l-volume of $f^{n-1}(W_n(t_y))$ with respect to $\|\cdot\|''_{f^{n-1}(y)}$ satisfies

$$\left|f^{n-1}(W_n(t_y))\right|''_{f^{n-1}(y)} \geq c_l \delta^l e^{-l^2(n-1)\gamma'}, \tag{10}$$

for some universal constant c_l. By applying again Lemma 2 in the Appendix we obtain :

$$\left|f^{n-1}(W_n(t_y))\right| \geq (4a_1 e^{(k+n-1)\gamma})^{-l} \left|f^{n-1}(W_n(t_y))\right|''_{f^{n-1}(y)}, \tag{11}$$

where $\left|f^{n-1}(W_n(t_y))\right|$ denotes the l-volume of $f^{n-1}(W_n(t_y))$ with respect to the Riemannian norm $\|\cdot\|$ on M. For $z \in \mathcal{A}^{cs}$ we let

$$\Gamma_z := \{y \in E'_n,\ z_y = z\}$$

and

$$\Delta_n^z := \mathcal{H}_f^n\left(\sigma_z, \sum_i \lambda_i^+(f,\nu), (2l+2)\gamma, C\right) \cap \sigma_z^{-1} B_n(x, 2\varepsilon, f).$$

As the sets $W_n(t_y)$, $y \in E'_n$ are pairwise disjoint, we have :

$$\left|(f^{n-1} \circ \sigma_z)_{|\Delta_n^z}\right| \geq \sum_{y \in \Gamma_z} \left|f^{n-1}(W_n(t_y))\right|. \tag{12}$$

By combining the inequalities (10), (11), (12) we obtain

$$\begin{aligned} \left|(f^{n-1} \circ \sigma_z)_{|\Delta_n^z}\right| &\geq (4a_1 e^{(k+n-1)\gamma})^{-l} \sum_{y \in \Gamma_z} \left|f^{n-1}_{|W_n(t_y)}\right|''_{f^{n-1}(y)} \\ &\geq c_l \delta^l e^{-l^2(n-1)\gamma'} (4a_1 e^{(k+n-1)\gamma})^{-l} \cdot \sharp\Gamma_z. \end{aligned}$$

With the notations introduced at the beginning of Section 3, we have therefore for some constant D independent of n and $\hat{x} \in F_\eta$,

$$\#\Gamma_z \leq D e^{n(l\gamma + l^2\gamma')} V_x^{n,2\varepsilon}\left(\sigma_z \middle| \sum_i \lambda_i^+(f,\nu), (2l+2)\gamma, C\right).$$

Now we are in a position to complete the proof of Proposition 2.

By letting $\mathcal{F}_{n,\delta} := \{\sigma_z,\ z \in \mathcal{A}^{cs}\}$, we get for all $\hat{x} \in \Lambda_k$ and some constants, all denoted by D and independent of n and $\hat{x} \in F_\eta$:

$$\begin{aligned} s(n, \delta, B_n(\hat{x}, \varepsilon, f)) &= \sharp E_n \\ &\leq D \sharp E'_n \\ &\leq D \sum_{z \in \mathcal{A}^{cs}} \sharp\Gamma_z \end{aligned}$$

$$
\begin{aligned}
&\leq De^{n(l\gamma+l^2\gamma')}\sharp\mathcal{A}^{cs}\sup_{\sigma\in\mathcal{F}_{n,\delta}}V_x^{n,2\varepsilon}\left(\sigma_z\middle|\sum_i\lambda_i^+(f,\nu),(2l+2)\gamma,C\right)\\
&\leq De^{n((\mathsf{d}-l)(4\gamma+l\gamma')+l\gamma+l^2\gamma')}\sup_{\sigma\in\mathcal{F}_{n,\delta}}V_x^{n,2\varepsilon}\left(\sigma_z\middle|\sum_i\lambda_i^+(f,\nu),(2l+2)\gamma,C\right).
\end{aligned}
$$

This concludes the proof of the Proposition 2 as γ and thus $\gamma'=\alpha^{-1}\gamma$ may be chosen arbitrarily small. □

4. Proof of Main Theorem

By Proposition 2 Newhouse local entropy of an ergodic measure with one positive Lyapunov exponent is bounded from above by the local volume growth of curves. This volume growth may be controled by using the Reparametrization Lemma of [6]. Following straightforwardly the proof of the Main Proposition in [6] we get :

Proposition 3. *Let f be a $\mathcal{C}^r$ diffeomorphism with $r>1$ on a Riemannian manifold M and $\mu\in\mathcal{M}_{inv}(f)$. For all $\gamma>0$, there exist $m_\mu,k_\mu\in\mathbb{N}^*$ such that for $\nu\in\mathcal{M}_{erg}(f)$ close enough to μ with $l_u(f,\nu)=1$, we have*

$$h^{New}_{m_\mu,k_\mu}(f,\nu)\leq\frac{\lambda_1^+(f,\mu)-\lambda_1^+(f,\nu)}{r-1}+\gamma.$$

From the criterion in Theorem 1, for proving the Main Theorem, we need to consider all ergodic measures with any possible l_u. Actually, the Main Theorem is obtained from the following Proposition by applying Theorem 1 with the upper semicontinuous affine function $E:=\frac{1}{r-1}\sum_{i=1,2}\lambda_i^+(f,\cdot)$.

Proposition 4. *Let f be a $\mathcal{C}^r$ diffeomorphism with $r>1$ on a 3-dimensional Riemannian manifold M and $\mu\in\mathcal{M}_{inv}(f)$. For all $\gamma>0$, there exist an entropy structure $(h_k)_k$ and $k_\mu\in\mathbb{N}$ such that for $\nu\in\mathcal{M}_{erg}(f)$ close enough to μ, we have*

$$h_{k_\mu}(f,\nu)\leq\frac{\sum_{i=1,2}\lambda_i^+(f,\mu)-\sum_{i=1,2}\lambda_i^+(f,\nu)}{r-1}+\gamma.$$

In other terms, $E:=\frac{1}{r-1}\sum_{i=1,2}\lambda_i^+(f,\cdot)$ satisfies Inequaliy (1) for a 3-dimensional $\mathcal{C}^r$ diffeomorphism f with $r>1$.

Proof of Proposition 4. Fix $\mu\in\mathcal{M}_{inv}(f)$. By the upper semicontinuity of $\sum_{i=1,2}\lambda_i^+(f,\cdot)$, lower semicontinuity of $\lambda_3(f,\cdot)$ and continuity of the integral of logarithm for Jacobian, when ν is close enough to μ, one has

$$\sum_{i=1,2}\lambda_i^+(f,\mu)-\sum_{i=1,2}\lambda_i^+(f,\nu)\geq-\frac{(r-1)\gamma}{2},\tag{13}$$

$$\lambda_3(f,\mu)-\lambda_3(f,\nu)\leq\frac{\gamma}{2},\tag{14}$$

$$\left|\int\log\mathrm{Jac}(f)\,d\nu-\int\log\mathrm{Jac}(f)\,d\mu\right|\leq(r-1)\gamma.\tag{15}$$

Hence, if $h_\nu(f) \leq \gamma/2$, from $h^{New}_{m,k}(f,\nu) \leq h_\nu(f)$ for any m,k, by (13) we get

$$h^{New}_{m,k}(f,\nu) \leq \frac{\sum_{i=1,2}\lambda_i^+(f,\mu) - \sum_{i=1,2}\lambda_i^+(f,\nu)}{r-1} + \gamma. \tag{16}$$

Next we assume $h_\nu(f) > \gamma/2$. By Ruelle inequality [23], it holds that $\min\left(l_u(f,\nu), l_u(f^{-1},\nu)\right) = 1$. Applying Proposition 3 to $f^{\pm}$, there exist $m^{\pm}_\mu, k^{\pm}_\mu \in \mathbb{N}$ such that for any $\nu \in \mathcal{M}_{erg}(f)$ close enough to μ with $l_u(f^{\pm},\nu) = 1$,

$$h^{New}_{m^{\pm}_\mu, k^{\pm}_\mu}(f^{\pm},\nu) \leq \frac{\lambda_1^+(f^{\pm},\mu) - \lambda_1^+(f^{\pm},\nu)}{r-1} + \gamma.$$

If $l_u(f,\nu) = 1$, then $\sum_{i=1,2}\lambda_i^+(f,\nu) = \lambda_1^+(f,\nu)$, thus by the above inequality, (16) holds with respect to m^+_μ, k^+_μ. If $l_u(f^{-1},\nu) = 1$, then $\lambda_3(f,\nu) \leq -h_\nu(f^{-1}) = -h_\nu(f) < -\gamma/2$, which implies $\lambda_3(f,\mu) < 0$ by (14). Thus,

$$\begin{aligned}\lambda_1^+(f^{-1},\mu) - \lambda_1^+(f^{-1},\nu) &= \lambda_3^-(f,\nu) - \lambda_3^-(f,\mu) \\ &= \lambda_3(f,\nu) - \lambda_3(f,\mu).\end{aligned}$$

Noting that $\int \log \mathrm{Jac}(f)\, d\tau = \sum_{i=1,2,3}\lambda_i(f,\tau)$ for any $\tau \in \mathcal{M}_{inv}(f)$, by (15) we finally get

$$\begin{aligned}\lambda_1^+(f^{-1},\mu) - \lambda_1^+(f^{-1},\nu) &= \sum_{i=1,2}\lambda_i(f,\mu) - \sum_{i=1,2}\lambda_i(f,\nu) + \int \log \mathrm{Jac}(f)\, d\nu - \int \log \mathrm{Jac}(f)\, d\mu \\ &\leq \sum_{i=1,2}\lambda_i^+(f,\mu) - \sum_{i=1,2}\lambda_i^+(f,\nu) + (r-1)\gamma\end{aligned}$$

and therefore

$$h^{New}_{m^-_\mu, k^-_\mu}(f^{-1},\nu) \leq \frac{\sum_{i=1,2}\lambda_i^+(f,\mu) - \sum_{i=1,2}\lambda_i^+(f,\nu)}{r-1} + \gamma. \tag{17}$$

By Lemma 2 in [6], the sequence $(\underline{h_k})_k := (\min(h^{New}_{m^+_\mu,k}(f,\cdot), h^{New}_{m^-_\mu,k}(f^{-1}\cdot)))$ defines an entropy structure. Combining (16) for $l_u(f,\nu) = 1$ and (17) for $l_u(f^{-1},\nu) = 1$, we conclude the proof of Proposition 4 and thus also of the Main Theorem by considering the entropy structure $(\underline{h_k})_k$. □

Remark 2. *For a local diffeomorphism $f : M \to M$, the following local Ruelle inequlity holds [7][10] : there exists a scale $\varepsilon > 0$ such that $h^*(f,\mu,\varepsilon) \leq \min\left(\sum_j \lambda_j^+(f,\mu), -\sum_j \lambda_j^-(f,\mu)\right)$ for any $\mu \in \mathcal{M}_{inv}(f)$. In particular in dimension 3, any invariant measure with positive Newhouse local entropy admits at least one positive and one negative Lyapunov exponent. As the proofs of Main Theorem and Proposition 2 are just local they apply verbatim in the context of a local 3-dimensional diffeomorphism.*

Appendix A.

Let E and F be two finite dimensional vector spaces of dimension k. We endow E (resp. F) with two Euclidean norms $\|\cdot\|_E$ and $\|\cdot\|'_E$ (resp. $\|\cdot\|_F$ and $\|\cdot\|'_F$). We consider the associated Euclidean structures on $\bigwedge^k E$ (resp. $\bigwedge^k F$). Let $A : E \to F$ be an invertible linear map and $\bigwedge^k A$ the induced map on the k-exterior powers. We denote by $\|\bigwedge^k A\|$ and $\|\bigwedge^k A\|'$ the associated subordinated norms.

Lemma 2. *With the above notations, assume that we have for some constants $C_E, C_F \geq 1$ and $D_E, D_F \leq 1$:*

$$\forall v \in E, \quad D_E\|v\|_E \leq \|v\|'_E \leq C_E\|v\|_E,$$
$$\forall w \in F, \quad D_F\|w\|_F \leq \|w\|'_F \leq C_F\|w\|_F,$$

then

$$(D_F/C_E)^k \| \bigwedge^k A\| \leq \| \bigwedge^k A\|' \leq (C_F/D_E)^k \| \bigwedge^k A\|.$$

Proof. By the singular value decomposition there exists an orthonormal family $(e_i)_{i=1,\cdots,k}$ of $(E, \|\cdot\|_E)$ such that $(Ae_i)_i$ is an orthogonal family in $(F, \|\cdot\|_F)$ with $\|\bigwedge^k A\| = \|Ae_1 \cdots \wedge Ae_k\|_F = \prod_{i=1}^k \|Ae_i\|_F$. Similarly we let $(e'_i)_{i=1,\cdots,k}$ be the corresponding orthonormal family for the norms $\|\cdot\|'_E$ and $\|\cdot\|'_F$. Let P be the change of basis matrix from $(e'_i)_i$ to $(e_i)_i$. Then the norms $\|e_1 \wedge \cdots \wedge e_k\|'_E$ and $\|e'_1 \wedge \cdots \wedge e'_k\|_E$ are just given by the absolute values of the determinants of P and P^{-1} respectively. Therefore we have

$$\begin{aligned} |\det(P^{-1})| &\leq \prod_i \|e'_i\|_E \\ &\leq D_E^{-k}, \end{aligned}$$

and

$$\begin{aligned} \|e_1 \wedge \cdots \wedge e_k\|'_E &= |\det(P)| \\ &= 1/|\det(P^{-1})| \\ &\geq D_E^k. \end{aligned}$$

We conclude that

$$\begin{aligned} \| \bigwedge^k A\|' &\leq \frac{\|Ae_1 \wedge \cdots \wedge Ae_k\|'_F}{\|e_1 \wedge \cdots \wedge e_k\|'_E} \\ &\leq D_E^{-k} \prod_i \|Ae_i\|'_F \\ &\leq D_E^{-k} C_F^k \prod_i \|Ae_i\|_F \\ &\leq (C_F/D_E)^k \| \bigwedge^k A\|. \end{aligned}$$

The other inequality is obtained symmetrically.

□

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