SRB MEASURES FOR C^∞ SURFACE DIFFEOMORPHISMS

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Abstract. A C^∞ surface diffeomorphism admits a SRB measure if and only if the set $\{x,\ \limsup_n \frac{1}{n}\log\|d_xf^n\|>0\}$ has positive Lebesgue measure. Moreover the basins of the ergodic SRB measures are covering this set Lebesgue almost everywhere. We also obtain similar results for C^r surface diffeomorphisms with $+\infty > r > 1$.

Contents

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1. Introduction

One fundamental problem in dynamics consists in understanding the statistical behaviour of the system. Given a topological system (X, f) we are more precisely interesting in the asymptotic distribution of the empirical measures $\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^kx}\right)_n$ for typical points x with respect to a reference measure. In the setting of differentiable dynamical systems the natural reference measure to consider is the Lebesgue measure on the manifold.

The basin of a f-invariant measure μ is the set $\mathcal{B}(\mu)$ of points whose empirical measures are converging to μ in the weak-$*$ topology. By Birkhoff's ergodic theorem the basin of an ergodic measure μ has full μ-measure. An invariant measure is said physical when its basin has positive Lebesgue measure. We may wonder when such measures exist and then study their basins.

In the works of Y. Sinai, D. Ruelle and R. Bowen [40, 11, 36] these questions have been successfully solved for uniformly hyperbolic systems. A SRB measure of a C^{1+} system is an invariant probability measure with at least one positive Lyapunov exponent almost everywhere, which has absolutely continuous conditional measures on unstable manifolds [43]. Physical measures may neither be SRB measures nor sinks (as in the famous figure-eight attractor), however hyperbolic ergodic SRB measures are physical measures. For uniformly hyperbolic systems, there is a finite number of such measures and their basins cover a full Lebesgue subset of the manifold. Beyond the uniformly hyperbolic case such a picture is also known for large classes of partially hyperbolic systems [10, 2, 1]. Corresponding results have been established for unimodal maps with negative Schwartzian derivative [25]. SRB measures have been also deeply investigated for parameter families such as the quadratic family and Henon maps [24, 5, 6, 7]. In his celebrated ICM's talk, M. Viana conjectured that a surface diffeomorphism admits a SRB measure, whenever the set of points with positive Lyapunov exponent has positive Lebesgue measure. In recent works some weaker versions of the conjecture (with some additional assumptions of recurrence and Lyapunov regularity) have been proved [20, 8]. Finally we mention that in the present context of C^∞ surface diffeomorphims J. Buzzi, S. Crovisier, O. Sarig have also recently shown the existence of a SRB measure when the set of points with a positive Lyapunov exponent has positive Lebesgue measure [18] (Corollary 2).

In this paper we define a general entropic approach to build SRB measures, which we apply to prove Viana's conjecture for C^∞ surface diffeomorphisms. We strongly believe the same approach may be used to recover the existence of SRB measures for weakly mostly expanding partially hyperbolic systems [1] and to give another proof of Ben Ovadia's criterion for C^{1+} diffeomorphisms in any dimension [8].

We state now the main results of our paper. Let $(M, \|\cdot\|)$ be a compact Riemannian surface and let Leb be a volume form on M, called Lebesgue measure. We consider a C^∞ surface diffeomorphism $f : M \circlearrowleft$. The maximal Lyapunov exponent at $x \in M$ is given by $\chi(x) = \limsup_n \frac{1}{n}\log\|d_xf^n\|$. When μ is a f-invariant probability measure, we let $\chi(\mu) = \int \chi(x)\, d\mu(x)$. For two Borel subsets A and B of M we write $A \overset{o}{\subset} B$ (resp. $A \overset{o}{=} B$) when we have $\mathrm{Leb}(A \setminus B) = 0$ (resp. $\mathrm{Leb}(A\Delta B) = 0$).

Theorem 1. *Let $f : M \circlearrowleft$ be a C^∞ surface diffeomorphism. There are countably many ergodic SRB measures $(\mu_i)_{i\in I}$, such that we have with $\Lambda = \{\chi(\mu_i),\ i \in I\} \subset \mathbb{R}_{>0}$:*

- $\{\chi > 0\} \overset{o}{=} \{\chi \in \Lambda\}$,

- $\{\chi = \lambda\} \overset{o}{\subset} \bigcup_{i,\chi(\mu_i)=\lambda} \mathcal{B}(\mu_i)$ *for all* $\lambda \in \Lambda$.

Corollary 1. *Let $f : M \circlearrowleft$ be a C^∞ surface diffeomorphism. Then*

$$\{\chi > 0\} \overset{o}{\subset} \bigcup_{\mu \text{ SRB ergodic}} \mathcal{B}(\mu).$$

Corollary 2 (Buzzi-Crovisier-Sarig [18])**.** *Let $f : M \circlearrowleft$ be a C^∞ surface diffeomorphism. If* $\mathrm{Leb}(\chi > 0) > 0$*, then there exists a SRB measure.*

In fact we establish a C^r, $1 < r < +\infty$, stronger version, which implies straightforwardly Theorem 1:

Main Theorem. *Let $f : M \circlearrowleft$ be a C^r, $r > 1$, surface diffeomorphism. Let $R(f) := \lim_n \frac{1}{n} \log^+ \sup_{x\in M} \|d_x f^n\|$. There are countably many ergodic SRB measures $(\mu_i)_{i\in I}$ with $\Lambda := \{\chi(\mu_i),\ i \in I\} \subset]\frac{R(f)}{r}, +\infty[$, such that we have :*

- $\left\{\chi > \frac{R(f)}{r}\right\} \overset{o}{=} \{\chi \in \Lambda\}$,
- $\{\chi = \lambda\} \overset{o}{\subset} \bigcup_{i,\chi(\mu_i)=\lambda} \mathcal{B}(\mu_i)$ *for all* $\lambda \in \Lambda$.

When f is a C^{1+} topologically transitive surface diffeomorphism, there is at most one SRB measure, i.e. $\sharp I \leq 1$ [23]. If moreover the system is topologically mixing, then the SRB measure when it exists is Bernoulli [16]. By the spectral decomposition of C^r surface diffeomorphisms for $1 < r \leq +\infty$ [16] there are at most finitely many ergodic SRB measures with entropy and thus maximal exponent larger than a given constant $b > \frac{R(f)}{r}$. Therefore, in the Main Theorem, the set $\Lambda = \{\chi(\mu_i),\ i \in I\}$ is either finite or a sequence decreasing to $\frac{R(f)}{r}$. When r is finite, there may also exist ergodic SRB measures μ with $\chi(\mu) \leq \frac{R(f)}{r}$.

We prove in a forthcoming paper [12] that the above statement is sharp by building for any finite $r > 1$ a C^r surface diffeomorphism (f, M) with a periodic saddle hyperbolic point p such that $\chi(x) = \frac{R(f)}{r} > 0$ for all $x \in U$ for some set $U \subset \mathcal{B}(\mu_p)$ with $\mathrm{Leb}(U) > 0$, where μ_p denotes the periodic measure associated to p (see [14] for such an example of interval maps).

In higher dimensions we let $\Sigma^k\chi(x) := \limsup_n \frac{1}{n}\|\Lambda^k d_x f^n\|$ where $\Lambda^k df$ denotes the action induced by f on the k^{th} exterior power of TM for $k = 1, \cdots, d$ with d being the dimension of M. By convention we also let $\Sigma^0\chi = 0$. For any C^1 diffeomorphism (M, f) we have $\mathrm{Leb}(\Sigma^d\chi > 0) = 0$ (see [3]). The product of a figure-eight attractor with a surface Anosov diffeomorphism does not admit any SRB measure whereas χ is positive on a set of positive Lebesgue measure. However we conjecture :

Conjecture. *Let $f : M \circlearrowleft$ be a C^∞ diffeomorphism on a compact manifold (of any dimension).*

If $\mathrm{Leb}\left(\Sigma^k\chi > \Sigma^{k-1}\chi \geq 0\right) > 0$*, then there exists an ergodic measure with at least k positive Lyapunov exponents, such that its entropy is larger than or equal to the sum of its k smallest positive Lyapunov exponents.*

In the present two-dimensional case the semi-algebraic tools used to bound the distorsion and the local volume growth of C^∞ curves are elementary. This is a challenging problem to adapt this technology in higher dimensions.

When the empirical measures from $x \in M$ are not converging, the point x is said to have historic behaviour [37]. A set U is contracting when the diameter of $f^n U$ goes to zero when $n \in \mathbb{N}$ goes to infinity. In a contracting set the empirical measures of all points have the same limit set, however they may not converge. P. Berger and S. Biebler have shown that C^∞ densely inside the Newhouse domains [9] there are contracting domains with historic behaviour. In intermediate smoothness, such domains have been previously built in [27]. As a consequence of the Main Theorem, Lebesgue almost every point x with historic behaviour satisfies $\chi(x) \leq 0$ for C^∞ surface diffeomorphisms. We also show the following statement.

Theorem 2. *Let f be a C^∞ diffeomorphism on a compact manifold (of any dimension). Then Lebesgue a.e. point x in a contracting set satisfies $\chi(x) \leq 0$.*

Question. Let f be a C^∞ surface diffeomorphism. Assume the set H of points with historic behaviour has positive Lebesgue measure. Does every Lebesgue density point of H belongs to a almost contracting* set with positive Lebesgue measure?

We explain now in few lines the main ideas to build a SRB measure under the assumptions of the Main Theorem. The geometric approach for uniformly hyperbolic systems consists in considering a weak limit of $\left(\frac{1}{n}\sum_{k=0}^{n-1} f^k_* \operatorname{Leb}_{D_u}\right)_n$, where D_u is a local unstable disc and Leb_{D_u} denotes the normalized Lebesgue measure on D_u induced by its inherited Riemannian structure as a submanifold of M. Here we take a smooth C^r embedded curve D such that

$$\chi(x, v_x) := \limsup_n \frac{1}{n} \log \|d_x f^n(v_x)\| > b > \frac{R(f)}{r}$$

for (x, v_x) in the unit tangent space $T^1 D$ of D with x in a subset B of D with positive Leb_D-measure. For x in B we define a subset $E(x)$ of positive integers, called the *geometric set*, such that the following properties hold for any $n \in E(x)$:

- the geometry of $f^n D$ around $f^n x$ is *bounded* meaning that for some uniform $\epsilon > 0$, the connected component $D_n^\epsilon(x)$ of $f^n D$ with the ball at $f^n x$ of radius $\epsilon > 0$ is a curve with bounded s-derivative for $s \leq r$,
- the distorsion of df^{-n} on the tangent space of $D_n^\epsilon(x)$ is controlled,
- for some $\tau > 0$ we have $\frac{\|d_x f^l(v_x)\|}{\|d_x f^k(v_x)\|} \geq e^{(l-k)\tau}$ for any $l > k \in E(x)$.

We show that $E(x)$ has positive upper asymptotic density for x in a subset A of B with positive Leb_D-measure. Let $F : \mathbb{P}TM \circlearrowleft$ be the map induced by f on the projective tangent bundle $\mathbb{P}TM$. We build a SRB measure by considering a weak limit μ of a sequence of the form $\left(\frac{1}{\sharp F_n}\sum_{k\in F_n} F^k_* \mu_n\right)_n$ such that :

- $(F_n)_n$ is a Fölner sequence, so that the weak limit μ will be invariant by F,
- for all n, the measure μ_n is the probability measure induced by Leb_D on $A_n \subset A$, the Leb_D-measure of A_n being not exponentially small,
- the sets $(F_n)_n$ are in some sense *filled with* the geometric set $E(x)$ for $x \in A_n$. Then the measure μ on $\mathbb{P}TM$ will be supported on the unstable Oseledec's bundle.

Finally we check with some Fölner Gibbs property that the limit empirical measure μ projects to a SRB measure on M by using the Ledrappier-Young entropic characterization.

*See Section 7 for the definition of almost contracting set.

The paper is organized as follows. In Section 2 we recall for general sequences of integers the notion of asymptotical density and we build for any sequence E with positive upper density a Fölner set F filled with E. Then we use a Borel-Cantelli argument to define our sets $(A_n)_n$ and the Fölner sequence $(F_n)_n$. In Section 3, we study the maximal Lyapunov exponent and the entropy of the generalized empirical measure μ assuming some Gibbs property. We introduce the geometric set in Section 4 by using the Reparametrization Lemma of [13]. We build then SRB measures in Section 5 by using the abstract formalism of Section 2 and 3. Then we prove the covering property of the basins in Section 6 by the standard argument of absolute continuity of Pesin stable foliation. The last section is devoted to the proof of Theorem 2.

Comment : In a first version of this work, by following [13] (incorrectly) the author claimed that, at b-hyperbolic times n of the sequence $\left(\|d_x f^k(v_x)\|\right)_k$ for some $b > 0$, the geometry of $f^n D$ at $f^n x$ was bounded. J. Buzzi, S. Crovisier and O. Sarig gave then in [18] another proof of Corollary 2 by using their analysis of the entropic continuity of Lyapunov exponents from [17]. But as realized recently, our claim on the geometry at hyperbolic times is wrong in general and we manage to show it only when $\chi(x) > \frac{R(f)}{2}$. In this last version, we correct our proof by showing directly that the set of times with bounded geometry has positive upper asymptotic density on a set of positive Leb_D-measure. Our proof is still based on the Reparametrization Lemma proved in [13].

2. Some asymptotic properties of integers

2.1. **Asymptotic density.** We first introduce some notations. In the following we let $\mathcal{P}_\mathbb{N}$ and $\mathcal{P}_n$ be respectively the power sets of $\mathbb{N}$ and $\{1, 2, \cdots, n\}$, $n \in \mathbb{N}$. The **boundary** ∂E of $E \in \mathcal{P}_\mathbb{N}$ is the subset of $\mathbb{N}$ consisting in the integers $n \in E$ with $n-1 \notin E$ or $n+1 \notin E$. We also let $E^- := \{n \in E,\ n+1 \in E\}$. For $a, b \in \mathbb{N}$ we write $[\![a,b]\!]$ (resp. $[\![a,b[\![$, $]\!]a,b]\!]$) the interval of integers k with $a \leq k \leq b$ (resp $a \leq k < b$, $a < k \leq b$). The **connected components** of E are the maximal intervals of integers contained in E. An interval of integers $[\![a,b[\![$ is said E-**irreducible** when we have $a, b \in E$ and $[\![a,b[\![\cap E = \{a\}$. For $E \in \mathcal{P}_\mathbb{N}$ we let $E_{(n)} := E \cap [\![1,n]\!] \in \mathcal{P}_n$ for all $n \in \mathbb{N}$. For $M \in \mathbb{N}$, we denote by E_M the union of the intervals $[\![a,b]\!]$ with $a, b \in E$ and $|a-b| \leq M$.

We let $\mathfrak{N}$ be the set of increasing sequences of natural integers, which may be identified with the subset of $\mathcal{P}(\mathbb{N})$ given by infinite subsets of $\mathbb{N}$. For $\mathfrak{n} \in \mathfrak{N}$ we define the **generalized power set of** $\mathfrak{n}$ as $\mathcal{Q}_\mathfrak{n} := \prod_{n \in \mathfrak{n}} \mathcal{P}_n$.

We recall now the classical notion of upper and lower asymptotic densities. For $n \in \mathbb{N}^*$ and $F_n \in \mathcal{P}_n$ we let $d_n(F_n)$ be the frequency of F_n in $[\![1,n]\!]$:

$$d_n(F_n) = \frac{\sharp F_n}{n}.$$

The **upper and lower asymptotic densities** $\overline{d}(E)$ and $\underline{d}(E)$ of $E \in \mathcal{P}_\mathbb{N}$ are respectively defined by

$$\overline{d}(E) := \limsup_{n \in \mathbb{N}} d_n(E_{(n)}) \text{ and}$$

$$\underline{d}(E) := \liminf_{n \in \mathbb{N}} d_n(E_{(n)}).$$

We just write $d(E)$ for the limit, when the frequencies $d_n(E_{(n)})$ are converging. For any $\mathfrak{n} \in \mathfrak{N}$ we let similarly $\overline{d}^{\mathfrak{n}}(E) := \limsup_{n \in \mathfrak{n}} d_n(E_{(n)})$ and $\underline{d}^{\mathfrak{n}}(E) := \liminf_{n \in \mathfrak{n}} d_n(E_{(n)})$.

The concept of upper and lower asymptotic densities of $E \in \mathcal{P}_{\mathbb{N}}$ may be extended to generalized power sets as follows. For $\mathfrak{n} \in \mathfrak{N}$ and $\mathcal{F} = (F_n)_{n\in\mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ we let

$$\overline{d}^{\mathfrak{n}}(\mathcal{F}) := \limsup_{n\in\mathfrak{n}} d_n(F_n) \text{ and}$$

$$\underline{d}^{\mathfrak{n}}(\mathcal{F}) := \liminf_{n\in\mathfrak{n}} d_n(F_n).$$

Again we just write $d^{\mathfrak{n}}(E)$ and $d^{\mathfrak{n}}(\mathcal{F})$ when the corresponding frequencies are converging.

2.2. **Fölner sequence and density along subsequences.** We say that $E \in \mathcal{P}_{\mathbb{N}}$ is **Fölner along** $\mathfrak{n} \in \mathfrak{N}$ when its boundary ∂E has zero upper asymptotic density with respect to $\mathfrak{n}$, i.e. $\overline{d}^{\mathfrak{n}}(\partial E) = 0$. More generally $\mathcal{F} = (F_n)_{n\in\mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ with $\mathfrak{n} \in \mathfrak{N}$ is Fölner when we have $\overline{d}^{\mathfrak{n}}(\partial\mathcal{F}) = 0$ with $\partial\mathcal{F} = (\partial F_n)_{n\in\mathfrak{n}}$. In general this property seems to be weaker than the usual Fölner property $\limsup_{n\in\mathfrak{n}} \frac{\sharp\partial F_n}{\sharp F_n} = 0$. But in the following we will work with sequences $\mathcal{F}$ with $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. In this case our definition coincides with the standard one.

Let $E, F \in \mathcal{P}_{\mathbb{N}}$ and $\mathfrak{n} \in \mathfrak{N}$. We say that F is **$\mathfrak{n}$-filled with** E or E is **dense in** F **along** $\mathfrak{n}$ when we have

$$\overline{d}^{\mathfrak{n}}(F \setminus E_M) \xrightarrow{M\to+\infty} 0.$$

Observe that $\left(\overline{d}(E_M)\right)_M$ is converging nondecreasingly to some $a \geq \overline{d}(E)$ when M goes to infinity. The limit a is in general strictly less than 1. For example if $E := \bigcup_n [\![2^{2n}, 2^{2n+1}]\!]$ one easily computes $\overline{d}(E_M) = \overline{d}(E) = 2/3$ for all M. In this case, the set E is moreover a Fölner set.

Also $\mathcal{F} = (F_n)_{n\in\mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ is said filled with E when we have with $\mathcal{F}\setminus E_M := (F_n \setminus E_M)_{n\in\mathfrak{n}}$:

$$\overline{d}^{\mathfrak{n}}(\mathcal{F} \setminus E_M) \xrightarrow{M\to+\infty} 0.$$

2.3. **Fölner set F filled with a given E with $\overline{d}(E) > 0$.** Given a set E with positive upper asymptotic density we build a Fölner set F filled with E by using a diagonal argument. More precisely we will build F by filling the holes in E of larger and larger size when going to infinity.

Lemma 1. *For any E with $\overline{d}(E) > 0$ there is a subsequence $\mathfrak{n} \in \mathfrak{N}$ and $F \in \mathcal{P}_{\mathbb{N}}$ with $\partial F \subset E$ such that*

- $d^{\mathfrak{n}}(F) \geq d^{\mathfrak{n}}(E \cap F) = \overline{d}(E)$;
- *F is Fölner along $\mathfrak{n}$;*
- *E is dense in F along $\mathfrak{n}$.*

Proof. We first consider a subsequence $\mathfrak{n}^0 = (\mathfrak{n}^0_k)_k$ satisfying $d^{\mathfrak{n}^0}(E) = \overline{d}(E)$. We can ensure that $\mathfrak{n}^0_k$ belong to E for all k. Observe that $\overline{d}(E \setminus E_M) \leq 1/M$ for all $M \in \mathbb{N}^*$. Therefore for $M > 2/\overline{d}(E)$, we have $\overline{d}^{\mathfrak{n}^0}(E_M) \geq \overline{d}^{\mathfrak{n}^0}(E_M \cap E) > \overline{d}(E)/2 > 0$. We fix such an integer M and we extract again a subsequence $\mathfrak{n}^M = (\mathfrak{n}^M_k)_k$ of $\mathfrak{n}^0$ such that $d^{\mathfrak{n}^M}(E_M)$ is a limit equal to $\Delta_M := \overline{d}^{\mathfrak{n}^0}(E_M)$. Then we put $\Delta_{M+1} = \overline{d}^{\mathfrak{n}^M}(E_{M+1})$ and we consider a subsequence $\mathfrak{n}^{M+1}$ of $\mathfrak{n}^M$ such that $d^{\mathfrak{n}^{M+1}}(E_{M+1})$ is a limit equal to $\Delta_{M+1} \geq \Delta_M$ and $d_l(E_{M+1}) \leq \Delta_{M+1} + 1/2^{M+1}$ for all $l \in \mathfrak{n}^{M+1}$. We define by induction in this way nested sequences $\mathfrak{n}^k$ for $k > M$ such that $d^{\mathfrak{n}^k}(E_k)$ is a limit equal to $\Delta_k = \overline{d}^{\mathfrak{n}^{k-1}}(E_k)$ and $d_l(E_k) \leq$

$\Delta_k + 1/2^k$ for all $l \in \mathfrak{n}^k$. We let $\Delta_\infty > \overline{d}(E)/2 > 0$ be the limit of the nondecreasing sequence $(\Delta_k)_k$. We consider the diagonal sequence $\mathfrak{n} = (\mathfrak{n}_k)_{k\geq M} = (\mathfrak{n}_k^k)_{k\geq M}$ and we let

$$F = \bigcup_{k>M} [\![\mathfrak{n}_{k-1}, \mathfrak{n}_k]\!] \cap E_k.$$

Clearly we have $\partial F \subset \mathfrak{n}^0 \cup E \subset E$.

On the one hand, $F \cap [\![1, \mathfrak{n}_k]\!]$ is contained in $E_k \cap [\![1, \mathfrak{n}_k]\!]$ so that

$$\begin{aligned} d_{\mathfrak{n}_k}(F) &\leq d_{\mathfrak{n}_k^k}(E_k), \\ &\leq \Delta_k + 1/2^k, \\ \overline{d}^{\mathfrak{n}}(F) &\leq \lim_k \Delta_k = \Delta_\infty, \end{aligned}$$

On the other hand, $F \cap [\![1, \mathfrak{n}_k]\!]$ contains $E_l \cap [\![\mathfrak{n}_{l-1}, \mathfrak{n}_k]\!]$ for all $M < l < k$. Therefore

$$\begin{aligned} d_{\mathfrak{n}_k^k}(E_l) - \frac{\mathfrak{n}_{l-1}}{\mathfrak{n}_k^k} &\leq d_{\mathfrak{n}_k}(F), \\ \Delta_l &\leq \underline{d}^{\mathfrak{n}}(F), \\ \Delta_\infty &\leq \underline{d}^{\mathfrak{n}}(F). \end{aligned}$$

We conclude $d^{\mathfrak{n}}(F) = \Delta_\infty$.

Similarly we have for all $l > M$:

$$\begin{aligned} \underline{d}^{\mathfrak{n}}(E \cap F) &\geq \underline{d}^{\mathfrak{n}}(E \cap E_l), \\ &\geq d^{\mathfrak{n}}(E) - 1/l, \\ &\geq \overline{d}^{\mathfrak{m}}(E) - 1/l, \end{aligned}$$

therefore $\underline{d}^{\mathfrak{n}}(E \cap F) \geq \overline{d}(E)$. Also $\overline{d}^{\mathfrak{n}}(E \cap F) \leq d^{\mathfrak{n}}(E) = \overline{d}^{\mathfrak{m}}(E)$. Consequently we get $d^{\mathfrak{n}}(E \cap F) = \overline{d}^{\mathfrak{m}}(E)$.

We check now that E is dense in F. For l fixed and for all $k \geq l$ we have

$$\begin{aligned} d_{\mathfrak{n}_k}(F \setminus E_l) &\leq d_{\mathfrak{n}_k^k}(E_k \setminus E_l), \\ &\leq d_{\mathfrak{n}_k^k}(E_k) - d_{\mathfrak{n}_k^k}(E_l), \\ &\leq \Delta_k + 1/2^k - d_{\mathfrak{n}_k^k}(E_l). \end{aligned}$$

By taking the limit in k, we get $\overline{d}^{\mathfrak{n}}(F \setminus E_l) \leq \Delta_\infty - \Delta_l \xrightarrow{l} 0$.

Let us prove finally the Fölner property of the set F. For $\mathfrak{n}_k < K \in \partial F$ either $[K-k, K[$ or $]K, K+k]$ lies in the complement of E. Therefore $\overline{d}^{\mathfrak{n}}(\partial F) \leq 2/k$. As it holds for all k, the set F is Fölner along $\mathfrak{n}$. □

2.4. **Borel-Cantelli argument.** Let $(X, \mathcal{A}, \lambda)$ be a measure space with λ being a finite measure. A map $E : X \to \mathcal{P}_{\mathbb{N}}$ is said measurable, when for all $n \in \mathbb{N}$ the set $\{x,\ n \in E(x)\}$ belongs to $\mathcal{A}$ (equivalently writing E as an increasing sequence $(n_i)_{i\in\mathbb{N}}$ the integers valued functions n_i are measurable). For such measurable maps E and $\mathfrak{n}$, the upper asymptotic density $\overline{d}^{\mathfrak{n}}(E)$ defines a measurable function.

Lemma 2. *Assume E is a measurable sequence of integers such that $\overline{d}(E(x)) > \beta > 0$ for x in a measurable set A of a positive λ-measure. Then there exist $\mathfrak{n} \in \mathfrak{N}$, measurable subsets $(A_n)_{n\in\mathfrak{n}}$ of X and $\mathcal{F} = (F_n)_{n\in\mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ with $\partial F_n \subset E(x)$ for all $x \in A_n$, $n \in \mathfrak{n}$ such that :*

- $\underline{d}^{\mathfrak{n}}(\mathcal{F}) \geq \beta$;
- $\lambda(A_n) \geq \frac{e^{-n\delta_n}}{n^2}$ *for all $n \in \mathfrak{n}$ with* $\delta_n \xrightarrow{\mathfrak{n}\ni n\to+\infty} 0$;

- *$\mathcal{F}$ is a Fölner sequence;*
- *E is dense in $\mathcal{F}$ uniformly on A_n, i.e.*

$$\limsup_{n\in\mathfrak{n}} \sup_{x\in A_n} d_n\left(F_n \setminus E_M(x)\right) \xrightarrow{M} 0.$$

- $$\liminf_{n\in\mathfrak{n}} \inf_{x\in A_n} d_n\left(E(x)\cap F_n\right) \geq \beta.$$

Proof. The sequences $\mathfrak{n}$ and F built in the previous lemma define measurable sequences on A. By taking a smaller subset A we may assume

- $\mathfrak{n}_k(x)$ is bounded on A for all k,
- $d_{\mathfrak{n}_k(x)}(\partial F(x)) \xrightarrow{k} 0$ uniformly in $x\in A$,
- $\limsup_k \sup_{x\in A} d_{\mathfrak{n}_k(x)}(F(x)\setminus E_M(x)) \xrightarrow{M} 0$,
- $d_{\mathfrak{n}_k(x)}(E(x)\cap F(x)) \xrightarrow{k} d^{\mathfrak{n}(x)}(E(x)\cap F(x)) \geq \beta$ uniformly in $x\in A$.

By Borel-Cantelli Lemma, the subset $A_n := \{x\in A,\ n\in\mathfrak{n}(x)\}$ has λ-measure larger than $1/n^2$ for infinitely many $n\in\mathbb{N}$. We let $\mathfrak{n}$ be this infinite subset of integers. By the (uniform in x) Fölner property of $F(x)$, the cardinality of the boundary of $(F(x))_{(n)} = F(x)\cap[\![1,n]\!]$ for $x\in A_n$ and $n\in\mathfrak{n}$ is less than $n\alpha_n$ for some sequence $(\alpha_n)_{n\in\mathfrak{n}}$ (independent of x) going to 0. Therefore there are at most $2\sum_{k=1}^{[n\alpha_n]}\binom{n}{k}$ choices for $(F(x))_{(n)}$ and thus it may be fixed by dividing the measure of A_n by $2\sum_{k=1}^{[n\alpha_n]}\binom{n}{k} = e^{n\delta_n}$ for some $\delta_n \xrightarrow{n} 0$. □

3. Empirical measures associated to Fölner sequences

Let (X,T) be a topological system, i.e. X is a compact metrizable space and $T: X \circlearrowleft$ is continuous. We denote by $\mathcal{M}(X)$ the set of Borel probability measures on X endowed with the weak-$*$ topology and by $\mathcal{M}(X,T)$ the compact subset of invariant measures. We will write δ_x for the Dirac measure at $x\in X$. We let T_* be the induced (continuous) action on $\mathcal{M}(X)$. For $\mu\in\mathcal{M}(X)$ and a finite subset F of $\mathbb{N}$, we let μ^F be the empirical measure $\mu^F := \frac{1}{\sharp F}\sum_{k\in F} T_*^k\mu$.

3.1. **Invariant measures.**

The following lemma is standard, but we give a proof for the sake of completeness. We fix $\mathfrak{n}\in\mathfrak{N}$ and $\mathcal{F}=(F_n)_{n\in\mathfrak{N}}\in\mathcal{Q}_{\mathfrak{n}}$.

Lemma 3. *Assume $\mathcal{F}$ is a Fölner sequence and $\underline{d}^{\mathfrak{n}}(\mathcal{F})>0$. Let $(\mu_n)_{n\in\mathfrak{n}}$ be a family in $\mathcal{M}(X)$ indexed by $\mathfrak{n}$. Then any limit of $\left(\mu_n^{F_n}\right)_{n\in\mathfrak{n}}$ is a T-invariant Borel probability measure.*

Proof. Let $\mathfrak{n}'$ be a subsequence of $\mathfrak{n}$ such that $\left(\mu_n^{F_n}\right)_{n\in\mathfrak{n}'}$ is converging to some μ'. It is enough to check that $\left|\int\phi\,d\mu_n^{F_n} - \int\phi\circ T\,d\mu_n^{F_n}\right|$ goes to zero for when $\mathfrak{n}'\ni n\to+\infty$ for any $\phi: X\to\mathbb{R}$ continuous.

This follows from

$$\int \phi\, d\mu_n^{F_n} - \int \phi\circ T\, d\mu_n^{F_n} = \frac{1}{\sharp F_n}\int \left(\sum_{\substack{k+1\in F_n\\ k\notin F_n}} \phi\circ T^k - \sum_{\substack{k+1\notin F_n\\ k\in F_n}} \phi\circ T^k \right) d\,\mu_n,$$

$$\left|\int \phi\, d\mu_n^{F_n} - \int \phi\circ T\, d\mu_n^{F_n}\right| \le \sup_{x\in X}|\phi(x)|\frac{\sharp\partial F_n}{\sharp F_n},$$

$$\liminf_{n\in\mathfrak{n}}\left|\int \phi\, d\mu_n^{F_n} - \int \phi\circ T\, d\mu_n^{F_n}\right| \le \sup_{x\in X}|\phi(x)|\liminf_{n\in\mathfrak{n}}\frac{\sharp\partial F_n}{\sharp F_n},$$

$$\le \sup_{x\in X}|\phi(x)|\frac{d^{\mathfrak{n}}(\partial\mathcal{F})}{\underline{d}^{\mathfrak{n}}(\mathcal{F})} = 0.$$

□

3.2. **Subadditive cocycles.** We fix a general continuous subadditive process $\Phi = (\phi_n)_{n\in\mathbb{N}}$ with respect to (X,T), i.e. $\phi_0 = 0$, $\phi_n : X\to\mathbb{R}$ is a continuous function for all n and $\phi_{n+m} \le \phi_n + \phi_m\circ T^n$ for all m,n. In the proof of the main theorem we will only consider additive cocycles, but we think it could be interesting to consider general subadditive cocycles in other contexts.

Observe that $\Phi^+ = (\phi_n^+)_n$, with $\phi_n^+ = \max(\phi_n,0)$ for all n, is also subadditive. For any $\mu\in\mathcal{M}(X,T)$, we let $\phi^+(\mu) = \lim_n \frac{1}{n}\int\phi_n^+\,d\mu = \inf_n\frac{1}{n}\int\phi_n^+\,d\mu$ (the existence of the limit follows from the subadditivity property). Recall also that by the subadditive ergodic theorem [26], the limit $\phi_*(x) = \lim_n\frac{\phi_n(x)}{n}$ exists for x in a set of full measure with respect to any invariant measure μ. When $\phi_*(x)\ge 0$ for μ-almost every point x with $\mu\in\mathcal{M}(X,T)$, then we have $\phi(\mu) := \int\phi_*(x)\,d\mu(x) = \phi+(\mu)$. If μ is moreover ergodic, then $\phi_*(x) = \phi^+(\mu)$ for μ almost every x.

Let $E : Y\to\mathcal{P}_{\mathbb{N}}$ be a measurable sequence of integers defined on a Borel subset Y of X. For a set $F_n\in\mathcal{P}_n$ with $\partial F_n\subset E(x)$ for some $x\in X$, we may write F_n^- uniquely as the finite union of $E(x)$-irreducible intervals $F_n^- = \bigcup_{\mathsf{k}\in\mathsf{K}}[\![a_{\mathsf{k}}, b_{\mathsf{k}}[\![$. Let $n_{\mathsf{k}} = b_{\mathsf{k}} - a_{\mathsf{k}}$ for any $\mathsf{k}\in\mathsf{K}$. Then we define

$$\forall x\in X,\ \phi_E^{F_n}(x) := \sum_{\mathsf{k}\in\mathsf{K}}\phi_{n_{\mathsf{k}}}(T^{a_{\mathsf{k}}}x)$$

When Φ is additive, i.e. $\Phi_n = \sum_{0\le k<n}\phi\circ T^k$ for some continuous function $\phi : X\to\mathbb{R}$, we always have $\phi_E^{F_n}(x) = \phi^{F_n}(x) := \sum_{k\in F_n^-}\phi(T^k x)$.

The set valued map E is said a**-large** with respect to Φ for some $a\ge 0$ when we have $\phi_{l-k}(T^k x)\ge (l-k)a$ for all consecutive integers $l>k$ in $E(x)$.

Lemma 4. *Let Φ, F_n and E as above. Assume E is 0-large. Then for all $x\in X$ and for all integers $n\ge N\ge M$ we have:*

$$\frac{\phi_E^{F_n}(x)}{\sharp F_n} \ge \int\frac{\phi_N^+}{N}\,d\delta_x^{F_n} - \frac{d_n\left(F_n\setminus E_M(x)\right) + N d_n(\partial F_n) + 4M/N}{d_n(F_n)}\sup_y|\phi_1(y)|$$

Proof. Let $k\in\{0,\cdots,N-1\}$ and $l\in\mathbb{N}$. The interval of integers $J_{k,l} = [\![k+lN, k+(l+1)N[\![$ may be written as

$$J_{k,l} = I_1\coprod I_2\coprod I_3\coprod I_4$$

where I_1 is the union of disjoint E-irreducible intervals of length less than M contained in $J_{k,l}$, $I_2\subset\mathbb{N}\setminus F_n^-$, $I_3\subset F_n^-\setminus E_M(x)$ and I_4 is the union of at most two subintervals of E-irreducible intervals of length less than M containing an extremal point of $J_{k,l}$.

Therefore for a fixed k, by summing over all l with $k+lN \in F_n$ we get as E is 0-large and Φ is subadditive:

$$\begin{aligned}
&\sum_{l,\ k+lN\in F_n} \phi_N^+(T^k x) \\
&\leq \sum_{\mathsf{k}\in\mathsf{K},\ [\![a_\mathsf{k},b_\mathsf{k}[\![\cap(k+N\mathbb{N})=\emptyset} \phi_{n_\mathsf{k}}(T^{a_\mathsf{k}}x) + \sup_y |\phi_1(y)| \left(N\sharp\partial F_n + \sharp\left(F_n \setminus E_M(x)\right) + 2M([n/N]+1)\right) \\
&\leq \phi_E^{F_n}(x) + \sup_y |\phi_1(y)| \left(N\sharp\partial F_n + \sharp\left(F_n \setminus E_M(x)\right) + 2M([n/N]+1)\right).
\end{aligned}$$

Then by summing over all $k \in \{0,\cdots,N-1\}$ and dividing by N, we conclude that

$$\sharp F_n \int \frac{\phi_N^+}{N}\, d\delta_x^{F_n} \leq \phi_E^{F_n}(x) + \sup_y |\phi_1(y)| \left(N\sharp\partial F_n + \sharp F_n \setminus E_M(x) + 2M([n/N]+1)\right).$$

□

3.3. **Positive exponent of empirical measures for additive cocycles.** We consider here an additive cocycle Φ associated to a continuous function $\phi : X \to \mathbb{R}$. With the notations of Lemma 2 and Lemma 3, we have :

Lemma 5. *Let $(\mu_n)_{n\in\mathfrak{n}}$ with $\mu_n(A_n) = 1$ for all $n \in \mathfrak{n}$. Assume E is a-large with $a > 0$. Then for any weak-$*$ limit μ of $\mu_n^{F_n}$ we have*

$$\phi_*(x) \geq a \text{ for } \mu \text{ a.e. } x.$$

Proof. We claim that for any $0 < \alpha < 1$ and any $\epsilon > 0$, there is arbitrarily large N_0 such that

$$\limsup_n \mu_n^{F_n}(\phi_{N_0}/N_0 \geq \alpha a) \geq 1 - \epsilon. \tag{3.1}$$

By weak-$*$ convergence of μ_n to μ, it will imply, the set $\{\phi_{N_0}/N_0 \geq \alpha a\}$ being closed :

$$\mu(\phi_{N_0}/N_0 \geq \alpha a) \geq 1 - \epsilon.$$

Then we may consider a sequence $(N_k)_k$ going to infinity such that

$$\mu(\phi_{N_k}/N_k \geq \alpha a) \geq 1 - \epsilon/2^k.$$

Therefore $\mu\left(\bigcap_k \{\phi_{N_k}/N_k \geq \alpha a\}\right) \geq 1 - 2\epsilon$. We conclude $\limsup_n \frac{\phi_n(x)}{n} \geq \alpha a$ for μ a.e. x by letting ϵ go to zero.

Let us show now our first claim (3.1). It is enough to show the inequality for $\mu_n = \delta_x$ uniformly in $x \in A_n$. We use the same notations as in the proof of Lemma 4. Fix $x \in A_n$. For k, l with $k + lN \in F_n$, the interval $J_{k,l}$ is said admissible, when $\phi_N(f^{k+lN}x)/N \geq \alpha a$. If $J_{k,l}$ is not admissible we have

$$\begin{aligned}
\phi_N(f^{k+lN}x) &\geq \sum_{i\in I_1} \phi(f^i x) - \sup_y |\phi(y)| \sharp(I_2 \cup I_3 \cup I_4), \\
\alpha a N &\geq\ a\sharp I_1 - \sup_y |\phi(y)| \sharp(I_2 \cup I_3 \cup I_4), \\
&\geq aN - (a + \sup_y |\phi(y)|)\sharp(I_2 \cup I_3 \cup I_4), \\
\sharp(I_2 \cup I_3 \cup I_4) &\geq \frac{(1-\alpha)aN}{a + \sup_y |\phi(y)|}.
\end{aligned}$$

If we sum over all l with $k+lN \in F_n$ and then over $k \in \{0, \cdots, N-1\}$, we get by arguing as in the proof of Lemma 4 :

$$N\Big(N\sharp\partial F_n + \sharp\left(F_n \setminus E_M(x)\right) + 2M([n/N]+1)\Big) \geq$$
$$\sharp\{J_{k,l} \text{ not admissible, } k+lN \in F_n\} \times \frac{(1-\alpha)aN}{a+\sup_y |\phi(y)|}.$$

Therefore by Lemma 2 (third and fourth items) we have for $\mathfrak{n} \ni n \gg N \gg M$ uniformly in $x \in A_n$,

$$\sharp\{J_{k,l} \text{ not admissible, } k+lN \in F_n\} \leq \epsilon \sharp F_n.$$

By definition of admissible intervals we conclude that

$$\limsup_n \delta_x^{F_n}(\phi_N/N \geq \alpha a) \geq 1-\epsilon.$$

□

3.4. **Entropy of empirical measures.** Following Misiurewicz's proof of the variational principle, we estimate the entropy of empirical measures from below. For a finite partition P of X and a finite subset F of $\mathbb{N}$, we let P^F be the iterated partition $P^F = \bigvee_{k\in F} f^{-k}P$. When $F = [\![0, n-1]\!]$, $n \in \mathbb{N}$, we just let $P^F = P^n$. We denote by $P(x)$ the element of P containing $x \in X$.

For a Borel probability measure μ on X, the static entropy $H_\mu(P)$ of μ with respect to a (finite measurable) partition P is defined as follows:

$$\begin{aligned} H_\mu(P) &= -\sum_{A\in P} \mu(A)\log\mu(A), \\ &= -\int \log\mu\left(P(x)\right)\, d\mu(x). \end{aligned}$$

When μ is T-invariant, we recall that the measure theoretical entropy of μ with respect to P is then

$$h_\mu(P) = \lim_n \frac{1}{n} H_\mu(P^n)$$

and the entropy $h(\mu)$ of μ is

$$h(\mu) = \sup_P h_\mu(P).$$

We will use the two following standard properties of the static entropy[21]:

- for a fixed partition P, the map $\mu \mapsto H_\mu(P)$ is concave on $\mathcal{M}(X)$,
- for two partitions P and Q, the joined partition $P \vee Q$ satisfies

$$H_\mu(P \vee Q) \leq H_\mu(P) + H_\mu(Q). \tag{3.2}$$

Lemma 6. *Let $\mathcal{F} = (F_n)_{n\in\mathfrak{n}}$ be a Fölner sequence with $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. For any measurable finite partition P and $m \in \mathbb{N}^*$, there exist a sequence $(\epsilon_n)_{n\in\mathfrak{n}}$ converging to 0 such that*

$$\forall n \in \mathfrak{n},\ \frac{1}{m} H_{\mu_n^{F_n}}(P^m) \geq \frac{1}{\sharp F_n} H_{\mu_n}(P^{F_n}) + \epsilon_n.$$

Proof. When F_n is an interval of integers, we have [31] :

$$\frac{1}{m} H_{\mu_n^{F_n}}(P^m) \geq \frac{1}{\sharp F_n} H_{\mu_n}(P^{F_n}) - \frac{3m\log\sharp P}{\sharp F_n}. \tag{3.3}$$

Consider a general set $F_n \in \mathcal{P}_n$. We decompose F_n into connected components $F_n = \coprod_{k=1,\cdots,K} F_n^k$. Observe $K \leq \sharp\partial F_n$. Then we get :

$$\begin{aligned}
\frac{1}{m}H_{\mu_n^{F_n}}(P^m) &\geq \sum_{k=1}^{K} \frac{\sharp F_n^k}{m\sharp F_n} H_{\mu_n^{F_n^k}}(P^m), \text{by concavity of } \mu \mapsto H_\mu(P^m) \\
&\geq \frac{1}{\sharp F_n}\sum_{k=1}^{K} H_{\mu_n}(P^{F_n^k}) - \frac{3mK\log\sharp P}{\sharp F_n}, \text{ by applying (3.3) to each } F_n^k, \\
&\geq \frac{1}{\sharp F_n} H_{\mu_n}(P^{F_n}) - 3m\log\sharp P\frac{\sharp\partial F_n}{\sharp F_n}, \text{ according to (3.2).}
\end{aligned}$$

This concludes the proof with $\epsilon_n = 3m\frac{\sharp\partial F_n}{\sharp F_n}\log\sharp P$, because $\mathcal{F}$ is a Fölner sequence with $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. □

With the notations of Lemma 2 we let μ_n be the probability measure induced by λ on A_n, i.e. $\mu_n = \frac{\lambda(A_n\cap\cdot)}{\lambda(A_n)}$. Let $\Psi = (\psi_n)_n$ be a continuous subadditive process such that E is 0-large with respect to Ψ. We assume that λ satisfies the following *Fölner Gibbs property* with respect to the subadditive cocycle Ψ :

$$\text{(G)} \qquad \begin{aligned}
&\text{There exists } \epsilon > 0 \text{ such that} \\
&\text{we have for any partition } P \text{ with diameter less than } \epsilon : \\
&\exists N\ \forall x \in A_n \text{ with } N < n \in \mathfrak{n}, \quad \frac{1}{\lambda\left(P^{F_n}(x) \cap A_n\right)} \geq e^{\psi_E^{F_n}(x)}.
\end{aligned}$$

Proposition 3. *Under the above hypothesis (H), any weak-$*$ limit μ of $(\mu_n^{F_n})_{n\in\mathfrak{n}}$ satisfies*

$$h(\mu) \geq \psi^+(\mu).$$

Proof. Without loss of generality we may assume $(\mu_n^{F_n})_{n\in\mathfrak{n}}$ is converging to μ. Take a partition P with $\mu(\partial P) = 0$ and with diameter less than ϵ. In particular we have for all fixed $m \in \mathbb{N}$:

$$\frac{1}{m}H_\mu(P^m) = \lim_n \frac{1}{m}H_{\mu_n^{F_n}}(P^m).$$

Then we get for $n \gg N \gg M \gg m$

$$\begin{aligned}
\frac{1}{m}H_\mu(P^m) \geq& \limsup_{n\in\mathfrak{n}} \frac{1}{\sharp F_n}H_{\mu_n}(P^{F_n}), \text{ by Lemma 6,} \\
\geq& \limsup_{n\in\mathfrak{n}} \frac{1}{\sharp F_n}\int \left(-\log\lambda\left(P^{F_n}(x)\cap A_n\right) + \log\lambda(A_n)\right)\, d\mu_n(x), \\
\geq& \limsup_{n\in\mathfrak{n}} \int \frac{\psi_E^{F_n}}{\sharp F_n}\, d\mu_n(x), \text{ by Hypothesis (G),} \\
\geq& \limsup_{n\in\mathfrak{n}} \Bigg(\int \frac{\psi_N^+}{N}\, d\mu_n^{F_n} \\
&- \frac{\sup_y|\psi(y)|\left(\sup_{x\in A_n} d_n(F_n\setminus E_M(x)) + Nd_n(\partial F_n) + 4M/N\right)}{d_n(F_n)}\Bigg), \text{ by Lemma 4,} \\
\geq& \int \frac{\psi_N^+}{N}\, d\mu - \frac{1}{\underline{d}^{\mathfrak{n}}(\mathcal{F})}\left(\sup_y|\psi(y)|\left(\limsup_{n\in\mathfrak{n}}\sup_{x\in A_n} d_n(F_n\setminus E_M(x)) + 4M/N\right)\right), \\
\geq& \psi^+(\mu) - \frac{1}{\underline{d}^{\mathfrak{n}}(\mathcal{F})}\left(\sup_y|\psi(y)|\left(\limsup_{n\in\mathfrak{n}}\sup_{x\in A_n} d_n(F_n\setminus E_M(x)) + 4M/N\right)\right).
\end{aligned}$$

Letting N, then M, then m go to infinity, we conclude that

$$h(\mu) \geq h_\mu(P) \geq \psi^+(\mu).$$

□

In the following we will also consider a general† additive cocycle $\Psi = (\psi_n)_n$ associated to a continuous function $\psi : X \to \mathbb{R}$. Then for any Fölner sequence $(F_n)_n$, the *Fölner Gibbs property* with respect to the additive cocycle Ψ may be simply written as follows:

$$\text{(H)} \qquad \begin{aligned} &\text{There exists } \epsilon > 0 \text{ such that} \\ &\text{we have for any partition } P \text{ with diameter less than } \epsilon : \\ &\exists N \ \forall x \in A_n \text{ with } N < n \in \mathfrak{n}, \quad \frac{1}{\lambda\left(P^{F_n}(x) \cap A_n\right)} \geq e^{\psi^{F_n}(x)}. \end{aligned}$$

In this additive setting we get :

Proposition 4. *Under the above hypothesis (H), any weak-$*$ limit μ of $(\mu_n^{F_n})_{n\in\mathfrak{n}}$ satisfies*

$$h(\mu) \geq \psi(\mu).$$

Proof. Let P as in the proof of Proposition 3. Then for $n \gg N \gg m$ we obtain by following this proof :

$$\begin{aligned} \frac{1}{m} H_\mu(P^m) &\geq \limsup_{n\in\mathfrak{n}} \int \frac{\psi^{F_n}}{\sharp F_n}\, d\mu_n(x), \text{ by Hypothesis (H)}, \\ &\geq \limsup_{n\in\mathfrak{n}} \int \psi\, d\mu_n^{F_n}, \\ &\geq \psi(\mu). \end{aligned}$$

Letting m go to infinity, we conclude that $h(\mu) \geq \psi(\mu)$. □

4. Geometric times

Let $r \geq 2$ be an integer and let $(M, \|\cdot\|)$ be a C^r smooth compact Riemannian manifold, not necessarily a surface for the moment. We denote by d the distance induced by the Riemannian structure on M. We also consider a distance $\hat{\mathrm{d}}$ on the projective tangent bundle $\mathbb{P}TM$, such that $\hat{\mathrm{d}}(\hat{x}, \hat{y}) \geq \mathrm{d}(\pi\hat{x}, \pi\hat{y})$ for all $\hat{x}, \hat{y} \in \mathbb{P}TM$ with $\pi : \mathbb{P}TM \to M$ being the natural projection. For a C^r map $f : M \to M$ or a C^r curve $\sigma : [0,1] \to M$ we may define the norm $\|d^s f\|_\infty$ and $\|d^s \sigma\|_\infty$ for $1 \leq s \leq r$ as the supremum norm of the s-derivative of the induced maps through the charts of a given atlas or through the exponential map exp. In the following, to simplify the presentation we lead the computations as M was an Euclidean space. For a C^1 curve $\sigma : I \to M$, I being a compact interval of $\mathbb{R}$, we let $\sigma_* = \sigma(I)$. The length of σ_* for the induced Riemannian metric is denoted by $|\sigma_*|$. For a fixed curve σ we also let $v_x \in \mathbb{P}TM$ be the line tangent to σ_* at x and we write $\hat{x} = (x, v_x)$.

We denote by F the projective action $F : \mathbb{P}TM \circlearrowleft$ induced by f and we consider: the additive derivative cocycle $\Phi = (\phi_k)_k$ for F on $\mathbb{P}TM$ given by $\phi(x,v) = \phi_1(x,v) = \log\|d_x f(v)\|$, where we have identified the line v_x with one of its unit generating vectors.

†The set E is not assumed here to be 0-large with respect to Ψ.

4.1. **Bounded curve.** Following [13] a C^r smooth curve $\gamma : [-1,1] \to M$ is said **bounded** when

$$\max_{s=2,\cdots,r} \|d^s\gamma\|_\infty \leq \frac{1}{6}\|d\gamma\|_\infty.$$

We first recall some basic properties of bounded curves (see Lemma 7 in [13]). A bounded curve has bounded distorsion meaning that

$$\forall t,s \in [-1,1],\ \frac{\|d\gamma(t)\|}{\|d\gamma(s)\|} \leq 3/2. \tag{4.1}$$

Indeed, if $t_* \in [-1,1]$ satisfies $\|d\gamma(t_*)\| = \|d\gamma\|_\infty$ then we have for all $s \in [-1,1]$,

$$\begin{aligned} \|d\gamma(t_*) - d\gamma(s)\| &\leq 2\|d^2\gamma\|_\infty, \\ &\leq \frac{1}{3}\|d\gamma(t_*)\|, \\ \text{therefore } \ \frac{2}{3}\|d\gamma(t_*)\| \leq \|d\gamma(s)\| &\leq \|d\gamma(t_*)\|, \end{aligned}$$

The projective component of γ oscillates also slowly. If we identify M with $\mathbb{R}^2$ [‡], we have

$$\|d\gamma(t_*)\| \cdot \sin\angle d\gamma(t_*), d\gamma(s) \leq \|d\gamma(t_*) - d\gamma(s)\| \leq \frac{1}{3}\|d\gamma(t_*)\|,$$

$$\angle d\gamma(t_*), d\gamma(s) \leq \pi/6. \tag{4.2}$$

When moreover $\|d\gamma\|_\infty \leq \epsilon$ we say that γ is **strongly ϵ-bounded**. In particular such a map satisfies $\|\gamma\|_r := \max_{1\leq s\leq r} \|d^s\gamma\| \leq \epsilon$, which is the standard C^r upper bound required for the reparametrizations in the usual Yomdin's theory. But this last condition does not allow to control the distorsion along the curve in general.

If γ is bounded then so is $\gamma_a = \gamma(a\cdot) : [-1,1] \to M$ for any $a \leq \frac{2}{3}$:

$$\begin{aligned} \forall s \geq 2,\ \|d^s\gamma_a\|_\infty &\leq \frac{1}{6}a^s\|d\gamma\|_\infty, \\ &\leq \frac{1}{6}a^s\frac{3}{2}\|d\gamma(0)\|, \\ &\leq \frac{1}{6}a^{s-1}\|d\gamma(0)\|, \\ &\leq \frac{1}{6}\|d\gamma_a\|_\infty. \end{aligned}$$

As $\|d\gamma_a\|_\infty \leq a\|d\gamma\|_\infty$, if γ is moreover strongly ϵ-bounded, then γ_a is $a\epsilon$-strongly bounded.

Lemma 7. *Let $\gamma : [-1,1] \to M$ be a C^r bounded curve with $\|d\gamma\|_\infty \geq \epsilon$. Then there is a family of affine maps $\iota_j : [-1,1] \circlearrowleft$, $j \in L := \underline{L} \cup \overline{L}$ such that:*

- *each $\gamma \circ \iota_j$ is ϵ-bounded and $\|d(\gamma \circ \iota_j)(0)\| \geq \frac{\epsilon}{6}$,*
- *$[-1,1]$ is the union of $\bigcup_{j\in\underline{L}} \iota_j([-1,1])$ and $\bigcup_{j\in\overline{L}} \iota_j([-\frac{1}{3},\frac{1}{3}])$,*
- *$\sharp\underline{L} \leq 2$ and $\sharp\overline{L} \leq 6\left(\frac{\|d\gamma\|_\infty}{\epsilon} + 1\right)$,*
- *for any $x \in \gamma_*$, we have $\sharp\{j \in L,\ (\gamma \circ \iota_j)_* \cap B(x,\epsilon) \neq \emptyset\} \leq 100$.*

Sketch of proof. For the first three items it is enough to consider affine reparametrizations of $[-1,1]$ with rate $\frac{2\epsilon}{3\|d\gamma\|_\infty}$. As the bounded map γ stays in a cone of opening angle $\pi/6$, its intersection with $B(x,\epsilon)$ has length less than 2ϵ. The last item follows then easily. $\square$

‡This will be always possible as we will only consider curves with diameter less than the radius of injectivity.

Fix a C^r smooth diffeomorphism $f : M \circlearrowleft$. A curve $\gamma : [-1,1] \to M$ is said n**-bounded** (resp. **strongly** (n,ϵ)**-bounded**) when $f^k \circ \gamma$ is bounded (resp. strongly ϵ-bounded) for $k = 0, \cdots, n$. A strongly ϵ-bounded curve γ is contained in the dynamical ball $B_n(x,\epsilon) := \{y \in M,\ \forall k = 0, \cdots, n-1,\ \mathrm{d}(f^k x, y) < \epsilon\}$ with $x = \gamma(0)$.

Fix a C^r curve $\sigma : I \to M$. For $x \in \sigma_*$, a positive integer n is called an (α,ϵ)**-geometric time** of x when there exists an affine map $\theta_n : [-1,1] \to I$ such that $\gamma_n := \sigma \circ \theta_n$ is strongly (n,ϵ)-bounded, $\gamma_n(0) = x$ and $\|d(f^n \circ \gamma_n)(0)\| \geq \frac{3}{2}\alpha\epsilon$. The concept of (α,ϵ)**-geometric time** is almost independent of ϵ. Indeed it follows from the above observations that, if n is a (α,ϵ)-geometric time of x, then it is also a (α,ϵ')-geometric time for $\epsilon' < \frac{2\epsilon}{3}$. Moreover if n is a (α,ϵ)-geometric time of x with γ_n the associated curve, then n is a $(\frac{2}{3}\alpha, \frac{2}{3}\epsilon)$-geometric time of any $y \in \gamma_n([1/3,1/3])$: if $y = \gamma_n(t)$ for $t \in [-1/3,1/3]$ then $\tilde{\gamma}_n := \gamma_n(t + \frac{2}{3}\cdot)$ is strongly $(n, \frac{2}{3}\epsilon)$-bounded and satisfies $\tilde{\gamma}_n(0) = y$ and $\|d(f^n \circ \tilde{\gamma}_n)(0)\| = \frac{2}{3}\|d(f^n \circ \gamma_n)(t)\| \geq \frac{4}{9}\|d(f^n \circ \gamma_n)(0)\| \geq \frac{2}{3}\alpha\epsilon$.

We let $D_n(x)$ and $H_n(x)$ be the images of $f^n \circ \gamma_n$ and γ_n respectively with γ_n as above of maximal length. We define the semi-length of $D_n(x)$ as the minimum of the lengths of $f^n \circ \gamma_n([0,1])$ and $f^n \circ \gamma_n([-1,0])$. The semi-length of $D_n(x)$ is larger than $\alpha\epsilon$ at a (α,ϵ)-geometric time n. One can also easily checks that the curvature of $f^n \circ \sigma$ at $f^n x$ is bounded from above by $\frac{1}{\alpha\epsilon}$. From the bounded distorsion property of bounded curves (4.1) we get

$$\forall y, z \in H_n(x)\, \forall 0 \leq l < n, \quad \frac{e^{\phi_{n-l}(F^l \hat{y})}}{e^{\phi_{n-l}(F^l \hat{z})}} \leq \frac{9}{4}. \tag{4.3}$$

4.2. **Reparametrization Lemma.** We consider a C^r smooth diffeomorphism $g : M \circlearrowleft$ and a C^r smooth curve $\sigma : I \to M$ with $\mathbb{N} \ni r \geq 2$. We state a global reparametrization lemma to describe the dynamics on σ_*. We will apply this lemma to $g = f^p$ for large p with f being the C^r smooth system under study. We denote by G the map induced by g on $\mathbb{P}TM$.

We will encode the dynamics of g on σ_* with a tree, in a similar way the symbolic dynamic associated to monotone branches encodes the dynamic of a continuous piecewise monotone interval map. A weighted directed rooted tree $\mathcal{T}$ is a directed rooted tree whose edges are labelled. Here the weights on the edges are pairs of integers. Moreover the nodes of our tree will be coloured, either in blue or in red.

We let $\mathcal{T}_n$ (resp. $\underline{\mathcal{T}_n}$, $\overline{\mathcal{T}_n}$) be the set of nodes (resp. blue, red nodes) of level n. For all $k \leq n-1$ and for all $\mathbf{i}^n \in \mathcal{T}_n$, we also let $\mathbf{i}^n_k$ be the node of level k leading to $\mathbf{i}^n$. For $\mathbf{i}^n \in \mathcal{T}_n$, we let $k(\mathbf{i}^n) = (k_1(\mathbf{i}^n), k'_1(\mathbf{i}^n), k_2(\mathbf{i}^n) \cdots, k'_n(\mathbf{i}^n))$ be the $2n$-uple of integers given by the sequence of labels along the path from the root $\mathbf{i}^0$ to $\mathbf{i}^n$, where $(k_l(\mathbf{i}^n), k'_l(\mathbf{i}^n))$ denotes the label of the edge joining $\mathbf{i}^n_{l-1}$ and $\mathbf{i}^n_l$.

For $x \in \sigma_*$, we let $k(x) \geq k'(\hat{x})$ be the following integers:

$$k(x) := [\log \|d_x g\|],$$

$$k'(\hat{x}) := [\log \|d_x g(v_x)\|].$$

Then for all $n \in \mathbb{N}^*$ we define

$$k^n(x) = (k(x), k'(\hat{x}), k(gx), \cdots k'(G^{n-2}\hat{x}), k(g^{n-1}x), k'(G^{n-1}\hat{x})).$$

For a $2n$-uple of integers $\mathbf{k}^n = (k_1, k_1', \cdots k_n', k_n)$ we consider then

$$\mathcal{H}(\mathbf{k}^n) := \{x \in \sigma_*,\ k^n(x) = \mathbf{k}^n\}.$$

We restate the Reparametrization Lemma (RL for short) proved in [13] in a *global* version. Let $\exp_x$ be the exponential map at x and let R_{inj} be the radius of injectivity of $(M, \|\cdot\|)$.

Reparametrization Lemma. Let $\frac{R_{inj}}{2} > \epsilon > 0$ satisfying $\|d^s g^x_{2\epsilon}\|_\infty \leq 3\epsilon \|d_x g\|$ for all $s = 1, \cdots, r$ and all $x \in M$, where $g^x_{2\epsilon} = g \circ \exp_x(2\epsilon \cdot) : \{w_x \in T_x M,\ \|w_x\| \leq 1\} \to M$ and let $\sigma : [-1,1] \to M$ be a strongly ϵ-bounded curve.

Then there is $\mathcal{T}$, a bicoloured weighted directed rooted tree, and $(\theta_{\mathbf{i}^n})_{\mathbf{i}^n \in \mathcal{T}_n}$, $n \in \mathbb{N}$, families of affine reparametrizations of $[-1,1]$, such that for some universal constant C_r depending only on r:

1. $\forall \mathbf{i}^n \in \mathcal{T}_n$, the curve $\sigma \circ \theta_{\mathbf{i}^n}$ is (n, ϵ)-bounded,
2. $\forall \mathbf{i}^n \in \mathcal{T}_n$, the affine map $\theta_{\mathbf{i}^n}$ may be written as $\theta_{\mathbf{i}^n_{n-1}} \circ \phi_{\mathbf{i}^n}$ with $\phi_{\mathbf{i}^n}$ being an affine contraction with rate smaller than $1/100$ and $\theta_{\mathbf{i}^n}([-1,1]) \subset \theta_{\mathbf{i}^n_{n-1}}([-1/3, 1/3])$ when $\mathbf{i}^n_{n-1}$ belongs to $\overline{\mathcal{T}_{n-1}}$,
3. $\forall \mathbf{i}^n \in \overline{\mathcal{T}_n}$, we have $\|d\left(f^n \circ \sigma \circ \theta_{\mathbf{i}^n}\right)(0)\| \geq \epsilon/6$,
4. $\forall \mathbf{k}^n \in (\mathbb{Z} \times \mathbb{Z})^n$, the set $\sigma^{-1}\mathcal{H}(\mathbf{k}^n)$ is contained in the union of
$$\bigcup_{\substack{\mathbf{i}^n \in \overline{\mathcal{T}_n} \\ k(\mathbf{i}^n) = \mathbf{k}^n}} \theta_{\mathbf{i}^n}([-1/3, 1/3]) \text{ and } \bigcup_{\substack{\mathbf{i}^n \in \underline{\mathcal{T}_n} \\ k(\mathbf{i}^n) = \mathbf{k}^n}} \theta_{\mathbf{i}^n}([-1,1]).$$
Moreover any term of these unions have a non-empty intersection with $\sigma^{-1}\mathcal{H}(\mathbf{k}^n)$,
5. $\forall \mathbf{i}^{n-1} \in \mathcal{T}_{n-1}$ and $(k_n, k_n') \in \mathbb{Z} \times \mathbb{Z}$ we have
$$\sharp \left\{\mathbf{i}^n \in \overline{\mathcal{T}_n},\ \mathbf{i}^n_{n-1} = \mathbf{i}^{n-1} \text{ and } (k_n(\mathbf{i}^n), k_n'(\mathbf{i}^n)) = (k_n, k_n')\right\} \leq C_r e^{\max\left(k_n', \frac{k_n - k_n'}{r-1}\right)},$$
$$\sharp \left\{\mathbf{i}^n \in \underline{\mathcal{T}_n},\ \mathbf{i}^n_{n-1} = \mathbf{i}^{n-1} \text{ and } (k_n(\mathbf{i}^n), k_n'(\mathbf{i}^n)) = (k_n, k_n')\right\} \leq C_r e^{\frac{k_n - k_n'}{r-1}}.$$

Proof. We argue by induction on n. For $n = 0$ we let $\overline{\mathcal{T}_0} = \underline{\mathcal{T}_0} = \{\mathbf{i}^0\}$ and we just take $\theta_{\mathbf{i}^0}$ equal to the identity map on $[-1,1]$. Assume the tree and the associated reparametrizations have been built till the level n.

Fix $\mathbf{i}^n \in \mathcal{T}_n$ and let

$$\hat{\theta}_{\mathbf{i}^n} := \begin{cases} \theta_{\mathbf{i}^n}(\frac{1}{3}\cdot) & \text{if } \mathbf{i}^n \in \overline{\mathcal{T}_n}, \\ \theta_{\mathbf{i}^n} & \text{if } \mathbf{i}^n \in \underline{\mathcal{T}_n}. \end{cases}$$

We will define the children $\mathbf{i}^{n+1}$ of $\mathbf{i}^n$, i.e. the nodes $\mathbf{i}^{n+1} \in \mathcal{T}_{n+1}$ with $\mathbf{i}^{n+1}_n = \mathbf{i}^n$. The label on the edge joining $\mathbf{i}^n$ to $\mathbf{i}^{n+1}$ is a pair (k_{n+1}, k_{n+1}') such that the $2(n+1)$-uple $\mathbf{k}^{n+1} = (k_1(\mathbf{i}^n), \cdots, k_n'(\mathbf{i}^n), k_{n+1}, k_{n+1}')$ satisfies $\mathcal{H}(\mathbf{k}^{n+1}) \cap \left(\sigma \circ \hat{\theta}_{\mathbf{i}^n}\right)_* \neq \emptyset$. We fix such a pair (k_{n+1}, k_{n+1}') and the associated sequence $\mathbf{k}^{n+1}$. We let $\eta, \psi : [-1,1] \to M$ be the curves defined as:

$$\eta := \sigma \circ \hat{\theta}_{\mathbf{i}^n},$$
$$\psi := g^n \circ \eta.$$

First step : **Taylor polynomial approximation.** One computes for an affine map $\theta : [-1,1] \circlearrowleft$ with contraction rate b precised later and with $y = \psi(t) \in g^n\mathcal{H}(\mathbf{k}^{n+1})$, $t \in \theta([-1,1])$:

$$\begin{aligned}\|d^r(g\circ\psi\circ\theta)\|_\infty &\leq b^r\left\|d^r\left(g^y_{2\epsilon}\circ\psi^y_{2\epsilon}\right)\right\|_\infty, \text{with } \psi^y_{2\epsilon}:=(2\epsilon)^{-1}\exp_y^{-1}\circ\psi,\\ &\leq b^r\left\|d^{r-1}\left(d_{\psi^y_{2\epsilon}}g^y_{2\epsilon}\circ d\psi^y_{2\epsilon}\right)\right\|_\infty,\\ &\leq b^r2^r\max_{s=0,\cdots,r-1}\left\|d^s\left(d_{\psi^y_{2\epsilon}}g^y_{2\epsilon}\right)\right\|_\infty\|\psi^y_{2\epsilon}\|_r.\end{aligned}$$

By assumption on ϵ, we have $\|d^sg^y_{2\epsilon}\|_\infty\leq 3\epsilon\|d_yg\|$ for any $r\geq s\geq 1$. Moreover $\|\psi^y_{2\epsilon}\|_r\leq (2\epsilon)^{-1}\|d\psi\|_\infty\leq 1$ as ψ is strongly ϵ-bounded. Therefore by Faá di Bruno's formula, we get for some[§] constants $C_r>0$ depending only on r:

$$\max_{s=0,\cdots,r-1}\|d^s\left(d_{\psi^y_{2\epsilon}}g^y_{2\epsilon}\right)\|_\infty\leq\epsilon C_r\|d_yg\|,$$

then ,

$$\begin{aligned}\|d^r(g\circ\psi\circ\theta)\|_\infty &\leq \epsilon C_rb^r\|d_yg\|\|\psi^y_{2\epsilon}\|_r,\\ &\leq C_rb^r\|d_yg\|\|d\psi\|_\infty,\\ &\leq (C_rb^{r-1}\|d_yg\|)\|d(\psi\circ\theta)\|_\infty,\\ &\leq (C_rb^{r-1}e^{k_{n+1}})\|d(\psi\circ\theta)\|_\infty, \text{ because } y \text{ belongs to } g^n\mathcal{H}(\mathbf{k}^{n+1}),\\ &\leq e^{k'_{n+1}-4}\|d(\psi\circ\theta)\|_\infty, \text{ by taking } b=\left(C_re^{k_{n+1}-k'_{n+1}+4}\right)^{-\frac{1}{r-1}}.\end{aligned}$$

The Taylor polynomial P at 0 of degree $r-1$ of $d(g\circ\psi\circ\theta)$ satisfies on $[-1,1]$:

$$\|P-d(g\circ\psi\circ\theta)\|_\infty\leq e^{k'_{n+1}-4}\|d(\psi\circ\theta)\|_\infty.$$

We may cover $[-1,1]$ by at most $b^{-1}+1$ such affine maps θ.

Second step : **Bezout theorem.** Let $a_n:=e^{k'_{n+1}}\|d(\psi\circ\theta)\|_\infty$. Note that for $s\in[-1,1]$ with $\eta\circ\theta(s)\in\mathcal{H}(\mathbf{k}^{n+1})$ we have $\|d(g\circ\psi\circ\theta)(s)\|\in[a_ne^{-2},a_ne^2]$, therefore $\|P(s)\|\in[a_ne^{-3},a_ne^3]$. Moreover if we have now $\|P(s)\|\in[a_ne^{-3},a_ne^3]$ for some $s\in[-1,1]$ we get also $\|d(g\circ\psi\circ\theta)(s)\|\in[a_ne^{-4},a_ne^4]$.

By Bezout theorem the semi-algebraic set $\{s\in[-1,1],\ \|P(s)\|\in[e^{-3}a_n,e^3a_n]\}$ is the disjoint union of closed intervals $(J_i)_{i\in I}$ with $\sharp I$ depending only on r. Let θ_i be the composition of θ with an affine reparametrization from $[-1,1]$ onto J_i.

Third step : **Landau-Kolmogorov inequality.** By the Landau-Kolmogorov inequality on the interval (see Lemma 6 in [14]), we have for some constants $C_r\in\mathbb{N}^*$ and for all $1\leq s\leq r$:

$$\begin{aligned}\|d^s(g\circ\psi\circ\theta_i)\|_\infty &\leq C_r\left(\|d^r(g\circ\psi\circ\theta_i)\|_\infty+\|d(g\circ\psi\circ\theta_i)\|_\infty\right),\\ &\leq C_r\frac{|J_i|}{2}\left(\|d^r(g\circ\psi\circ\theta)\|_\infty+\sup_{t\in J_i}\|d(g\circ\psi\circ\theta)(t)\|\right),\\ &\leq C_ra_n\frac{|J_i|}{2}.\end{aligned}$$

We cut again each J_i into $1000C_r$ intervals $\tilde{J}_i$ of the same length with $(\eta\circ\theta)(\tilde{J}_i)\cap\mathcal{H}(\mathbf{k}^{n+1})\neq\emptyset$. Let $\tilde{\theta}_i$ be the affine reparametrization from $[-1,1]$ onto $\theta(\tilde{J}_i)$. We check

[§]Although these constants may differ at each step, they are all denoted by C_r.

that $g\circ\psi\circ\tilde{\theta}_i$ is bounded:

$$\begin{aligned}\forall s=2,\cdots,r,\ \|d^s(g\circ\psi\circ\tilde{\theta}_i)\|_\infty &\leq (1000C_r)^{-2}\|d^s(g\circ\psi\circ\theta_i)\|_\infty,\\ &\leq \frac{1}{6}(1000C_r)^{-1}\frac{|J_i|}{2}a_ne^{-4},\\ &\leq \frac{1}{6}(1000C_r)^{-1}\frac{|J_i|}{2}\min_{s\in J_i}\|d(g\circ\psi\circ\theta)(s)\|,\\ &\leq \frac{1}{6}(1000C_r)^{-1}\frac{|J_i|}{2}\min_{s\in \tilde{J}_i}\|d(g\circ\psi\circ\theta)(s)\|,\\ &\leq \frac{1}{6}\|d(g\circ\psi\circ\tilde{\theta}_i)\|_\infty.\end{aligned}$$

Last step : **ϵ-bounded curve.** Either $g\circ\psi\circ\tilde{\theta}_i$ is ϵ-bounded and $\hat{\theta}_{\mathbf{i}^n}\circ\tilde{\theta}_i=\theta_{\mathbf{i}^{n+1}}$ for some $\mathbf{i}^{n+1}\in\underline{\mathcal{T}}_{n+1}$. Or we apply Lemma 7 to $g\circ\psi\circ\tilde{\theta}_i$: the new affine parametrizations $\hat{\theta}_{\mathbf{i}^n}\circ\tilde{\theta}_i\circ\iota_j$, $j\in\underline{L}$ (resp. $j\in\overline{L}$) then define $\theta_{\mathbf{i}^{n+1}}$ for a node $\mathbf{i}^{n+1}$ in $\underline{\mathcal{T}}_{n+1}$ (resp. $\overline{\mathcal{T}}_{n+1}$). Note finally that:

$$\begin{aligned}\sharp\overline{L} &\leq 6\left(\frac{\|d(g\circ\psi\circ\tilde{\theta}_i)\|_\infty}{\epsilon}+1\right),\\ &\leq 100\max(e^{k'_{n+1}}b,1),\ \text{ as } \psi \text{ is } \epsilon\text{-bounded and } \|d\tilde{\theta}_i\|_\infty\leq b,\\ &\leq C_r\max\left(\frac{e^{k'_{n+1}}}{e^{\frac{k_{n+1}-k'_{n+1}}{r-1}}},1\right),\end{aligned}$$

therefore

$$\begin{aligned}\sharp\left\{\mathbf{i}^{n+1}\in\overline{\mathcal{T}_{n+1}}\ \middle|\ \overset{\mathbf{i}_n^{n+1}=\mathbf{i}^n \text{ and}}{(k_{n+1}(\mathbf{i}^{n+1}),k'_{n+1}(\mathbf{i}^{n+1}))=(k_{n+1},k'_{n+1})}\right\} &\leq \sum_{\tilde{\theta}_i}C_r\max\left(\frac{e^{k'_{n+1}}}{e^{\frac{k_{n+1}-k'_{n+1}}{r-1}}},1\right),\\ &\leq C_re^{\max\left(k'_{n+1},\frac{k_{n+1}-k'_{n+1}}{r-1}\right)}.\end{aligned}$$

□

As a corollary of the proof of RL we state a *local* reparametrization lemma, i.e. we only reparametrize the intersection of σ_* with some given dynamical ball. For $x\in\sigma_*$, $n\in\mathbb{N}$ and $\epsilon>0$ we let

$$B^G_\sigma(x,\epsilon,n):=\left\{y\in\sigma_*,\ \forall k=0,\cdots,n-1,\ \hat{\mathrm{d}}(G^k\hat{x},G^k\hat{y})<\epsilon\right\}.$$

For all $(x,v)\in\mathbb{P}TM$, we also let $w(x,v)=w_g(x,v):=\log\|d_xg\|-\log\|d_xg(v)\|$ and for all $n\in\mathbb{N}$ we let $w^n(x,v)=w^n_g(x,v):=\sum_{k=0}^{n-1}w(G^k(x,v))$. We consider $\epsilon>0$ as in the Reparametrization Lemma. We assume moreover that

$$[\hat{\mathrm{d}}((x,v),(y,w))<\epsilon]\Rightarrow[|\log\|d_xg(v)\|-\log\|d_yg(w)\||<1 \text{ and } |\log\|d_xg\|-\log\|d_yg\||<1].$$

Corollary 3. *For any strongly ϵ-bounded curve $\sigma:[-1,1]\to M$ and for any $x\in\sigma_*$, we have for some constant C_r depending only on r:*

$$\forall n\in\mathbb{N},\ \ \sharp\left\{\mathbf{i}^n\in\mathcal{T}_n,\ (\sigma\circ\theta_{\mathbf{i}^n})_*\cap B^G_\sigma(x,\epsilon,n)\neq\emptyset\right\}\leq C_r^n e^{\frac{w^n(\hat{x})}{r-1}}. \tag{4.4}$$

Sketch of proof. The Corollary follows from the Reparametrization Lemma together with the two following facts :

- for $y\in B^G_\sigma(x,\epsilon,n)$ we have $k^n(x)\simeq k^n(y)$ up to 1 on each coordinate,

- for any $\mathbf{i}^{n-1}$ there is at most $C_r e^{\frac{k(g^n x)-k'(G^n\hat{x})}{r-1}}$ nodes $\mathbf{i}^n \in \overline{\mathcal{T}}_n$ with $\mathbf{i}^n_{n-1} = \mathbf{i}^{n-1}$ and $\theta_{\mathbf{i}^n}([-1,1]) \cap \sigma^{-1} B^G_\sigma(x,\epsilon,n+1) \neq \emptyset$.

This last point is a consequence of the last item of Lemma 7 applied to the bounded map $g \circ \psi \circ \tilde{\theta}_i$ introduced in the third step of the proof of the reparametrization lemma. □

4.3. **The geometric set E.** We apply the Reparametrization Lemma to $g = f^p$ for some positive integer p. For $x \in \sigma_*$ we define the set $E_p(x) \subset p\mathbb{N}$ of integers mp such that there is $\mathbf{i}^m \in \overline{\mathcal{T}_m}$ with $k(\mathbf{i}^m) = k^m(x)$ and $x \in \sigma \circ \theta_{\mathbf{i}^m}([-1/3, 1/3])$. In particular, any integer $n = mp \in E_p(x)$ is a (α_p, ϵ_p)-geometric time of x for f with α_p and ϵ_p depending only on p by item (3) of RL. Observe that if $k < m$ we have with $x = \sigma \circ \theta_{\mathbf{i}^m}(t)$ and $\theta_{\mathbf{i}^m}(t) = \theta_{\mathbf{i}^m_k}(s)$:

$$\begin{aligned}
\phi_{mp-kp}(F^{kp}\hat{x}) &= \frac{\|d\left(f^{mp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(t)\|}{\|d\left(f^{kp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(t)\|},\\
&\geq \frac{2}{3}\frac{\|d\left(f^{mp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(0)\|}{\|d\left(f^{kp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(t)\|}, \text{since } \sigma \circ \theta_{\mathbf{i}^m} \text{ is } m\text{-bounded},\\
&\geq \frac{2}{3}\frac{\|d\left(f^{mp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(0)\|}{\left\|d\left(f^{kp} \circ \sigma \circ \theta_{\mathbf{i}^m_k}\right)(s)\right\|}100^{m-k}, \text{by item (2) of RL},\\
&\geq \frac{2}{3\epsilon}\|d\left(f^{mp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(0)\|\, 100^{m-k}, \text{as } \sigma \circ \theta_{\mathbf{i}^n_k} \text{ is strongly } (k,\epsilon)\text{-bounded},\\
&\geq \frac{1}{9}100^{m-k} \geq \left(\frac{1}{10}\right)^{m-k}, \text{ by item (3) of RL.}
\end{aligned}$$

Therefore E_p is τ_p-large with $\tau_p = \frac{\log 10}{p}$-large.

Proposition 5. *Let $f : M \circlearrowleft$ is a C^r diffeomorphism and $b > \frac{R(f)}{r}$. For p large enough there exists $\beta_p > 0$ such that*

$$\limsup_n \frac{1}{n} \log Leb_{\sigma_*}\left(\left\{x \in A,\ d_n(E_p(x)) < \beta_p \text{ and } \|d_x f^n(v_x)\| \geq e^{nb}\right\}\right) < 0.$$

Proof. It is enough to consider $n = mp \in p\mathbb{N}$. We apply the Reparametrization Lemma to $g = f^p$ with $\epsilon > 0$ being the scale. Let $\mathcal{T}$ be the corresponding tree and $(\theta_{\mathbf{i}^m})_{\mathbf{i}^m \in \mathcal{T}_m}$ its associated affine reparametrizations. By a standard argument the number of sequences of positive integers $(k_1, \cdots, k_m)$ with $k_i < M \in \mathbb{N}$ for all i is less than $e^{mMH(M)}$ where $H(t) := -\frac{1}{t}\log\frac{1}{t} - (1-\frac{1}{t})\log\left(1-\frac{1}{t}\right)$ for $t > 0$. Therefore we can fix the sequence $\mathbf{k}^m = k^m(x)$ up to a factor combinatorial term equal to $e^{2mpA_f H(pA_f)}$ with $A_f := \log\|df\|_\infty + \log\|df^{-1}\|_\infty + 1$. Assume there is $x = \sigma \circ \theta_{\mathbf{i}^m}(t)$ with $\|d_x f^n(v_x)\| \geq e^{nb}$. Then by the distorsion property of the bounded maps $f^n \circ \sigma \circ \theta_{\mathbf{i}^m}$ and $\sigma \circ \theta_{\mathbf{i}^m}$ we have

$$\begin{aligned}
|(\sigma \circ \theta_{\mathbf{i}^m})_*| &\leq 2\|d(\sigma \circ \theta_{\mathbf{i}^m})\|_\infty,\\
&\leq 3\|d(\sigma \circ \theta_{\mathbf{i}^m})(t)\|, \text{ as } \sigma \circ \theta_{\mathbf{i}^m} \text{ is bounded},\\
&\leq 3\frac{\|d(f^n \circ \sigma \circ \theta_{\mathbf{i}^m})(t)\|}{\|d_x f^n(v_x)\|},\\
&\leq 3\epsilon e^{-nb}, \text{as } f^n \circ \sigma \circ \theta_{\mathbf{i}^m} \text{ is } \epsilon\text{-bounded.}
\end{aligned}$$

Moreover when x belongs to $(\sigma \circ \theta_{\mathbf{i}^m})_*$ for some $\mathbf{i}^m \in \mathcal{T}_m$ and satisfies $d_n(E_p(x)) < \beta_p$, then we have $\sharp\left\{0 < k < m,\ \mathbf{i}^m_k \in \overline{\mathcal{T}_k}\right\} \leq n\beta_p$. But, by the estimates on the valence of $\mathcal{T}$ given in the last item of RL, the number of m-paths from the root labelled with $\mathbf{k}^m$ and with at most $n\beta_p$ red nodes are less than $2^m C_r^m e^{\sum_i \frac{k_i - k'_i}{r-1}}\|df\|_\infty^{\beta_p p^2 m}$ for some constant C_r depending only on r. Then if $x \in \mathcal{H}(\mathbf{k}^m)$ satisfies $\|d_x f^n(v_x)\| \geq e^{nb}$, we have $e^{\sum_i \frac{k_i - k_i'}{r-1}} \leq e^m e^{m\frac{\log \|df^p\|_\infty - bp}{r-1}}$. But, as b is larger than $\frac{\log \|df^p\|_\infty}{pr}$ for large p, we get for such values of p : $\frac{\log \|df^p\|_\infty - bp}{r-1} \leq \frac{1-\frac{1}{r}}{r-1} \cdot \log \|df^p\|_\infty = \frac{\log \|df^p\|_\infty}{r}$. Therefore :

$$\limsup_n \frac{1}{n} \log \mathrm{Leb}_{\sigma_*}\left(\left\{x \in A,\ d_n(E_p(x)) < \beta_p \text{ and } \|d_x f^n(v_x)\| \geq e^{nb}\right\}\right)$$
$$\leq \frac{\log C_r}{p} + \left(H(pA_f) + p\beta_p\right) A_f - b + \frac{\log \|df^p\|_\infty}{pr}.$$

As b is larger than $\frac{R(f)}{r}$ one can choose firstly $p \in \mathbb{N}^*$ large then $\beta_p > 0$ small in such a way the right member is negative.

□

From now we fix p and the associated quantities satisfying the conclusion of Proposition 5 and we will simply write $E, \tau, \alpha, \epsilon, \beta$ for $E_p, \tau_p, \alpha_p, \epsilon_p, \beta_p$. The set $E(x)$ is called the **geometric set** of x.

4.4. **Cover of F-dynamical balls by bounded curves.** As a consequence of Corollary 3, we give now an estimate of the number of strongly (n, ϵ')-bounded curves reparametrizing the intersection of a given strongly ϵ'-bouded curve with a F-dynamical ball of length n and radius ϵ'. This estimate will be used in the proof of the Fölner Gibbs property (Proposition 7).

For any $q \in \mathbb{N}^*$ we let $\omega_q : \mathbb{P}TM \to \mathbb{R}$ be the map defined for all $(x, v) \in \mathbb{P}TM$ by

$$\omega_q(x, v) := \frac{1}{q} \sum_{k=0}^{q-1} \log \|d_{f^k x} f^q\| - \log \|d_x f(v)\|.$$

Note that $\omega_1 = w$. We also write $(\omega_q^n)_n$ the additive associated F-cocycle, i.e.

$$\omega_q^n(x, v) = \sum_{0 \leq k < n} \omega_q(F^k(x, v)).$$

Lemma 8. *For any $q \in \mathbb{N}^*$, there exists $\epsilon_q' > 0$ and $B_q > 0$ such that for any strongly ϵ_q'-bounded curve $\sigma : [-1, 1] \to M$, for any $x \in \sigma_*$ and for any $n \in \mathbb{N}^*$ there exists a family $(\theta_i)_{i \in I_n}$ of affine maps of $[-1, 1]$ such that :*

- $B_\sigma^F(x, \epsilon_q', n) \subset \bigcup_{i \in I_n} (\sigma \circ \theta_i)_*$,
- *$\sigma \circ \theta_i$ is (n, ϵ_q')-bounded (with respect to f) for any $i \in I_n$,*
- $\sharp I_n \leq B_q C_r^{\frac{n}{q}} e^{\frac{\omega_q^n(\hat{x})}{r-1}}$, *with C_r a universal constant depending only on r.*

Proof. Fix q. Let $\epsilon_q' = \epsilon/2$ with ϵ as in Corollary 3 for $g = f^q$. There is a family Θ of affine maps of $[-1, 1]$ such that for any strongly ϵ_q'-bounded map $\gamma : [-1, 1] \to M$, the map $\gamma \circ \theta$ is (q, ϵ_q')-bounded and $\bigcup_{\theta \in \Theta} \theta_* = [-1, 1]$.

Fix now a strongly ϵ_q'-bounded curve $\sigma : [-1, 1] \to M$ and let $x \in \sigma_*$. We consider only the map $\theta \in \Theta$ such that $B_\sigma^F(x, \epsilon_q', n) \cap (\sigma \circ \theta)_* \neq \emptyset$. For such a map θ we let $x_\theta \in B_\sigma^F(x, \epsilon_q', n) \cap (\sigma \circ \theta)_*$.

Take any $0 \leq k < q$. By applying Corollary 3 to "$g = f^q$", "$\sigma = f^k \circ \sigma \circ \theta$", "$x = f^k(x_\theta)$" and "$n = [\frac{n-k}{q}]$", we get a family $\Psi_{\theta,k}$ of affine maps of $[-1, 1]$ with

$$\bigcup_{\psi \in \Psi_{\theta,k}} (\sigma \circ \theta \circ \psi)_* \supset f^{-k} B_{f^k \circ \sigma \circ \theta}^{F^q}\left(f^k(x_\theta), \epsilon, \left[\frac{n-k}{q}\right]\right) \supset B_{\sigma \circ \theta}^F(x, \epsilon_q', n)$$

such that $f^{mq+k}\circ\sigma\circ\theta\circ\psi$ is ϵ-bounded for $\psi\in\Psi_{\theta,k}$ and integers m with $0\leq mq+k\leq n$. Then $\Theta_k=\{\theta\circ\psi\circ\theta',\ \psi\in\Psi_{\theta,k}$ and $(\theta,\theta')\in\Theta^2\}$ satisfies the two first items of the conclusion. Moreover by letting $m_k=\left[\frac{n-k}{q}\right]$ we have:

$$\sharp\Theta_k\leq C_r^{m_k}\sharp\Theta^2 e^{w_{f^q}^{m_k}(F^k\hat{x})}.$$

But for some constant A_q depending only on q, we have

$$\min_{0\leq k<q} e^{w_{f^q}^{m_k}(F^k\hat{x})}\leq\left(\prod_{0\leq k<q} e^{w_{f^q}^{m_k}(F^k\hat{x})}\right)^{1/q}\leq A_q e^{\omega_q^n(\hat{x})}.$$

This concludes the proof of the lemma, as $\sharp\Theta$ depends only on q.

□

5. Existence of SRB measures

5.1. **Entropy formula.** By Ruelle's inequality [38], for any C^1 system, the entropy of an invariant measure is less than or equal to the integral of the sum of its positive Lyapunov exponents. For C^{1+} systems, the following entropy characterization of SRB measures was obtained by Ledrappier and Young :

Theorem 6. [30] *An invariant measure of a C^{1+} diffeomorphism on a compact manifold is a SRB measure if and only it has a positive Lyapunov exponent almost everywhere and the entropy is equal to the integral of the sum of its positive Lyapunov exponents.*

As the entropy is harmonic (i.e. preserves the ergodic decomposition), the ergodic components of a SRB measures are also SRB measures.

5.2. **Lyapunov exponents.** We consider in this susection a C^1 diffeomorphism $f:M\circlearrowleft$. Let $\|\|$ be a Riemaninan structure on M. The (forward upper) Lyapunov exponent of (x,v) for $x\in M$ and $v\in T_xM$ is defined as follows (see [33] for an introduction to Lyapunov exponents):

$$\chi(x,v):=\limsup_{n\to+\infty}\frac{1}{n}\log\|d_xf^n(v)\|.$$

The function $\chi(x,\cdot)$ admits only finitely many values $\chi_1(x)>...>\chi_{p(x)}(x)$ on $T_xM\setminus\{0\}$ and generates a flag $0\subsetneq V_{p(x)}(x)\subsetneq\cdots\subsetneq V_1=T_xM$ with $V_i(x)=\{v\in T_xM,\ \chi(x,v)\leq\chi_i(x)\}$. In particular, $\chi(x,v)=\chi_i(x)$ for $v\in V_i(x)\setminus V_{i+1}(x)$. The function p as well the functions χ_i and the vector spaces $V_i(x)$, $i=1,...,p(x)$ are invariant and depend Borel measurably on x. One can show the maximal Lyapunov exponent χ introduced in the introduction coincides with χ_1 (see Appendix A).

A point x is said **regular** when there exists a decomposition

$$T_xM=\bigoplus_{i=1}^{p(x)}H_i(x)$$

such that

$$\forall v\in H_i(x)\setminus\{0\},\ \lim_{n\to\pm\infty}\frac{1}{|n|}\log\|d_xf^n(v)\|=\chi_i(x)$$

with uniform convergence in $\{v\in H_i(x),\ \|v\|=1\}$ and

$$\lim_{n\to\pm\infty}\frac{1}{|n|}\log\mathrm{Jac}\,(d_xf^n)=\sum_i\dim(H_i(x))\chi_i(x).$$

In particular we have $V_i(x) = \bigoplus_{j=1}^{i} H_j(x)$ for all i. The set $\mathcal{R}$ of regular points is an invariant measurable set of full measure for any invariant measure [29]. The invariant sub-bundles H_i are called the Oseledec's bundles. We also let $\mathcal{R}^* := \{x \in \mathcal{R},\ \forall i\ \chi_i(x) \neq 0\}$. For $x \in \mathcal{R}^*$ we put $E_u(x) = \bigoplus_{i,\ \chi_i(x)>0} H_i(x)$ and $E_s(x) = \bigoplus_{i,\ \chi_i(x)<0} H_i(x)$.

In the following we only consider surface diffeomorphisms. Therefore we always have $p(x) \leq 2$ and when $p(x)$ is equal to 1, we let $\chi_2(x) = \chi_1(x)$. When ν is f-invariant we let $\chi_i(\nu) = \int \chi_i \, d\nu$.

5.3. **Building SRB measures.** Assume f is a C^r, $r > 1$, surface diffeomorphism and $\limsup_n \frac{1}{n} \log \|d_x f^n\| > b > \frac{R(f)}{r}$ on a set of positive Lebesgue measure as in the Main Theorem. Take $\epsilon > 0$ as in Proposition 5 (it depends only on $b - \frac{R(f)}{r} > 0$). By using Fubini's theorem as in [14] there is a C^r smooth embedded curve $\sigma : I \to M$, which can be assumed to be ϵ-bounded, and a subset A of σ_* with $\mathrm{Leb}_{\sigma_*}(A) > 0$, such that we have $\limsup_n \frac{1}{n} \log \|d_x f^n(v_x)\| > b$ for all $x \in A$. Here Leb_{σ_*} denotes the Lebesgue measure on σ_* induced by its inherited Riemannian structure as a submanifold of M. This a finite measure with $\mathrm{Leb}_{\sigma_*}(M) = |\sigma_*|$. We can also assume that the countable set of periodic sources has an empty intersection with σ_*.

Let $\mathfrak{m}$ be the measurable sequence given by the set $\mathfrak{m}(x) := \{n,\ \|d_x f^n(v_x)\| \geq e^{nb}\} \in \mathfrak{N}$ for $x \in A$. It follows from Proposition 5 that for x in a subset of A of positive Leb_{σ_*} measure we have $d_n(E(x)) \geq \beta > 0$ for $n \in \mathfrak{m}(x)$ large enough, i.e. we have by denoting again this subset by A:

$$\forall x \in A,\ \overline{d}(E(x)) \geq \overline{d}^{\mathfrak{m}}(E(x)) \geq \beta.$$

For any $q \in \mathbb{N}^*$ we let

$$\psi^q = \phi - \frac{\omega_q}{r-1}.$$

Proposition 7. *There exists an infinite sequence of positive real numbers $(\delta_q)_q$ with $\delta_q \xrightarrow{q\to\infty} 0$ such that the property (H) holds with respect to the additive cocycle on $\mathbb{P}TM$ associated to the observable $\psi^q + \delta_q$.*

We prove now the existence of a SRB measure assuming Proposition 7, whose proof is given in the next section. This is a first step in the proof of the Main Theorem. We will apply the results of the first sections to the projective action $F : \mathbb{P}TM \circlearrowleft$ induced by f, where we consider:

- the additive derivative cocycle $\Phi = (\phi_k)_k$ given by $\phi_k(x, v) = \log \|d_x f^k(v)\|$,
- the measure $\lambda = \lambda_\sigma$ on $\mathbb{P}TM$ given by $\mathfrak{s}^* \mathrm{Leb}_{\sigma_*}$ with $\mathfrak{s} : x \in \sigma_* \mapsto (x, v_x)$,
- the geometric set E, which is τ-large with respect to Φ,
- the additive cocycles Ψ^q associated to $\psi^q + \delta_q$.

The topological extension $\pi : (\mathbb{P}TM, F) \to (M, f)$ is principal¶ by a straightforward application of Ledappier-Walters variational principle [30] and Lemma 3.3 in [39]. In fact this holds in any dimension and more generally for any finite dimensional vector bundle morphism instead of $df : TM \circlearrowleft$.

Let $\mathcal{F} = (F_n)_{n\in\mathfrak{n}}$ and $(A_n)_{n\in\mathfrak{n}}$ be the sequences associated to E given by Lemma 2. Rigorously E should be defined on the projective tangent bundle, but as π is one-to-one on $\mathbb{P}T\sigma_*$ there is no confusion. In the same way we see the sets A_n, $n \in \mathbb{N}$, as subsets of $A \subset \sigma_*$. Any weak-$*$ limit μ of $\mu_n^{F_n}$ is invariant under F and thus supported by Oseledec's bundles. Let $\nu = \pi\mu$. By Lemma 5, μ is supported by the unstable bundle E_u and

¶i.e. $h_f(\pi\mu) = h_F(\mu)$ for all F-invariant measure μ.

$\phi_*(\hat{x}) \geq \tau > 0$ for μ a.e. $\hat{x} \in \mathbb{P}TM$. Note also that $\phi_*(\hat{x}) \in \{\chi_1(\pi\hat{x}), \chi_2(\pi\hat{x})\}$ for μ-almost every $\hat{x}$. We claim that $\phi_*(\hat{x}) = \chi_1(\pi\hat{x})$. If not ν would have an ergodic component with two positive exponents. It is well known such a measure is necessarily a periodic measure associated to a periodic source S. But there is an open neighborhood U of the orbit of S with $f^{-1}U \subset U$ and $\sigma_* \cap U = \emptyset$. In particular we have $\pi\mu_n^{F_n}(U) = 0$ for all n because $\pi\mu_n^{F_n}\left(\bigcup_{N\in\mathbb{N}} f^N\sigma_*\right) = 1$ and $f^N\sigma_* \cap U = f^N(\sigma_* \cap f^{-N}U) \subset f^N(\sigma_* \cap U) = \emptyset$. By taking the limit in n we would obtain $\nu(S) = 0$. Therefore $\phi_*(\hat{x}) = \chi_1(\pi\hat{x}) > \tau$ for μ-almost every x and $\chi_1(x) > \tau > 0 \geq \chi_2(x)$ for ν-almost every x.

Then by Proposition 4 and Proposition 7 we obtain:

$$\begin{aligned} h(\nu) = h(\mu) &\geq \int \psi^q \, d\mu + \delta_q, \\ &\geq \int \phi \, d\mu - \frac{1}{r-1}\int \omega_q \, d\mu + \delta_q, \\ &\geq \chi_1(\nu) - \frac{1}{r-1}\left(\frac{1}{q}\sum_{k=0}^{q-1}\int \log\|d_{f^kx}f^q\| \, d\nu(x) - \chi_1(\nu)\right) + \delta_q, \\ &\geq \chi_1(\nu) - \frac{1}{r-1}\left(\frac{1}{q}\int \log\|d_xf^q\| \, d\nu(x) - \chi_1(\nu)\right) + \delta_q. \end{aligned}$$

By a standard application of the subadditive ergodic theorem, we have

$$\frac{1}{q}\int \log\|d_xf^q\| \, d\nu(x) \xrightarrow{q\to+\infty} \int \chi_1(x) \, d\nu(x) = \chi_1(\nu).$$

Therefore $h(\nu) \geq \chi_1(\nu)$, since $\delta_q \xrightarrow{q\to\infty} 0$. Then Ruelle's inequality implies $h(\mu) = \chi_1(\nu)$. According to Ledrappier-Young characterization (Theorem 6), the measure ν is a SRB measure of f. Note also that any ergodic component ξ of ν is also a SRB measure, therefore $h(\xi) = \chi_1(\xi) > \tau$. But by Ruelle inequality applied to f^{-1}, we get also $h(\xi) \leq -\chi_2(\xi)$. In particular we have $\chi_1(x) > \tau > 0 > -\tau > \chi_2(x)$ for ν-almost every x.

Remark 8. *We have assumed that $r \geq 2$ is an integer. The proof can be adapted without difficulties to the non integer case $r > 1$.*

5.4. **Proof of the Fölner Gibbs property (H).** In this subsection we prove Proposition 7. We will show that for any $\delta > 0$ there is q arbitrarily large and $\epsilon_q' > 0$ such that we have for any partition P of $\mathbb{P}TM$ with diameter less than ϵ_q':

$$\exists n_* \ \forall x \in A_n \subset \sigma_* \text{ with } n_* < n \in \mathfrak{n}, \quad \frac{1}{\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1}A_n\right)} \geq e^{\delta\sharp F_n}e^{\psi_q^{F_n}(\hat{x})}, \tag{5.1}$$

where we denote $\psi_q^{F_n}(\hat{x}) := \sum_{k\in F_n}\psi^q(F^k\hat{x})$ to simplify the notations.

For $G \subset \mathbb{N}$ we let A^G be the set of points $x \in A$ with $G \subset E(x)$. When $G = \{k\}$ or $\{k,l\}$ with $k,l \in \mathbb{N}$, we just let $A^G = A^k$ or $A^{k,l}$. We recall that $\partial F_n \subset E(x)$ for all $x \in A_n$, in others terms $A_n \subset A^{\partial F_n}$. We will show (5.2) for $A^{\partial F_n}$ in place of A_n.

Fix the error term $\delta > 0$. Let q be so large that $C_r^{1/q} < e^{\delta/3}$ and ϵ_q' as in Lemma 8. Without loss of generality we may assume $\epsilon_q' < \frac{\alpha\epsilon}{4}$. Recall $\epsilon, \alpha > 0$ corresponds to the fixed scales in the definition of the geometric set E. We can also ensure that

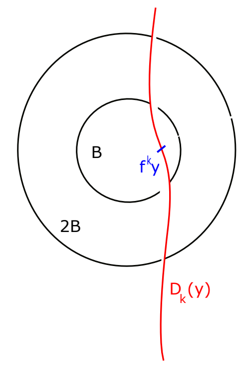
$$\forall \hat{x}, \hat{y} \in \mathbb{P}TM \text{ with } \hat{\mathrm{d}}(\hat{x},\hat{y}) < \epsilon_q', \quad |\phi(\hat{x}) - \phi(\hat{y})| < \delta/3. \tag{5.2}$$

In the next triplets lemmas we consider a strongly ϵ-bounded curve σ.

Lemma 9. *For any subset N of M, any $k \in \mathbb{N}$ and any ball B_k of radius less than ϵ_q', there exists a finite family $(y_j)_{j\in J}$ of $\sigma_* \cap A^k \cap f^{-k}B_k \cap N$ such that :*

- $B_k \cap f^k(\sigma_* \cap A^k \cap N) \subset \bigcup_{j\in J} D_k(y_j)$,
- $B_k \cap D_k(y_j)$, $j \in J$, *are pairwise disjoint.*

Proof. Let $y \in \sigma_* \cap A^k \cap N$ with $f^k y \in B = B_k$. Let $2B$ be the ball of same center as B with twice radius. The curve $D_k(y)$ lies in a cone with opening angle $\pi/6$ by (4.2). Moreover its length is larger than $\alpha\epsilon > 4\epsilon'_q$. By elementary euclidean geometric arguments, the set $D_k(y) \cap 2B$ is a curve crossing $2B$, i.e. its two endpoints lies in the boundary of $2B$. Two such subcurves of $f^k \circ \sigma$ if not disjoint are necessarily equal. $\square$



As the distorsion is bounded on $D_k(y_i)$ by (4.3) and the semi-length of $D_k(y_i)$ is larger than $\alpha\epsilon$ (because y_i belongs to A^k), we have

$$\begin{aligned}\sum_{i\in I} \frac{4}{9} e^{-\phi_k(\hat{y}_i)} |D_k(y_i)| &\leq \sum_{i\in I} \left| f^{-k} D_k(y_i) \right|,\\ &\leq |\sigma_*| \leq 2\epsilon,\\ \sum_{i\in I} 2\alpha\epsilon e^{-\phi_k(\hat{y}_i)} &\leq \frac{9}{2}\epsilon,\\ \sum_{i\in I} e^{-\phi_k(\hat{y}_i)} &\leq \frac{9}{4\alpha}. \qquad (5.3)\end{aligned}$$

Lemma 10. *For any subset N of M and any dynamical ball $B_{[\![0,k]\!]} := B^F_\sigma(x, \epsilon'_q, k+1)$, there exists a finite family $(z_i)_{i\in I}$ of $\sigma_* \cap A^k \cap B_{[\![0,k]\!]} \cap N$ such that*

- $f^k\left(\sigma_* \cap A^k \cap B_{[\![0,k]\!]} \cap N\right) \subset \bigcup_{i\in I} D_k(z_i)$,
- $B(f^k x, \epsilon'_q) \cap D_k(z_i)$, $i \in I$, *are pairwise disjoint,*
- $\sharp I \leq B_q e^{\delta k/3} e^{\frac{\omega_q^k(\hat{x})}{r-1}}$ *for some constant B_q depending only on q.*

Proof. As in the previous lemma we consider the subcurves $D_k(z)$ for $z \in \sigma_* \cap A^k \cap B_{[\![0,k]\!]} \cap N$. By Lemma 8 we can reparametrize $B_{[\![0,k]\!]}$ by a family of strongly (k, ϵ'_q)-bounded curves with cardinality less than $B_q C_r^{\frac{k}{q}} e^{\omega_q^k(\hat{x})}$. Each of this curve is contained in some $D_k(z)$. But as already mentioned, the sets $B(f^k x, \epsilon'_q) \cap D_k(z)$, $z \in \sigma_* \cap A^k \cap B_{[\![0,k]\!]}$, are either disjoint or equal. $\square$

Lemma 11. *For any dynamical ball $B_{[\![k,l]\!]} := f^{-k} B^F_\sigma(f^k x, \epsilon'_q, l-k+1)$, there exists a finite family $(y_i)_{i\in I}$ of $\sigma_* \cap A^{k,l} \cap B_{[\![k,l]\!]}$ and a partition $I = \coprod_{j\in J} I_j$ of I with $j \in I_j$ for all $j \in J \subset I$ such that*

- $f^l(\sigma_* \cap A^{k,l} \cap B_{[\![k,l]\!]}) \subset \bigcup_{i\in I} D_l(y_i)$,
- $B(f^l x, \epsilon'_q) \cap D_l(y_i)$, $i \in I$, *are pairwise disjoint,*
- $\forall j \in J\ \forall i, i' \in I_j,\ D_k(y_i) \cap B(f^k x, \epsilon'_q) = D_k(y_{i'}) \cap B(f^k x, \epsilon'_q)$,
- $\forall j \in J,\ \sharp I_j \leq B_q e^{\delta(l-k)/3} e^{\frac{\omega_q^{l-k}(F^k\hat{x})}{r-1}}$ *for some constant B_q depending only on q.*

Proof. We first apply Lemma 9 to σ and $N = A^{k,l} \cap B_{[\![k,l]\!]}$ to get the collection of strongly ϵ-bounded curve $(D_k(y_j))_{j\in J}$. Then we apply Lemma 10 to each $D_k(y_j)$ for $j \in J$ and $N = f^k(B_{[\![k,l]\!]} \cap A_k \cap \sigma_*)$ to get a family $(z_i)_{i\in I_j}$ of $D_k(y_j) \cap A^{l-k} \cap f^k(B_{[\![k,l]\!]} \cap A_k \cap \sigma_*)$ satisfying:

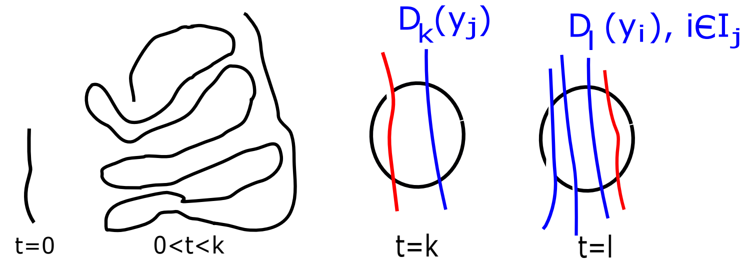


Figure 1: *For $0 \le t < k$ the image of $f^t \circ \sigma$ in black may be large and the disks $D_t(y_i)$ are scattered through the surface. For $t = k$, the sets $D_k(y_j)$ for $j \in J$ are covering $(f^t \circ \sigma)_* \cap B_k$. For $t = l$, we drew in blue the sets $D_l(y_i) \subset f^{l-k} D_k(y_j)$ for $i \in I_j$.*

- $f^{l-k}\left(D_k(y_j) \cap A^{l-k} \cap f^k(B_{[\![k,l]\!]} \cap A_k \cap \sigma_*)\right) \subset \bigcup_{j\in J} D_k(z_i)$,
- $B_l \cap D_{l-k}(z_i)$, $i \in I_j$, are pairwise disjoint,
- $\sharp I_j \le B_q e^{\delta(l-k)/3} e^{\frac{\omega_q^{l-k}(F^k \hat{x})}{r-1}}$.

For all $j \in J$ we can take $j \in I_j$ and $z_j = f^k(y_j)$. We conclude the proof by letting $y_i = f^{-k} z_i \in \sigma_* \cap A^{k,l} \cap B_{[\![k,l]\!]}$ for all $i \in I := \coprod_{j\in J} I_j$.

□

We prove now (H). Recall that $\lambda = \lambda_\sigma$ is the push-forward on $\mathbb{P}TM$ of the Lebesgue measure on σ_*. As $\sharp\partial F_n = o(n)$ it is enough to show there is a constant C such that for any strongly ϵ-bounded curve σ we have

$$\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n}\right) \le C^{\sharp\partial F_n} e^{2\delta\sharp F_n/3} e^{-\psi_q^{F_n}(\hat{x})}. \tag{5.4}$$

To prove (5.4) we argue by induction on the number of connected components of F_n. Let $[\![k,l]\!]$, $0 \le k \le l$, be the first connected component of F_n and write $G_{n-l} = \mathbb{N}^* \cap (F_n - l)$. Then with the notations of Lemma 11 we get

$$\begin{aligned}\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n}\right) &\le \lambda_\sigma\left(\coprod_{i\in I} F^{-l}\left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \cap D_l(y_i)\right)\right),\\ &\le \lambda_\sigma\left(\coprod_{j\in J} F^{-k}\left(\coprod_{i\in I_j} F^{-(l-k)}\left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \cap D_l(y_i)\right)\right)\right)\end{aligned}$$

For $j \in J$ we let σ_j^k be the strongly ϵ-bounded curve σ given by $D_k(y_j)$. By the bounded distorsion property (4.3) we get

$$\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n}\right) \le 3\sum_{j\in J} e^{-\phi_k(\hat{y}_j)} \lambda_{\sigma_j^k}\left(\coprod_{i\in I_j} F^{-(l-k)}\left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \cap D_l(y_i)\right)\right).$$

By using again the bounded distorsion property (now between the times k and l) we get with σ_i^l being the curve associated to $D_l(y_i)$:

$$\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n}\right) \le 9\sum_{j\in J} e^{-\phi_k(\hat{y}_j)} \sum_{i\in I_j} e^{-\phi_{l-k}(F^k \hat{y}_i)} \lambda_{\sigma_i^l}\left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x})\right).$$

We may assume that any $\hat{y}_i$, $i \in I$, lies in $P^{F_n}(\hat{x})$. In particular we have $|\phi_{l-k}(F^k \hat{y}_i) - \phi_{l-k}(F^k \hat{x})| < (l-k)\delta/3$ by (5.2). Then

$$\lambda_\sigma \left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n}\right) \leq 9 \left(\sum_{j \in J} e^{-\phi_k(\hat{y}_j)}\right) e^{\delta(l-k)/3} e^{-\phi_{l-k}(F^k \hat{x})} \sup_j \sharp I_j \times \sup_{i \in I} \lambda_{\sigma_i^l}\left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x})\right).$$

By (5.3) and the last item of Lemma 11 we obtain

$$\begin{aligned}\lambda_\sigma \left(P^{F_n}(\hat{x}) \cap \pi^{-1} \partial A^{F_n}\right) &\leq \frac{50 B_q e^{2\delta(l-k)/3}}{\alpha} e^{-\phi_{l-k}(F^k \hat{x}) + \frac{\omega_q^{l-k}(F^k \hat{x})}{r-1}} \sup_{i \in I} \lambda_{\sigma_i^l}\left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x})\right), \\ &\leq \frac{50 B_q e^{2\delta(l-k)/3}}{\alpha} e^{-\psi_q^{[\![k,l]\!]}(\hat{x})} \sup_{i \in I} \lambda_{\sigma_i^l}\left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x})\right).\end{aligned}$$

By induction hypothesis (5.4) applied to G_{n-l} for each σ_i^l, we have for all $i \in I$:

$$\lambda_{\sigma_i^l}\left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x})\right) \leq C^{\sharp \partial G_{n-l}} e^{2\delta \sharp \partial G_{n-l}/3} e^{-\psi_q^{G_{n-l}}(F^l \hat{x})}.$$

Note that $\sharp \partial F_n = \sharp \partial G_{n-l} + 2$. We conclude by taking $C = \sqrt{\frac{50 B_q}{\alpha}}$ that

$$\begin{aligned}\lambda_\sigma \left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n}\right) &\leq \frac{50 B_q e^{2\delta \sharp F_n/3}}{\alpha} C^{\sharp \partial G_{n-l}} e^{-\psi_q^{F_n}(\hat{x})}, \\ &\leq C^{\sharp \partial F_n} e^{2\delta \sharp F_n/3} e^{-\psi_q^{F_n}(\hat{x})}.\end{aligned}$$

This completes the proof of (5.1).

6. End of the proof of the Main Theorem

6.1. **The covering property of the basins.** For $x \in M$ the stable/unstable manifold $W^{s/u}(x)$ at x are defined as follows :

$$W^s(x) := \{y \in M,\ \limsup_{n \to +\infty} \frac{1}{n} \log d(f^n x, f^n y) < 0\},$$

$$W^u(x) := \{y \in M,\ \limsup_{n \to +\infty} \frac{1}{n} \log d(f^{-n} x, f^{-n} y) < 0\}.$$

For a subset Γ of M we let $W^s(\Gamma) = \bigcup_{x \in \Gamma} W^s(x)$. According to Pesin's theory, there are a nondecreasing sequence of compact, a priori non-invariant, sets $(K_n)_n$ (called the Pesin blocks) with $\mathcal{R}^* = \bigcup_n K_n$ and two families of embedded C^∞ discs $(W^s_{loc}(x))_{x \in K}$ and $(W^u_{loc}(x))_{x \in K}$ (called the local stable and unstable manifolds) such that :

- $W^{s/u}_{loc}(x)$ are tangent to $E_{s/u}$ at x,
- the splitting $E_u(x) \oplus E_s(x)$ and the discs $W^{s/u}_{loc}(x)$ are continuous on $x \in K_n$ for each n.

For $\gamma > 0$ and $x \in K$ we let $W^{s/u}_\gamma(x)$ be the connected component of $B(x, \gamma) \cap W^{s/u}_{loc}(x)$ containing x.

Proposition 9. *The set $\left\{\chi > \frac{R(f)}{r}\right\}$ is covered by the basins of ergodic SRB measures μ_i, $i \in I$, up to a set of zero Lebesgue measure.*

In fact we prove a stronger statement by showing that $\left\{\chi > \frac{R(f)}{r}\right\}$ is contained Lebesgue a.e. in $W^s(\Gamma)$ where Γ is any f-invariant subset of $\bigcup_{i\in I}\mathcal{B}(\mu_i)_{i\in I}$ with $\mu_i(\Gamma)=1$ for all i.

So far we only have used the characterization of SRB measure in terms of entropy (Theorem 6). In the proof of Proposition 9 we will use the absolutely continuity property of SRB measures. Let μ be a Borel measure on M. We recall a measurable partition ξ in the sense of Rokhlin [35] is said μ-subordinate to W^u when $\xi(x)\subset W^u(x)$ and $\xi(x)$ contains an open neighborhood of x in the topology of $W^u(x)$ for μ-almost every x. The measure μ is said to have **absolutely continuous conditional measures on unstable manifolds** if for every measurable partition ξ μ-subordinate to W^u, the conditional measures μ_x^ξ of μ with respect to ξ satisfy $\mu_x^\xi \ll Leb_{W^u(x)}$ for μ-almost every x.

Proof. We argue by contradiction. Take Γ as above. Assume there is a Borel set B with positive Lebesgue measure contained in the complement of $W^s(\Gamma)$ such that we have $\chi(x) > b > \frac{R(f)}{r}$ for all $x\in B$. Then we follow the approach of Section 5.3. We consider a C^r smooth disc σ with $\chi(x,v_x)>b$ for $x\in B'\subset B$, $\mathrm{Leb}_{\sigma_*}(B')>0$. One can then define the geometric set E on a subset B'' of B with $\mathrm{Leb}_{\sigma_*}(B'')>0$. We also let τ, β, α and ϵ be the parameters associated to E. Recall that :

- E is τ-large with respect to the derivative cocycle Φ,
- $\overline{d}(E(x))\geq \beta>0$ for $x\in B''$,
- $D_k(y)=f^k(H_k(y))$ has semi-length larger than $\alpha\epsilon$ when $k\in E(y)$, $y\in B''$.

Let B''' be the subset of B'' given by density points of B'' with respect to Leb_{σ_*}. In particular, we have

$$\forall x\in B''',\quad \frac{\mathrm{Leb}_{\sigma_*}(H_k(x)\cap B'')}{\mathrm{Leb}_{\sigma_*}(H_k(x))}\xrightarrow{k\to+\infty} 1.$$

We choose a subset A of B''' with $\mathrm{Leb}_{\sigma_*}(A)>0$ such that the above convergence is uniform in $x\in A$. Then from this set A and the geometric set E on A we may build $\mathfrak{n}$, $(F_n)_{n\in\mathfrak{n}}$ and $(\mu_n^{F_n})_{n\in\mathfrak{n}}$ as in Sections 2 and 3. As proved in Section 5.3 any limit measure μ of $\mu_n^{F_n}$ is supported on the unstable bundle and projects to a SRB measure ν with $\chi_1(x)\geq\tau>0>-\tau\geq\chi_2(x)$ for ν a.e. x. The measure ν is a barycenter of ergodic SRB measures with such exponents. Take $P=K_N$ a Pesin block with $\nu(P)\sim 1>1-\beta$. We let θ and l be respectively the minimal angle between E_u and E_s and the minimal length of the local stable and unstable manifolds on P.

Let ξ be a measurable partition subordinate to W^u with diameter less then $\gamma>0$. We have $\nu(\Gamma\cap P)=\int \nu_x^\xi(\Gamma\cap P)\,d\nu(x)\sim 1$ and $\nu_x^\xi\ll Leb_{W^u_\gamma(x)}$ for ν a.e. x. Therefore we get for some $c>0$

$$\nu\left(x,\ \mathrm{Leb}_{W^u_\gamma(x)}(\Gamma\cap P)>c\right)\sim 1.$$

We let $F=\{x\in\Gamma\cap P,\ \mathrm{Leb}_{W^u_\gamma(x)}(\Gamma\cap P)>c\}$. Observe that we have again $\nu(F)\sim 1$. For $x\in\sigma_*$ and $y\in P$ we use the following notations :

$$\hat{x}_\sigma=(x,v_x)\in\mathbb{P}T\sigma_*\qquad \hat{y}_u=(y,v_y^u)\in\mathbb{P}TM,$$

where v_y^u is the element of $\mathbb{P}TM$ representing the line $E_u(y)$. Let $\hat{F}_u^\gamma$ be the open $\gamma/8$-neighborhood of $\hat{F}_u:=\{\hat{y}_u,\ y\in F\}$ in $\mathbb{P}TM$. Recall $E(x)$ denotes the set of geometric times of x. We let for $n\in\mathfrak{n}$:

$$\zeta_n:=\int\frac{1}{\sharp F_n}\sum_{k\in E(x)\cap F_n}\delta_{F^k\hat{x}_\sigma}\,d\mu_n(\hat{x}_\sigma).$$

Observe that $\zeta_n(\mathbb{P}TM) \geq \inf_{x\in A_n} d_n(E(x)\cap F_n)$. By the last item in Lemma 2, we have $\liminf_{n\in\mathfrak{n}} \inf_{x\in A_n} d_n(E(x)\cap F_n) \geq \beta$. Therefore there is a weak limit $\zeta = \lim_k \zeta_{p_k}$ with $\zeta \leq \mu$ and $\zeta(\mathbb{P}TM) \geq \beta$. From $\mu(\hat{F}_u^\gamma) \sim 1 > 1-\beta$ we get $0 < \zeta(\hat{F}_u^\gamma) \leq \lim_k \zeta_{p_k}(\hat{F}_u^\gamma)$. Note also $\hat{A}_\sigma := \{\hat{y}_\sigma,\ y\in A\}$ has full μ_n-measure for all n. In particular, for infinitely many $n\in\mathbb{N}$ there is $(x^n, v_{x^n}) = \hat{x}^n_\sigma \in \hat{A}_\sigma$ with $F^n\hat{x}^n_\sigma \in \hat{F}_u^\gamma$ and $n\in E(x^n)$. Let $\hat{y}^n_u = (y^n, v^u_{y^n}) \in \hat{F}_u$ which is $\gamma/8$-close to $F^n\hat{x}^n_\sigma$. Then for $\gamma \ll \delta \ll \min(\theta, l)$ independent of n, the curve $D_n^\delta(x^n) := D_n(x^n)\cap B(f^n x^n, \delta)$ is transverse to $W^s(P\cap\Gamma\cap W^u_\gamma(y^n))$ and may be written as the graph of a C^r smooth function $\psi : E\subset E_u(y^n)\to E_s(y^n)$ with $\|d\psi\| < L$ for a universal constant L.

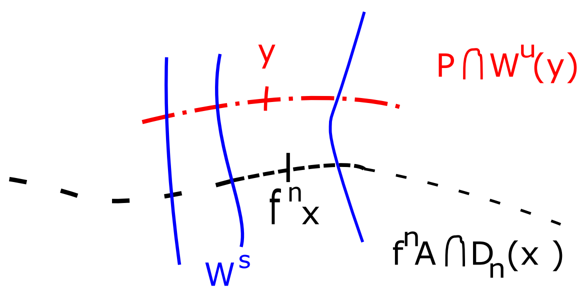


Figure 2: ***Holonomy of the local stable foliation between the transversals*** $D_n(x)$ ***and*** $W^u_\gamma(y)$.

By Theorem 8.6.1 in [33] the associated holonomy map $h : W^u_\gamma(y) \to D_n^\delta(x^n)$ is absolutely continuous and its Jacobian is bounded from below by a positive constant depending only on the Pesin block $P = K_N$ (not on x^n and y^n). In particular $\mathrm{Leb}_{D_n(x^n)}(W^s(\Gamma\cap P)) \geq c'$ for some constant c' independent of n. The distorsion of df^n on $H_n(x^n)$ being bounded by 3, we get (recall $f^nH_n(x^n) = D_n(x^n)$):

$$(\alpha\epsilon)^{-1}\,\mathrm{Leb}\,(D_n(x^n)\setminus f^nB) \leq \frac{\mathrm{Leb}\,(D_n(x^n)\setminus f^nB)}{\mathrm{Leb}\,(D_n(x^n))} \leq 9\frac{\mathrm{Leb}\,(H_n(x^n)\setminus B)}{\mathrm{Leb}\,(H_n(x^n))} \xrightarrow{n\to\infty} 0.$$

Therefore for n large enough, there exists $x\in f^nB\cap W^s(\Gamma\cap P)$, in particular $B\cap f^{-n}W^s(\Gamma) = B\cap W^s(\Gamma)\neq\emptyset$. This contradicts the definition of B.

□

6.2. **The maximal exponent.** Let $\mathcal{R}^{+*}$ denote the invariant subset of Lyapunov regular points x of (M,f) with $\chi_1(x) > 0 > \chi_2(x)$. Such a point admits so called regular neighborhoods (or ϵ-Pesin charts):

Lemma 12. [32] *For a fixed $\epsilon > 0$ there exists a measurable function $q = q_\epsilon : \mathcal{R}^{+*}\to(0,1]$ with $e^{-\epsilon} < q(fx)/q(x) < e^\epsilon$ and a collection of embeddings $\Psi_x : B(0,q(x))\subset T_xM = E_u(x)\oplus E_s(x)\sim\mathbb{R}^2\to M$ with $\Psi_x(0) = x$ such that $f_x = \Psi_{fx}^{-1}\circ f\circ\Psi_x$ satisfies the following properties :*

- $$d_0f_x = \begin{pmatrix} a^1_\epsilon(x) & 0 \\ 0 & a^2_\epsilon(x), \end{pmatrix}$$
  with $e^{-\epsilon}e^{\chi_i(x)} < a^i_\epsilon(x) < e^\epsilon e^{\chi_i(x)}$ for $i = 1,2$,
- *the C^1 distance between f_x and d_0f_x is less than ϵ,*
- *there exists a constant K and a measurable function $A = A_\epsilon : \mathcal{R}^{+*}\to\mathbb{R}$ such that for all $y,z\in B(0,q(x))$*
  $$Kd(\Psi_x(y),\Psi_x(z)) \leq \|y-z\| \leq A(x)d(\Psi_x(y),\Psi_x(z)),$$

with $e^{-\epsilon}A(fx)/A(x) < e^{\epsilon}$.

For any $i \in I$ we let

$$E_i := \{x,\ \chi(x) = \chi(\mu_i)\}.$$

The set E_i has full μ_i-measure by the subadditive ergodic theorem. Put $\Gamma_i = \mathcal{B}(\mu_i) \cap E_i \cap \mathcal{R}^{+*}$ and $\Gamma = \bigcup_i \Gamma_i$. Clearly Γ is f-invariant. We finally check that $\chi(x) = \chi(\mu_i)$ for $x \in W^s(\Gamma_i)$.

For uniformly hyperbolic systems, we have

$$\Sigma\chi(x) = \lim_n \frac{1}{n} \log \mathrm{Jac}(d_x f^n_{E_u}) = \lim_n \int \log \mathrm{Jac}(d_y f_{E_u})\, d\delta^n_x.$$

As the geometric potential $y \mapsto \log \mathrm{Jac}\, d_y f_{E_u}$ is continuous in this case, any point in the basin of a SRB measure μ satisfies $\Sigma\chi(x) = \int \Sigma\chi(y)\, d\mu(y)$.

Lemma 13. *If* $y \in W^s(x)$ *with* $x \in \mathcal{R}^{+*}$, *then* $\chi(y) = \chi(x)$.

Proof. Fix $x \in \mathcal{R}^{+*}$ and $\delta > 0$. We apply Lemma 12 with $\epsilon \ll \chi_1(x)$. For $\alpha > 0$ we let $\mathcal{C}_\alpha$ be the cone $\mathcal{C}_\alpha = \{(u, v) \in \mathbb{R}^2,\ \alpha\|u\| \geq \|v\|\}$. We may choose $\alpha > 0$ and $\epsilon > 0$ so small that for all $k \in \mathbb{N}$ we have $d_z f_{f^k x}(\mathcal{C}_\alpha) \subset \mathcal{C}_\alpha$ and $\|d_z f_{f^k x}(v)\| \geq e^{\chi_1(x) - \delta}$ for all $v \in \mathcal{C}_\alpha$ and all $z \in B(0, q_\epsilon(f^k x))$.

Let $y \in W^s(x)$. There is $C > 0$ and λ such that $d(f^n x, f^n y) < C\lambda^n$ holds for all $n \in \mathbb{N}$. We can choose $\epsilon \ll \lambda$. In particular there is $N > 0$ such that $f^n y$ belongs to $\Psi_{f^n x} B(0, q(f^n x))$ for $n \geq N$ since we have $A(f^n x) < e^{\epsilon n} A(x)$ and $q(f^n x) > e^{\epsilon n} q(x)$. Let $z \in B(0, q(f^N x))$ with $\Psi_{f^N x}(z) = y$. Then for all $v \in \mathcal{C}_\alpha$ and for all $n \geq N$ we have $\|d_z \left(\Psi^{-1}_{f^{n-N}x} \circ f^{n-N} \circ \Psi_{f^N x}\right)(v)\| \geq e^{(n-N)(\chi_1(x) - \delta)}$. As the conorm of $d_{f^{n-N}y}\psi_{f^n x}$ is bounded from above by $A(f^n x)^{-1}$ for all n we get

$$\begin{aligned}
\chi(y) &= \limsup_n \frac{1}{n} \log \|d_y f^{n-N}\|, \\
&= \limsup_n \frac{1}{n} \log \|d_z \left(f^{n-N} \circ \Psi_{f^N x}\right)\|, \\
&\geq \limsup_{n \to +\infty} \frac{1}{n} \log \left(A(f^n x)^{-1} \left\|d_z \left(\Psi^{-1}_{f^n x} \circ f^n \circ \Psi_{f^N x}\right)\right\|\right), \\
&\geq \chi_1(x) - \delta - \epsilon.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\left\|d_z \left(\Psi^{-1}_{f^n x} \circ f^n \circ \Psi_{f^N x}\right)\right\| &\leq \prod_{k=N}^{n-1} \sup_{t \in B(0, q(f^k x))} \|d_t f_{f^k x}\|, \\
&\leq \left(e^{\chi_1(x) + \epsilon} + \epsilon\right)^{n-N}, \\
&\leq e^{(n-N)(\chi_1(x) + 2\epsilon)}.
\end{aligned}$$

Then it follows from $\|d_{f^{n-N}y}\psi_{f^n x}\| \leq K$:

$$\begin{aligned}
\chi(y) &\leq \limsup_{n \to +\infty} \frac{1}{n} \log \left(\left\|d_z \left(\Psi^{-1}_{f^n x} \circ f^n \circ \Psi_{f^N x}\right)\right\|\right), \\
&\leq \chi_1(x) + 2\epsilon.
\end{aligned}$$

As it holds for arbitrarily small ϵ and δ we get $\chi(y) = \chi_1(x) = \chi(x)$.

$\square$

We conclude with $\Lambda = \{\chi(\mu_i),\ i \in I\}$ that for Lebesgue a.e. point x, we have $\chi(x) \in \left]-\infty, \frac{R(f)}{r}\right] \cup \Lambda$ and that $\{\chi = \lambda\} \overset{o}{\subset} \bigcup_{i\in I,\, \chi(\mu_i)=\lambda} \mathcal{B}(\mu_i)$ for all $\lambda \in \Lambda$. The proof of the Main Theorem is now complete. It follows also from Lemma 13, that the converse statement of Corollary 2 holds : if (f, M) admits a SRB measure then $\mathrm{Leb}(\chi > 0) > 0$.

In the proof of Lemma 13 we can choose the cone $\mathcal{C}_\alpha$ to be contracting, so that any vector in a small cone at y will converge to the unstable direction $E_u(x)$. In other terms if we endow the smooth manifold $\mathbb{P}TM$ with a smooth Riemanian structure, then the lift μ to the unstable bundle of an ergodic SRB measure ν is a physical measure of $(\mathbb{P}TM, F)$. Conversely if μ is a physical measure of $(\mathbb{P}TM, F)$ supported on the unstable bundle above $\mathcal{R}^{+*}$, we can reproduce the scheme of the proof of the Main Theorem to show μ projects to a SRB measure ν. Indeed we may consider a C^∞ smooth curve σ such that $\hat{x} = (x, v_x)$ lies in the basin of μ for $\hat{x}$ in a positive Lebesgue measure set A of $\hat{\sigma}_*$. Then by following the above construction of SRB measures, we obtain that $\mu = \lim_n \frac{1}{n}\sum_{k=0}^{n-1} F_*^k \mathrm{Leb}_A$ project to a SRB measure (or one can directly use the approach of [13]). This converse statement is very similar to a result of Tsujii (Theorem A in [41]) which states in dimension two that for C^{1+} surface diffeomorphism an ergodic hyperbolic measure ν, such that the set of **regular** points $x \in \mathcal{B}(\nu)$ with $\chi(x) = \int \chi\, d\nu$ has positive Lebesgue measure, is a SRB measure. Indeed if μ is a physical measure of $(\mathbb{P}TM, F)$ supported on the unstable bundle above $\mathcal{R}^{+*}$, its projection ν satisfies $\chi(\nu) = \chi(x)$ for any $x \in \pi(\mathcal{B}(\nu))$. In the present paper we are working with the stronger C^∞ assumption, but, in return, points in the basin are not supposed to be regular contrarily to Tsujii's theorem.

From the above discussion, we may restate the Main Theorem as follows:

Theorem 10. *Let $f : M \circlearrowleft$ be a C^∞ surface diffeomorphism and $F : \mathbb{P}TM$ be the induced map on the projective tangent bundle. Then the basins of the physical measures of $(\mathbb{P}TM, F)$ are covering Lebesgue almost everywhere the set $\{(x,v) \in \mathbb{P}TM,\ \chi(x,v) > 0\}$.*

7. Nonpositive exponent in contracting sets

In this last section we show Theorem 2. For a dynamical system (M, f) a subset U of M is said **almost contracting** when for all $\epsilon > 0$ the set $E_\epsilon = \{k \in \mathbb{N},\ \mathrm{diam}(f^k U) > \epsilon\}$ satisfies $\overline{d}(E_\epsilon) = 0$. In [22] the authors build subsets with historic behaviour and positive Lebesgue measure which are almost contracting but not contracting. We will show Theorem 2 for almost contracting sets.

We borrow the next lemma from [13] (Lemma 4 therein), which may be stated with the notations of Section 5 as follows :

Lemma 14. *Let $f : M \circlearrowleft$ be a C^∞ diffeomorphism admitting and let U be a subset of M with $\mathrm{Leb}(\{\chi > a\} \cap U) > 0$ for some $a > 0$. Then for all $\gamma > 0$ there is a C^∞ smooth embedded curve σ_* and $I \subset \mathbb{N}$ with $\sharp I = \infty$ such that*

$$\forall n \in I,\ |\{x \in U \cap \sigma_*,\ \|d_x f^n(v_x)\| > e^{na}\}| > e^{-n\gamma}.$$

We are now in a position to prove Theorem 2 for almost contracting sets.

Proof of Theorem 2. We argue by contradiction by assuming $\mathrm{Leb}(\{\chi > a\} \cap U) > 0$ for some $a > 0$ with U being a almost contracting set. By Yomdin's Theorem on one-dimensional local volume growth for C^∞ dynamical systems [42] there is $\epsilon > 0$ so small that

$$v^*(f,\epsilon) = \sup_\sigma \limsup_{n\to\infty} \frac{1}{n} \sup_{x\in M} \log|f^n(B(x,\epsilon,n) \cap \sigma_*| < a/2, \tag{7.1}$$

where the supremum holds over all C^∞ smooth embedded curves $\sigma : [0,1] \to M$. As U is almost contracting, there are subsets $(C_n)_{n\in\mathbb{N}}$ of M with $\lim_n \frac{\log \sharp C_n}{n} = 0$ satisfying for all n

$$U \subset \bigcup_{x\in C_n} B(x,\epsilon,n). \tag{7.2}$$

Fix an error term $\gamma \in]0, \frac{a}{2}[$. Then by Lemma 14 there is a C^∞-smooth curve $\sigma_* \subset U$ and an infinite subset I of $\mathbb{N}$ such that for all $n \in I$

$$\begin{aligned}
\sum_{x\in C_n} |f^n(B(x,\epsilon,n)\cap\sigma_*)| &\geq |f^n(U\cap\sigma_*)|, \\
&\geq e^{na}\left|\{x\in U\cap\sigma_*,\ \|d_xf^n(v_x)\| > e^{na}\}\right|, \\
&\geq e^{n(a-\gamma)} \text{ by (7.1)}, \\
\sharp C_n \sup_{x\in M} \log |f^n(B(x,\epsilon,n)\cap\sigma_*)| &\geq e^{n(a-\gamma)} \text{ by (7.2)}.
\end{aligned}$$

Therefore we get the contradiction $v^*(f,\epsilon) > a-\gamma > a/2$. □

Appendix A

Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ be a sequence in $M_d(\mathbb{R}^d)$. For any $n \in \mathbb{N}$ we let $A^n = A_{n-1}\cdots A_1A_0$. We define the Lyapunov exponent $\chi(\mathcal{A})$ of $\mathcal{A}$ with respect to $v \in \mathbb{R}^d\setminus\{0\}$ as

$$\chi(\mathcal{A},v) := \limsup_n \frac{1}{n}\log\|A^n(v)\|,$$

Lemma 15.

$$\sup_{v\in\mathbb{R}^d\setminus\{0\}} \chi(\mathcal{A},v) = \limsup_n \frac{1}{n}\log |||A^n|||.$$

Proof. The inequality $\leq$ is obvious. Let us show the other inequality. Let $v_n \in \mathbb{R}^d$ with $\|v_n\| = 1$ and $\|A^n(v_n)\| = |||A^n|||$. Then take $v = \lim_k v_{n_k}$ with $\lim_k \frac{1}{n_k}\log|||A^{n_k}||| = \limsup_n \frac{1}{n}\log|||A^n|||$. We get

$$\begin{aligned}
\|A^{n_k}(v)\| &\geq \|A^{n_k}(v_k)\| - \|A^{n_k}(v-v_k)\|, \\
&\geq |||A^{n_k}|||(1-\|v-v_k\|), \\
\limsup_k \frac{1}{n_k}\log\|A^{n_k}(v)\| &\geq \limsup_n \frac{1}{n}\log|||A^n|||.
\end{aligned}$$

□

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