MEAN DIMENSION OF CONTINUOUS CELLULAR AUTOMATA

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Abstract. We investigate the mean dimension of a cellular automaton (CA for short) with a compact non-discrete space of states. A formula for the mean dimension is established for *(near) strongly permutative*, *permutative algebraic* and *unit* one-dimensional automata. In higher dimensions, a CA permutative algebraic or having a spaceship has infinite mean dimension. However, building on Meyerovitch's example [Mey08], we give an example of algebraic surjective cellular automaton with positive finite mean dimension.

Contents

1. Introduction

The most basic topological invariant of topological dynamical systems is the topological entropy which has been studied for a long time. Gromov [Gro99] introduced a new topological invariant of dynamical systems called *mean topological dimension* as a dynamical analogue of topological covering dimension. It was further developed by Lindenstrauss and Weiss [LW00] with their work on the *metric mean dimension*, the dynamical quantity associated to the Minkowski dimension. A dynamical system of finite entropy or finite dimension has zero mean dimension.

Let X be a compact metric space and let f be a continuous function given by $f : X^I \to X$ with a finite set $I \subset \mathbb{Z}^d$. A *cellular automaton (CA for short)* on $X^{\mathbb{Z}^d}$ with local rule f is the continuous map $F : X^{\mathbb{Z}^d} \to X^{\mathbb{Z}^d}$ defined by

$$F\left((x_n)_{n\in\mathbb{Z}}\right) = \left(f((x_{n+j})_{j\in I})\right)_{n\in\mathbb{Z}^d}.$$

We also denote sometimes the cellular automaton F associated to the local rule f by T_f. The space X is called the *set of states* of the cellular automaton T_f. When X is finite, the topological entropy of a one-dimensional cellular automaton (i.e. $d = 1$) is known to be finite. Moreover, the explicit value of the topological entropy in some cases may be computed. For example, when d is equal to 1, the set of states X is the finite field $\mathbb{F}_p$ with a prime number p and the local rule f is linear, i.e. $f\left((x_j)_{j\in I}\right) = \sum_{j\in I} a_j x_j$ for some $a_j \in \mathbb{F}_p^*$, the topological entropy of the cellular automaton T_f on $\mathbb{F}_p^{\mathbb{Z}}$ is equal to $\mathrm{diam}(I \cup \{0\}) \cdot \log p$ where $\mathrm{diam}(I \cup \{0\})$ stands for the diameter

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of $I \cup \{0\}$ with respect to the Euclidean distance on $\mathbb{R}$ [War00]. For $d > 1$, the topological entropy of a linear cellular automaton F is always infinite unless $F = \pm \mathrm{Id}$ [MW98, LL04]. An example of a multidimensional surjective cellular automaton with nonzero finite entropy was constructed in [Mey08].

In this paper, we investigate cellular automata with a non-discrete compact finite dimensional set of states X. We involve the theory of mean dimension to study these cellular automata. We show in Proposition 4.1 that the mean dimension of a one-dimensional cellular automaton is bounded from above by $\mathrm{diam}(I \cup \{0\}) \cdot \mathrm{stabdim} X$, where $\mathrm{stabdim} X$ denotes the stable topological dimension of X. As a consequence, the mean dimension of a one-dimensional automaton is finite. Then one may wonder whether this upper bound is the value of its mean dimension. The *permutative* property of cellular automaton implies that it has a topological factor with mean dimension $\mathrm{diam}(I \cup \{0\}) \cdot \mathrm{stabdim} X$. However, the permutative property does not ensure that such an upper bound is achieved as the value of its mean dimension. Several examples are discussed in Section 3.1.2. Nevertheless, when a one-dimensional cellular automaton is *(near) strongly permutative* or permutative *algebraic*, the mean dimension is equal to $\mathrm{diam}(I \cup \{0\}) \cdot \mathrm{stabdim} X$ (Theorem 5.3, Corollary 5.10 and Lemma 7.3). We summarize the aforementioned results in the following statement :

Theorem A. *Let $F : X^{\mathbb{Z}} \circlearrowleft$ be a one-dimensional CA with local rule $f : X^I \to X$. Then we have*

$$\mathrm{mdim}(F) \leq \mathrm{diam}(I \cup \{0\}) \cdot \mathrm{stabdim} X. \tag{1·1}$$

The equality holds in (1·1) when F is near strongly permutative or permutative algebraic, but the inequality may be strict for a general permutative CA.

On the other hand, when considering a unit (i.e. $I = \{1\}$) one-dimensional cellular automaton (X, T), we show that its natural extension is topologically conjugate to a full shift and the mean dimension of it is then given by that of its natural extension. To this end, we also investigate the relation between a dynamical system and its natural extension: whereas the topological entropy is preserved by natural extension, the mean dimension of a dynamical system is always larger than or equal to that of its natural extension (Proposition 3.5) but may differ. In Proposition 3.7 we give an example of a topological system, whose mean dimension is strictly larger than that of its natural extension.

Furthermore, we study multidimensional cellular automata with a compact non-discrete set of states. We prove that nontrivial multidimensional algebraic permutative cellular automata have infinite mean dimension (Lemma 7.5). Building on Meyerovitch's example we construct a multidimensional algebraic non-permutative surjective cellular automaton with nonzero finite mean dimension (Proposition 7.7). Beyond the algebraic property, we also show that if a multidimensional cellular automaton has a *spaceship* (see the definition in Section 8), then its mean dimension is infinite.

Theorem B. *Let $F : X^{\mathbb{Z}^d} \circlearrowleft$ be a CA with $d > 1$. When F is a nontrivial permutative algebraic CA or F has a spaceship, the mean dimension of F is infinite. But there are algebraic non-permutative F surjective cellular automata with nonzero finite mean dimension.*

2. Background on mean dimension

Let X be a compact space. For two finite open covers $\mathcal{A}$ and $\mathcal{B}$ of X, we say that the cover $\mathcal{B}$ is *finer* than the cover $\mathcal{A}$, and write $\mathcal{B} \succ \mathcal{A}$, if for every element of $\mathcal{B}$, one can find an element of $\mathcal{A}$ which contains it. For a finite open cover $\mathcal{A}$ of X, we define the quantities

$$\mathrm{ord}(\mathcal{A}) := \sup_{x \in X} \sum_{A \in \mathcal{A}} 1_A(x) - 1,$$

and

$$D(\mathcal{A}) := \min_{\mathcal{B} \succ \mathcal{A}} \mathrm{ord}(\mathcal{B}).$$

Clearly, if $\mathcal{B} \succ \mathcal{A}$ then $D(\mathcal{B}) \geq D(\mathcal{A})$. The *(topological) dimension* of X is defined by

$$\dim(X) := \sup_{\mathcal{A}} D(\mathcal{A}),$$

where $\mathcal{A}$ runs over all finite open covers of X. For a non-empty compact X, the *stable topological dimension* of X is given by

$$\mathrm{stabdim}(X) := \lim_{n\to\infty} \frac{\dim(X^n)}{n} = \inf_{n\to\infty} \frac{\dim(X^n)}{n}.$$

The limit above exists by sub-additivity of the sequence $\{\dim(X^n)\}_{n\geq 1}$. Moreover, if X is finite dimensional, then we have either $\mathrm{stabdim}(X) = \dim(X)$ (the set X is then said of *basic type*) or $\mathrm{stabdim}(X) = \dim(X) - 1$ (and X is said of *exceptional type*) [**?**].

For finite open covers $\mathcal{A}$ and $\mathcal{B}$, we set the joint $\mathcal{A} \vee \mathcal{B} := \{U \cap V : U \in \mathcal{A}, V \in \mathcal{B}\}$. It is easy to check that $D(\mathcal{A} \vee \mathcal{B}) \leq D(\mathcal{A}) + D(\mathcal{B})$. Let (X, T) be a topological dynamical system, i.e. X is a compact metrizable space and $T : X \circlearrowleft$ is a continuous map. The *mean dimension* of (X, T) is defined by

$$\mathrm{mdim}(X, T) = \sup_{\alpha} \lim_{n\to\infty} \frac{D(\bigvee_{i=0}^{n-1} T^{-i}\alpha)}{n},$$

where α runs over all finite open covers of X. The existence of the limit follows from the sub-additivity of the sequence $\left(D(\bigvee_{i=0}^{n-1} T^{-i}\alpha)\right)_n$. We write also $\mathrm{mdim}(X, T, \alpha) = \lim_{n\to\infty} \frac{D(\bigvee_{i=0}^{n-1} T^{-i}\alpha)}{n}$.

For a set Z and $\epsilon > 0$, a map $f : X \to Z$ is called (ρ, ϵ)*-injective* if $\mathrm{diam}(f^{-1}(z)) < \epsilon$ for all $z \in Z$. The ϵ*-dimension* $\dim_\epsilon(X, \rho)$ is defined by

$$\dim_\epsilon(X, \rho) = \inf_Y \dim(Y),$$

where Y runs over all compact metrizable spaces for which there exists a (ρ, ϵ)-injective continuous map $f : X \to Y$.

We mention some basic properties of mean dimension. We refer to the book [Coo05] for the proofs and further properties.

- $$\mathrm{mdim}(X, T) = \sup_\epsilon \lim_{n\to\infty} \frac{\dim_\epsilon(X, d_n)}{n},$$
  where d is a metric on X compatible with the topology and $d_n(x, y) = \max_{0\leq i\leq n-1} d(T^i x, T^i y)$. We write sometimes $\mathrm{mdim}(X, T, d, \epsilon) = \lim_{n\to\infty} \frac{\dim_\epsilon(X, d_n)}{n}$.
- If (Y, T) is a subsystem of (X, T), i.e. Y is a closed T-invariant subset of X, then $\mathrm{mdim}(Y, T) \leq \mathrm{mdim}(X, T)$.
- For $n \in \mathbb{N}$, $\mathrm{mdim}(X, T^n) = n \cdot \mathrm{mdim}(X, T)$.
- For dynamical systems (X_i, T_i), $1 \leq i \leq n$, we have
  $$\mathrm{mdim}(X_1 \times X_2 \times \cdots \times X_n, T_1 \times T_2 \times \cdots \times T_n) \leq \sum_{i=1}^n \mathrm{mdim}(X_i, T_i).$$

Let (X, d, T) be a topological dynamical system where d is a metric compatible with the topology of X. Let $K \subset X$ and $\epsilon > 0$. A subset E of X is said to be (n, ϵ)*-separated* if we have $d_n(x, y) > \epsilon$ for any $x \neq y \in E$. Denote by $s_n(d, T, K, \epsilon)$ the largest cardinality of any (n, ϵ)-separated subset of K. Define

$$h_d(K, T, \epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(d, T, K, \epsilon).$$

We sometimes write $h_d(T, \epsilon)$ when $K = X$. The topological entropy is the limit of $h_d(T, \epsilon)$ when ϵ goes to zero. The *upper metric mean dimension* of the system $(\mathcal{X}, d, T)$ is defined by

$$\overline{\mathrm{mdim}}_M(X, d, T) = \limsup_{\epsilon\to 0} \frac{h_d(T, \epsilon)}{\log \frac{1}{\epsilon}}.$$

Similarly, the *lower metric mean dimension* is defined by

$$\underline{\mathrm{mdim}}_M(X, d, T) = \liminf_{\epsilon\to 0} \frac{h_d(T, \epsilon)}{\log \frac{1}{\epsilon}}.$$

If the upper and lower metric mean dimensions coincide, then we call their common value the metric mean dimension of (X, d, T) and denote it by $\mathrm{mdim}_M(X, d, T)$. Unlike the topological entropy, the metric mean dimension depends on the metric d. An important property of metric mean dimension is

$$\mathrm{mdim}(X, T) \leq \underline{\mathrm{mdim}}_M(X, d, T),$$

for any metric d [LW00].

3. Natural extension $(\widetilde{X_T}, \widetilde{T})$ and skew-product

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of compact metric spaces and let $\mathbf{f} = (f_i)_{i\in\mathbb{N}}$ be a sequence of continuous maps $f_i : X_{i+1} \to X_i$. Then the inverse limit of $\mathbf{f}$ denoted by $\varprojlim \mathbf{f}$ is the set of sequences $(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i$ satisfying $f_i(x_{i+1}) = x_i$ for all $i \in \mathbb{N}$. The maps f_i, $i \in \mathbb{N}$ are called the *bonding maps* of the inverse limit.

Let (X, T) be a topological dynamical system. The inverse limit $\varprojlim \mathbf{f}$ with $f_i = T$ and $X_i = \bigcap_{n\geq 0} T^n X$ for all $i \in \mathbb{N}$ is denoted by $\widetilde{X_T}$. Notice that $\widetilde{X_T}$ is a closed shift-invariant subspace of $X^{\mathbb{N}}$. The *natural extension* of (X, T) is the dynamical system $(\widetilde{X_T}, \widetilde{T})$, where $\widetilde{T}$ is the continuous maps on $\widetilde{X_T}$ which sends $(x_k)_{k\in\mathbb{N}}$ to $(Tx_0, x_0, x_1, \cdots)$. The map $\widetilde{T}$ is an homeomorphism of $\widetilde{X_T}$. Moreover the map $\pi : (x_k)_{k\in\mathbb{N}} \mapsto x_0$ semi-conjugates $\widetilde{T}$ and T, i.e. $\pi \circ \widetilde{T} = T \circ \pi$, and π is surjective when T is surjective. Moreover, if T is a homeomorphism, then (X, T) and $(\widetilde{X_T}, \widetilde{T})$ are topologically conjugated. The invertible topological system $(\widetilde{X_T}, \widetilde{T})$ satisfies the following universal property : for any extension $\psi : (Y, S) \to (X, T)$ by an invertible topological system (Y, S) there is a unique extension $\widetilde{\psi} : (Y, S) \to (\widetilde{X_T}, \widetilde{T})$ satisfying $\psi = \pi \circ \widetilde{\psi}$, which is given by $\widetilde{\psi}(y) = \big(\psi(S^{-k}y)\big)_{k\in\mathbb{N}}$ for all $y \in Y$.

3.1. Dimension of $\widetilde{X_T}$.

The inverse limits of topological spaces and their topological aspects have been highly investigated (see [IM12] and the references therein). We are interested in dimension properties of $\widetilde{X_T}$. An inverse limit of compact metric spaces does not raise dimension : with the above notations, the dimension of $\varprojlim \mathbf{f}$ is less than or equal to $\liminf_{i\to\infty} \dim(X_i)$ (Corollary 182 [IM12]). But in general the inverse limit may have lower dimension, even when one consider a single bonding surjective map, i.e. the dimension of $\widetilde{X_T}$ may be less than the dimension of X for a surjective dynamical topological system (X, T).

We give two general constructions of surjective maps $T : X \circlearrowleft$ with $\dim(\widetilde{X_T}) < \dim X$. Moreover we may assume X is connected for $\dim(\widetilde{X_T}) > 1$. The first one is inspired from Example 183 in [IM12], whereas the second construction seems to be new. Contrarily to the first one, the second one allows to build finite-to-one examples. In particular we will show the following proposition.

Proposition 3.1. *For any integers $0 \leq k \leq n$, there exists a surjective finite-to-one topological system (X, T) with $\dim \widetilde{X_T} = k$ and $\dim(X) = n$. For $k \geq 1$ we may assume X is connected.*

We recall that the inverse limit of continua (i.e. compact connected space) is also a continuum. In particular when $\widetilde{X_T}$ is not reduced to a point, its topological dimension is positive. Therefore in the above proposition, the set of states X can not be connected for $k = 0$.

3.1.1. *Factor of a lower dimensional space.*

Lemma 3.2. *Let (X, T) be a topological dynamical system. Assume there are subsets Y, Z of X, such that*

- *$X = Y \cup Z$,*
- *$T(Z) = Y$ and $T(Y) = Z$,*
- *Y and Z are closed subsets of X.*

Then we have

$$\dim(\widetilde{X_T}) \leq \min(\dim Y, \dim Z).$$

Proof. We define a sequence of topological spaces X_i and maps $f_i : X_{i+1} \to X_i$ as follows. Let $X_i = Y$ for i even and $X_i = X$ for i odd and take f_i be the restriction of T to X_{i+1}. Let $\varprojlim \mathbf{f}$ be the induced inverse limit. We let $\mathbf{f}'$ be the family obtained by inverting the role of odd and even numbers. Then $\widetilde{X_T}$ is contained in the union of $\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{f}'$. By [IM12, Corcollary 182], the dimensions of $\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{f}'$ are less than or equal to the dimensions of Y and Z. This completes the proof. □

In the settings of Lemma 3.2 we have $\dim(X) = \max(\dim Y, \dim Z)$. Therefore to get an example with $\dim(\widetilde{X_T}) < \dim X$ it is enough to take surjective continuous maps $R : Y \to Z$ and $S : Z \to Y$ with compact metric spaces Y, Z satisfying $\dim Y < \dim Z$ and consider T on the disjoint union $X = Y \coprod Z$ with $T|_Y = R$ and $T|_Z = S$. It is not difficult to produce such examples satisfying Proposition 3.1, but the space X is not connected (because it is a disjoint union of non empty closed sets). This problem may be avoided thanks to the following corollary.

Corollary 3.3 (Example 183 in [IM12])**.** *Let (X, T) be a topological system. Assume there is a compact metric space Y and surjective maps $R : X \to Y$ and $S : Y \to X$ with $T = S \circ R$. Then we have*

$$\dim(\widetilde{X_T}) \leq \dim Y.$$

Proof. Consider the surjective map T' on the disjoint union $X \coprod Y$ with $T'|_X = R$ and $T'|_Y = S$. Then (X, T) is the restriction of T'^2 on X, therefore we get $\dim(\widetilde{X_T}) \leq \dim(\widetilde{X_{T'^2}})$. By the subsequence theorem [IM12], we have $\dim(\widetilde{X_{T'^2}}) = \dim(\widetilde{X_{T'}})$, so that we finally get by Lemma 3.2 that $\dim(\widetilde{X_T}) \leq \dim(\widetilde{X_{T'}}) \leq \dim Y$. □

In [IM12, Example 183] the authors consider more precisely $X = [0, 1]^n$ and $Y = [0, 1] \times \{0\}$ with S being the projection on the first coordinate and R be a space-filling curve, then $\dim \widetilde{X_T} = 1 < \dim X = n$. In general, we may generalize the above example to general $k \geq 1$ by replacing X and Y by $[0, 1]^{n+k-1}$ and $[0, 1]^k \times \{0\}$ respectively.

We remark that the examples produced as above are never finite-to-one. Indeed, in the settings of Lemma 3.2 for example, we have $\dim Z \leq \dim Y + \sup_{z \in Z} \dim T^{-1}z$ (see for example [Eng95, Theorem 1.12.4]). Then if T is finite-to-one, its fibers are zero-dimensional and we have necessarily $\dim Z \leq \dim Y$ and then $\dim Y = \dim Z$ by symmetry.

3.1.2. *Examples with a lower dimensional subsystem.*

Lemma 3.4. *Let (X, T) be a topological system. Assume there are subsets Y, Z of X, such that*

- *$X = Y \cup Z$,*
- *$T(Z) \subset Z$,*
- *Y and Z are closed subsets of X.*

Then we have

$$\dim(\widetilde{X_T}) \leq \max\left(\dim Y, \dim\left(\bigcap_{n \in \mathbb{N}} T^n Z\right)\right).$$

Proof. For each $k \in \mathbb{N}$ we let $\widetilde{Y_k}$ be the subset of $\widetilde{X_T}$ consisting in $(x_l)_{l \in \mathbb{N}}$ with $x_l \in Y$ for $l \geq k$. Then we have

$$\widetilde{X_T} = \widetilde{Z_T} \cup \bigcup_{k \in \mathbb{N}} \widetilde{Y_k}.$$

The sets $\widetilde{Y_k}$, $k \in \mathbb{N}$, are compact subsets of X and the inverse limit $\widetilde{Z_T}$ has topological dimension less than or equal to $\dim\left(\bigcap_{n \in \mathbb{N}} T^n Z\right)$. Moreover we have $\dim(\widetilde{Y_k}) \leq \dim Y$ for any k. By the countable union theorem (e.g. Theorem 1.7.1 in [Coo15]), we conclude that

$$\begin{aligned}\dim(\widetilde{X_f}) &\leq \max(\dim Y, \dim \widetilde{Z_T}),\\ &\leq \max(\dim Y, \dim \bigcap_{n \in \mathbb{N}} T^n Z).\end{aligned}$$

□

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Let $n > k \in \mathbb{N}$. We choose Y equal to the standard Cantor set $C \subset [0,1]$ and Z be the unit Euclidean cube $[0,1]^n$. Write Y as the union of $Y_1 := C \cap [0,1/3]$ and $Y_2 = C \cap [2/3,1]$, which are both homeomorphic to C. There is a finite-to-one continuous surjective map $\phi : Y_2 \simeq C \to Z$. For example, for $n = 1$, we may take $\phi : \{0,1\}^{\mathbb{N}^*} \to [0,1]$, $(a_k) \mapsto \sum_{k\geq 1} \frac{a_k}{2^k}$. Then we let $T : Y \to Y \coprod Z$ be equal to $x \mapsto 3x$ on Y_1 and $T = \phi$ on C_2. Then we have $T(Y) = Y \coprod Z$. On Z we let $T(y_1, \cdots, y_n) = (y_1, \cdots, y_k, y_{k+1}/2, \cdots, y_n/2)$, therefore $\bigcap_{n\geq 0} T^n Z = [0,1]^k \times \{0^{n-k}\}$. Let $X = Y \coprod Z$. By Proposition 3.4, we get $\dim(\widetilde{X_T}) \leq k$. Finally as T is the identity map on $[0,1]^k \times \{0^{n-k}\}$, this cube embeds in $\widetilde{X_T}$ and $\dim(\widetilde{X_T}) \geq k$. The space X is not connected but for $k > 0$ we may arrange the construction to ensure the connectedness of X. Take $Y = [-1,0] \times \{0^{n-1}\}$ and let $T : [-1,-1/2] \times \{0^{n-1}\} \to Z = [0,1]^n$ be a finite-to-one space filling curve (e.g. the Hilbert space filling curve) with $T(-1/2,0) = 0^n$. On $[-1/2,0] \times \{0^{n-1}\}$ we let $T(x) = (F_2(x_1), 0^{n-1})$,where F_2 denotes the continuous "tent" map affine on $[-1/2,-1/4]$ and $[-1/4,0]$ with $F_2(-1/2) = F_2(0) = 0$ and $F_2(-1/4) = -1$. By taking T on Z as above, we get the desired example.

□

3.2. **Mean dimension of $\widetilde{T}$.** Let (X,T) be a topological dynamical system and $(\widetilde{X_T}, \widetilde{T})$ be its natural extension. We study now the relation between the mean dimension of $\widetilde{T}$ and T. In this section, we show that the mean dimension of the natural extension is always less than or equal to the mean dimension of the system, but they may differ.

3.2.1. *Inequality.* Given a sequence $(X_i, T_i)_{i\in\mathbb{N}}$ of topological systems and a family of continuous maps $\mathbf{f} = (f_i : X_{i+1} \to X_i)_{i\in\mathbb{N}}$, we may define the inverse limit system as the map $\varprojlim \mathbf{T} : \varprojlim \mathbf{f} \circlearrowleft$ which maps $(x_i)_{i\in\mathbb{N}}$ to $(T_i x_i)_{i\in\mathbb{N}}$. The second author proved $\mathrm{mdim}(\varprojlim \mathbf{f}, \varprojlim \mathbf{T}) \leq \liminf_{i\to\infty} \mathrm{mdim}(X_i, T_i)$ in [Shi21, Proposition 5.8] [1]. For a topological system (X,T), we consider $(X_i, T_i) = (X,T)$ and $f_i = T$ for all i. Then $(\varprojlim \mathbf{f}, \varprojlim \mathbf{T})$ is just the natural extension $(\widetilde{X_T}, \widetilde{T})$ of (X,T). As a consequence we have in particular :

Proposition 3.5. *Let (X,T) be a topological dynamical system and $(\widetilde{X_T}, \widetilde{T})$ be its natural extension. Then we have*

$$\mathrm{mdim}(X,T) \geq \mathrm{mdim}(\widetilde{X_T}, \widetilde{T}). \tag{3·1}$$

We remark that the equality of (3·1) can be achieved: for example, the mean dimension of a unilateral full-shift is equal to the mean dimension of its natural extension, which is the corresponding bilateral shift.

Question 3.6. *Does the mean dimension of a general CA coincide with the mean dimension of its natural extension?*

In some cases we answer positively to the above question in the next sections.

3.2.2. *A counterexample (X_C, T_C) to the equality of (3·1).* The inequality in Proposition 3.5 may be strict. We present below an example.

Let $f : [0,1] \to \mathbb{R}$ be the $\times 3$-map., i.e. $x \mapsto 3x$ mod 1. For each $n \in \mathbb{N}$ we let C_n be the n^{th} standard Cantor set, i.e. $C_n := \bigcap_{0\leq l\leq n} f^{-l}([0,1/3] \cup [2/3,1])$. Observe that $f_n = f|_{C_n} : C_{n+1} \to C_n$ is surjective and $\bigcap_{n\in\mathbb{N}} C_n$ is the standard Cantor set C. We consider the compact metrizable space $X_C = \prod_{n\in\mathbb{N}} C_n$ and the surjective map $T_C : X_C \circlearrowleft$ defined by

$$\forall x = (x_n)_{n\in\mathbb{N}} \in X_C, \; T_C(x) = (3x_{n+1})_{n\in\mathbb{N}}.$$

Proposition 3.7. *We have*

$$\mathrm{mdim}(X_C, T_C) \geq 1 > \mathrm{mdim}(\widetilde{X_{T_C}}, \widetilde{T_C}) = 0$$

[1] Even though it is stated in [Shi21, Proposition 5.8] that $\mathrm{mdim}(\varprojlim \mathbf{f}, \varprojlim \mathbf{T}) \leq \sup_{i\in\mathbb{N}} \mathrm{mdim}(X_i, T_i)$, it is indeed shown that $\mathrm{mdim}(\varprojlim \mathbf{f}, \varprojlim \mathbf{T}) \leq \liminf_{i\to\infty} \mathrm{mdim}(X_i, T_i)$ by carefully checking its proof.

Proof. 1. We first show $\mathrm{mdim}(X_C, T_C) \geq 1$. For each $n \in \mathbb{N}$, we let I_n be the connected component of C_n containing 0. Then $f|_{I_{n+1}} : I_{n+1} \to I_n$ is an homeomorphism. The product $Y = \prod_{n\in\mathbb{N}} I_n \subset X_C$ satisfies $T_C(Y) \subset Y$ and the induced subsystem $(Y, (T_C)_{|Y})$ of (X_C, T_C) is topologically conjugated to the full shift $([0,1]^{\mathbb{Z}}, \sigma)$ via the conjugacy

$$\begin{aligned}\phi :& Y \to [0,1]^{\mathbb{N}},\\ &(x_n)_{n\in\mathbb{N}} \mapsto (3^n x_n)_{n\in\mathbb{N}}.\end{aligned}$$

Therefore $\mathrm{mdim}(X_C, T_C) \geq \mathrm{mdim}([0,1]^{\mathbb{N}}, \sigma) = 1$.

2. We check now that $\mathrm{mdim}(\widetilde{X_{T_C}}, \widetilde{T_C}) = 0$. In fact we will show that $(\widetilde{X_{T_C}}, \widetilde{T_C})$ is topologically conjugated to $(C^{\mathbb{Z}}, \sigma)$. An element x of $\widetilde{X_{T_C}}$ may be written under the form $x = (x_n^k)_{\substack{n\in\mathbb{N}\\ k\in\mathbb{N}}}$ with $x^k = (x_n^k)_{n\in\mathbb{N}} \in X_C = \prod_{n\in\mathbb{N}} C_n$ and $x = (x^k)_{k\in\mathbb{N}} \in \widetilde{X_{T_C}}$. Moreover the Cantor set is the inverse limit of the family $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$ (recall $f_n : C_{n+1} \to C_n$ is the the $\times 3$-map). We will use this identification $C = \varprojlim \mathbf{f}$. For $x \in \widetilde{X_{T_C}}$ we let $x^k = T_C^{-k}(x^0)$ for $k < 0$ so that we have $T_C(x^{k+1}) = x^k$ for all $k \in \mathbb{Z}$, i.e. $3x_{n+1}^{k+1} = x_n^k$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

We consider the map $\phi : \widetilde{X_{T_C}} \to C^{\mathbb{Z}}$ defined by

$$\begin{aligned}\forall x = (x_n^k)_{k,n} \in \widetilde{X_{T_C}},\ & \phi(x) = (y^k)_{k\in\mathbb{Z}}\\ &\text{with } y^k = (x_n^{n-k})_{n\in\mathbb{N}} \in C.\end{aligned}$$

This map takes value in $C^{\mathbb{Z}}$ because $3y_n^k = 3x_n^{n-k} = x_{n-1}^{n-k-1} = y_{n-1}^k$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Clearly ϕ is continuous and bijective with inverse $\phi^{-1} = \phi$. Finally we check easily that

$$\begin{aligned}\phi \circ \widetilde{T_C}(x) &= \phi((x^{k-1})_{k\in\mathbb{N}}),\\ &= (y^{k+1})_{k\in\mathbb{Z}},\\ &= \sigma \circ \phi(x).\end{aligned}$$

This completes the proof. □

3.2.3. *Metric mean dimension of* (X_C, T_C). For a compact metric space (X, d) and a sequence $\delta = (\delta_i)_{i\in\mathbb{N}}$ of positive numbers going to zero, we let d_δ be the distance on $X^{\mathbb{N}}$ defined by

$$\forall x, y \in X^{\mathbb{N}},\ d_\delta(x, y) = \sup_{i\in\mathbb{N}} \delta_i d(x_i, y_i).$$

A metric equivalent to some d_δ will be called a product metric and will be denoted by $d^{\mathbb{N}}$. Such a product metric is compatible with the product topology on $X^{\mathbb{N}}$.

We show in the Appendix A that for any zero-dimensional system (X, T), there is a metric on D on X with $\mathrm{mdim}_M(X, T, D) = \mathrm{mdim}(X, T) = 0$. In particular there is such a metric D_C for the system $(\widetilde{X_{T_C}}, \widetilde{T_C})$. One may wonder if D_C is a product metric. The following lemma applied to (X_C, T_C) shows that it is not the case.

Lemma 3.8. *Let* (X, d, T) *be a surjective topological dynamical system. Then for any product metric* $d^{\mathbb{N}}$

$$\mathrm{mdim}_M(\widetilde{X_T}, \widetilde{T}, d^{\mathbb{N}}) = \mathrm{mdim}_M(X, T, d).$$

Proof. One only needs to consider the case of $d^{\mathbb{N}} = d_\delta$ for some sequence $\delta = (\delta_i)_{i\in\mathbb{N}}$ with $\lim_{i\to\infty} \delta_i = 0$ and $0 < \delta_i \leq 1$ for all i, because any product metric is equivalent to a metric of this form.

1. $\underline{\mathrm{mdim}_M(\widetilde{X_T}, \widetilde{T}, d^{\mathbb{N}}) \geq \mathrm{mdim}_M(X, T, d)}$. Let $\epsilon > 0$ and $n \in \mathbb{N}$. Let E be a (n, ϵ)-separated set of (X, d, T). By surjectivity of T, there is for each $x \in E$ a point $\tilde{x} = (x_k)_{k\in\mathbb{N}} \in \widetilde{X_f} \subset X^{\mathbb{N}}$ with $x_0 = x$. Let $\widetilde{E} = \{\tilde{x},\ x \in E\}$. Then for all $x \neq y \in E$, there is $0 \leq k < n$ with $d(f^k x, f^k y) \geq \epsilon$ so

that we have

$$d_\delta(\widetilde{T}^k\tilde{x},\widetilde{T}^k\tilde{y}) \geq \delta_0 d(f^k x_0, f^k y_0),$$
$$\geq \delta_0 d(f^k x, f^k y) \geq \delta_0\epsilon.$$

Therefore $\widetilde{E}$ is a $(n,\delta_0\epsilon)$-separated subset of $(\widetilde{X_T}, d_\delta, \widetilde{T})$. One easily concludes that $\mathrm{mdim}_M(\widetilde{X_T},\widetilde{T},d^{\mathbb{N}}) \geq \mathrm{mdim}_M(X,T,d)$.

2. $\underline{\mathrm{mdim}_M(\widetilde{X_T},\widetilde{T},d^{\mathbb{N}}) \leq \mathrm{mdim}_M(X,T,d).}$ Fix $\epsilon>0$. Let $N>0$, such that $\sup_{i\geq N}\delta_i \mathrm{diam}_d(X)<\epsilon$. For $n\in\mathbb{N}$, we let Y be a (ϵ,n)-separated set of $(\widetilde{X_T}, d_\delta, \widetilde{T})$. Denote by $\pi:(\widetilde{X_T},\widetilde{T})\to(X,T)$, $(x_k)_{k\in\mathbb{N}}\mapsto x_0$ the natural extension. For $x\in Y$ we let E_x be the set of points $y\in Y$ such that $\pi(x)$ and $\pi(y)$ are not (ϵ,n)-separated with respect to d. Observe that for any $y\in E_x$, there is $i,j\in[0,N[$ with $(d(x_i,y_i)\geq)\,\delta_j d(x_i,y_i)\geq\epsilon$ (if not we would have $d_\delta(\widetilde{T}^k x,\widetilde{T}^k y)<\epsilon$ for all $0\leq k<n$).

Therefore the cardinality of E_x is bounded from above by some constant C depending only on N and ϵ (namely the maximal cardinality $C=C(\epsilon,N)$ of ϵ-separated set in X^N for the usual finite product distance d^N on X^N). Consequently there is a (ϵ,n)-separated subset $Z\subset\pi(Y)$ of (X,d,T) with $C\sharp Z\geq\sharp Y$. For example we may take any $z_0=\pi(x_0)$ in $\pi(Y)$, then $z_1=\pi(x_1)$ in $\pi(Y\setminus E_{x_0})$, $z_2=\pi(x_2)$ in $\pi\left(Y\setminus(E_{x_0}\cup E_{x_1})\right)$, etc. The process stop at some N with $CN\geq\sharp Y$ and we may then let $Z=\{z_1,z_2,\cdots z_N\}$. Consequently $h_{d^{\mathbb{Z}}}(\widetilde{T},\epsilon)\leq h_d(T,\epsilon)$ for all $\epsilon>0$. □

3.3. **Mean dimension of a skew-product.** Let X and Y be compact spaces. Let $R:X\to X$ and $S:X\times Y\to Y$ be continuous maps. Define the *skew-product* $T:X\times Y\circlearrowleft$ over (X,R) by $(x,y)\mapsto(R(x),S(x,y))$.

Lemma 3.9. *Let α be an open cover of X and $\beta=\alpha\times Y$ be the induced cover of $X\times Y$. Then*

$$\mathrm{mdim}(X\times Y,T,\beta)\geq\mathrm{mdim}(X,R,\alpha).$$

In particular, we have $\mathrm{mdim}(X\times Y,T)\geq\mathrm{mdim}(X,R)$.

Proof. For any $n\in\mathbb{N}$, we let γ_n be a finite open cover of $X\times Y$ finer $\bigvee_{k=0}^{n-1}T^{-k}\beta$ with $D(\bigvee_{k=0}^{n-1}T^{-k}\beta)=\mathrm{ord}(\gamma_n)$. Fix $y\in Y$ and consider the open cover γ_n' of X given by the sets $\pi_X\left(O\cap(X\times\{y\})\right)$ over $O\in\gamma_n$, where $\pi_X:X\times Y\to X$ denotes the projection on the X-coordinate. Clearly γ_n' is finer then $\bigvee_{k=0}^{n-1}R^{-k}\alpha$ and $\mathrm{ord}(\gamma_n')\leq\mathrm{ord}(\gamma_n)$. Therefore $\frac{D(\bigvee_{k=0}^{n-1}T^{-k}\beta)}{n}\geq\frac{D(\bigvee_{k=0}^{n-1}R^{-k}\alpha)}{n}$ and we conclude by taking the limit in n. □

Remark 3.10. *The above proof still applies in the wider context of a topological system $T:E\circlearrowleft$ with $E\subset X\times Y$, $\pi_X(E)=X$ (where π_X denotes the coordinate projection on X) and $R\circ\pi_X=\pi_X\circ T$, such that there exists y with $X\times\{y\}\subset E$.*

Remark 3.11. *In general the natural extension of a skew-product is not a skew-product. When the skew-product is trivial, i.e. with S depending only on x, then the natural extension $(\widetilde{X_R},\widetilde{R})$ and $(\widetilde{(X\times Y)}_T,\widetilde{T})$ are topologically conjugated via the map $(x_k)_k\in\widetilde{X_R}\mapsto(x_k,S(x_{k+1}))_k\in\widetilde{(X\times Y)}_T$, in particular these systems have the same mean dimension.*

4. General one-dimensional cellular automata

Let X be a compact metrizable space. Let F be a cellular automaton on $X^{\mathbb{Z}}$ with a continuous local rule $f:X^I\to X$ for $I\subset\mathbb{Z}$. For $x=(x_k)_{k\in\mathbb{Z}}\in X^{\mathbb{Z}}$ and a finite subset K of $\mathbb{Z}$ we denote by x_K the tuple given by the j-coordinate of x for $j\in K$, i.e.

$$x_K=(x_{j_1},x_{j_2},\ldots,x_{j_k})\text{ for }K=\{j_1<j_2<\cdots<j_k\}.$$

Let $J=J_F$ be the integers in the convex hull of $I\cup\{0\}$ and $J^*=J\setminus\{0\}$. Let $J_-=\min\{J\}$ and $J_+=\max\{J\}$. Then $\mathrm{diam}(I\cup\{0\})=\mathrm{diam}(J)=\sharp J^*$. We denote again by f the function from X^J to X, mapping $(x_j)_{j\in J}$ to $f\left((x_j)_{j\in I}\right)$.

4.1. **Upper bound for the mean dimension.** We first generalize the upper bound of the mean dimension w.r.t. the shift map obtained in [LW00, Proposition 3.1] to cellular automata.

Proposition 4.1. *Let X be a compact metric space. Let F be a cellular automaton on $X^{\mathbb{Z}}$ with a continuous local rule $f : X^I \to X$ for $I \subset \mathbb{Z}$. Then* $\mathrm{mdim}(X^{\mathbb{Z}}, F) \leq \mathrm{stabdim}(X) \cdot \mathrm{diam}(I \cup \{0\})$.

Proof. For every $K \subset \mathbb{Z}$, let $\pi_K : X^{\mathbb{Z}} \to X^K$ be the natural projection. Let J be the integers in the convex hull of $I \cup \{0\}$. Let $\mathcal{A}$ be a finite open cover of $X^{\mathbb{Z}}$. There is a finite open cover $\mathcal{B}$ finer than $\mathcal{A}$ such that for some positive integer N

$$\mathcal{B} \subset \mathcal{O}(-N, N) := \{\{(x_n)_{n\in\mathbb{Z}},\ x_{-N} \cdots x_N \in O\}\ :\ O \text{ is open in } X^{2N+1}\}.$$

By assumption of F, we get that

$$\bigvee_{k=0}^{n-1} F^{-k}\mathcal{B} \subset \mathcal{O}(-N + nJ_-, N + nJ_+).$$

Let K_n be the integers in $[-N+nJ_-, N+nJ_+]$. It follows that there is a cover $\mathcal{C} \succ \pi_{K_n}(\bigvee_{k=0}^{n-1} F^{-k}\mathcal{B})$ of X^{K_n} such that $\mathrm{ord}(\mathcal{C}) \leq \dim(X^{K_n})$. As $\pi_{K_n}^{-1}\mathcal{C} \succ \bigvee_{k=0}^{n-1} F^{-k}\mathcal{B} \succ \bigvee_{k=0}^{n-1} F^{-k}\mathcal{A}$, we obtain that

$$\frac{D(\bigvee_{k=0}^{n-1} F^{-k}\mathcal{A})}{n} \leq \frac{\dim(X^{K_n})}{n} = \frac{\dim(X^{K_n})}{\sharp K_n} \cdot \frac{2N + n \cdot \mathrm{diam}(J)}{n}.$$

Therefore, we conclude that $\mathrm{mdim}(X^{\mathbb{Z}}, F) \leq \mathrm{stabdim}(X) \cdot \mathrm{diam}(J)$. □

A cellular automaton on $X^{\mathbb{Z}}$ with local rule $f : X^I \to X$ is also a cellular automaton with local rule $f' : X^K \to X$ for $K \supset I$ by letting $f'(x_K) = f(x_I)$. Thus we need some extra conditions on local rules in order to calculate the value of mean dimension.

4.2. **A factor of CA.** Define $\phi : X^{\mathbb{Z}} \to X \times \left(X^{J^*}\right)^{\mathbb{N}}$ by

$$x = (x_n)_{n\in\mathbb{Z}} \mapsto (x_0, (x_{J^*}, F(x)_{J^*}, \cdots, F^k(x)_{J^*}, \cdots)).$$

and $g : X \times \left(X^{J^*}\right)^{\mathbb{N}} \circlearrowleft$ by

$$(x, (y^n)_{n\in\mathbb{N}}) \mapsto (f(z), \sigma y)$$

where z is the point of X^J defined as $z_0 = x, (z_i)_{i\in J^*} = y^0$. For any $x \in X^{\mathbb{Z}}$, we have then

$$\phi \circ F(x) = (f(x_J), (F(x)_{J^*}, F^2(x)_{J^*}, \cdots, F^k(x)_{J^*}, \cdots) = g \circ \phi(x).$$

In general ϕ is not surjective, but when this is the case, the continuous map ϕ defines a factor map from $(X^{\mathbb{Z}}, F)$ to $(X \times X^{\mathbb{N}}, g)$.

Corollary 4.2.

$$\mathrm{mdim}\left(X \times \left(X^{J^*}\right)^{\mathbb{N}}, g\right) \geq \mathrm{stabdim}(X) \cdot \mathrm{diam}(I \cup \{0\}).$$

Proof. By Lemma 3.9 and [Tsu19, Theorem 1.1], we have

$$\mathrm{mdim}\left(X \times \left(X^{J^*}\right)^{\mathbb{N}}, g\right) \geq \mathrm{mdim}\left(\left(X^{J^*}\right)^{\mathbb{N}}, \sigma\right) = \sharp J^* \cdot \mathrm{stabdim}(X).$$

□

For higher dimensional CA, we may generalize the above semi-conjugacy as follows. Let J be a subset of $\mathbb{Z}^d$. Then for any J' contained in J satisfying $k + I \subset J$ for all $k \in J'$, we define $\phi = \phi_{J,J'}$ by $\phi : X^{\mathbb{Z}^d} \to X^{J'} \times \left(X^{J\setminus J'}\right)^{\mathbb{N}}$ by

$$x = (x_n)_{n\in\mathbb{Z}} \mapsto (x_{J'}, (x_{J\setminus J'}, F(x)_{J\setminus J'}, \cdots, F^k(x)_{J\setminus J'}, \cdots)).$$

and $g = g_{J,J'} : X^{J'} \times \left(X^{J\setminus J'}\right)^{\mathbb{N}} \circlearrowleft$ by

$$(x, (y^n)_{n\in\mathbb{N}}) \mapsto \left((f(z_{k+I}))_{k\in J'}, \sigma y\right)$$

where $z_{k+I} := (z_{k+i})_{i\in I}$ is the point of X^I defined as $z_{k+i} = x_{k+i}$ for $k+i \in J'$ and $z_{k+i} = y^0_{k+i}$ for $k+i \in J \setminus J'$. Then we have again $\phi \circ F(x) = g \circ \phi(x)$ for all $x \in X^{\mathbb{Z}^d}$.

4.3. **CA with surjective local rule.** For a topological dynamical system (X,T) we let $NW(T)$ be the set of non-wandering points of (X,T). Let F be a cellular automaton with local rule $f : X^I \to X$. For positive integers n we define by induction the subsets X_n of X by $X_n = f(X^I_{n-1})$ and $X_0 = X$. Finally we let $X_\infty := \bigcap_{n\in\mathbb{N}} X_n$, which is a compact subspace of X. Clearly $f(X^I_\infty) = X_\infty$. Therefore the restriction of F to $X^{\mathbb{Z}}_\infty$ is a CA with a surjective local rule.

Proposition 4.3. *With the above notations,*

$$\mathrm{mdim}(X^{\mathbb{Z}}, F) = \mathrm{mdim}(X^{\mathbb{Z}}_\infty, F|_{X^{\mathbb{Z}}_\infty}).$$

Proof. Since $X_\infty \subset X$, $\mathrm{mdim}(X^{\mathbb{Z}}, F) \geq \mathrm{mdim}(X^{\mathbb{Z}}_\infty, F|_{X^{\mathbb{Z}}_\infty})$. It remains to show the other direction. Notice that for a general topological system (X,T) we always have $NW(T) \subset \bigcap_{n\in\mathbb{N}} T^n X$. Moreover $\mathrm{mdim}(X,T) = \mathrm{mdim}(NW(T), T|_{NW(T)})$ by [Gut17, Lemma 7.2]. But $F^n(X^{\mathbb{Z}}) \subset X^{\mathbb{Z}}_n$ for all n, therefore $NW(F) \subset X^{\mathbb{Z}}_\infty$, implying that $\mathrm{mdim}(X^{\mathbb{Z}}, F) \leq \mathrm{mdim}(X^{\mathbb{Z}}_\infty, F|_{X^{\mathbb{Z}}_\infty})$. □

When considering the mean dimension, by Proposition 4.3 and the argument above, we could restrict to CA's with surjective local rule.

5. Permutative one-dimensional CA

Let X be a compact metric space. Let f be a continuous function $f : X^I \to X$ with $I \subset \mathbb{Z}$. For any $j \in I$ and for any $x^j = (x_i)_{i\in I\setminus\{j\}} \in X^{I\setminus\{j\}}$, we denote by $f_{x^j} : X \circlearrowleft$ the continuous function $x_j \mapsto f((x_i)_{i\in I})$.

A cellular automaton F on $X^{\mathbb{Z}}$ with local rule f is said to be *permutative* (resp. *strongly permutative*) when for $j \in \{\max I, \min I\} \setminus \{0\}$ and for all $x^j \in X^{I\setminus\{j\}}$ the map f_{x^j} is surjective (resp. bijective [2]).

If F is permutative (resp. stongly permutative) then so is F^k and $J_{F^k} = kJ_F$ for any $k \in \mathbb{N}^*$ (see Lemma 16 in [Bur]). Note that in the discrete case, any permutative CA is strongly permutative as the set of states is finite.

5.1. **Maximal mean dimension for strongly permutative CA's.** When there are no negative integers (resp. positive) in the domain I, the local rule f induces a cellular automaton on $X^{\mathbb{N}}$ (resp. $X^{-\mathbb{N}}$) which we denote respectively by F^+ and F^-. We let

$$(\mathbb{Y}, G) = \begin{cases} (X^{\mathbb{Z}}, F), & \text{if } I \text{ contains both negative and positive integers,} \\ (X^{\mathbb{N}}, F^+) & \text{if } I \text{ contains no negative integers,} \\ (X^{-\mathbb{N}}, F^-) & \text{if } I \text{ contains no positive integers.} \end{cases}$$

Lemma 5.1. *Let F be a permutative cellular automaton. The dynamical system $(\mathbb{Y}, G)$ is a topological extension of $(X \times X^{\mathbb{N}}, g)$ via ϕ.*

Proof. By Subsection 4.2 we only need to show the surjectivity of ϕ. Let $(x, (y^k)_{k\in\mathbb{N}}) \in X \times (X^{J^*})^{\mathbb{N}}$. Let us show that there is $z \in X^{\mathbb{Z}}$ with $\phi(z) = (x, (y^k)_{k\in\mathbb{N}})$. We prove by induction on k that there exist $z \in \mathbb{Y}$ such that we have $z_0 = x$, $(z_j)_{j\in J^*} = y^1$, ..., $F^k(z)_{J^*} = y^k$. Assume it holds for k and $1 \in J = J_F$. Recall that F^{k+1} is permutative and $J_{F^{k+1}} = (k+1)J_F$. Therefore we may change only the K-coordinate z_K of z with $K = (k+1)J_+ + 1$ to ensure $(F^{k+1}z)_1 = y^{k+1}_1$. Then we argue similarly for the other j-coordinates of $F^{k+1}z$ increasingly in $j \in J^* \cap \mathbb{N}$. Finally we may consider in the same way negative $j \in J^*$ starting from -1 and going decreasingly to $j = J_-$. □

Lemma 5.2. *Let $f : X^I \to X$ be a strongly permutative local rule. Then $(\mathbb{Y}, G)$ is topological conjugated to $(X \times X^{\mathbb{N}}, g)$ via ϕ.*

Proof. It is enough to notice that for a strongly permutative CA, the sequence $y \in \mathbb{Y}$ in the proof of Lemma 5.1 is uniquely defined, so that $\phi : \mathbb{Y} \to X \times X^{\mathbb{N}}$ is a homeomorphism. □

[2] By compactness of X, the map f_{x^j} is then an homeomorphism.

We obtain the following formula concerning about the mean dimension of strongly permutative cellular automata.

Theorem 5.3. *Let F be a strongly permutative cellular automaton as above, then* $\mathrm{mdim}(X^{\mathbb{Z}}, F) = \mathrm{stabdim}(X) \cdot \mathrm{diam}(I \cup \{0\})$.

Proof. When I contains both negative and positive integers, the result follows from Proposition 4.1, Lemma 5.2 and Corollary 4.2. Now assume $I \subset \mathbb{N}$ (one deals similarly for the remaining case). Then $(\mathbb{Y}, G) = (X^{\mathbb{N}}, F^+)$. Notice that $(X^{\mathbb{Z}}, F)$ is a skew-product over $(X^{\mathbb{N}}, F^+)$, so that by Lemma 3.9 we have $\mathrm{mdim}(X^{\mathbb{Z}}, F) \geq \mathrm{mdim}(X^{\mathbb{N}}, F^+) = \mathrm{mdim}(X \times X^{\mathbb{N}}, g)$. Then we conclude as in the previous case by using Proposition 4.1 and Corollary 4.2. □

Remark 5.4. *When the local rule does not depend on the zero coordinate, i.e. $0 \notin I$, then by Remark 3.11 the natural extension of the strongly permutative CA F is topologically conjugated to the bilateral shift on X^{J^*}. In particular F and its natural extension have the same mean dimension.*

We will show in the next section that Theorem 5.3 does not holds true anymore for general permutative CA by building examples of permutative CA with intermediate mean dimension, i.e. with mean dimension strictly less than $\mathrm{stabdim}(X) \cdot \mathrm{diam}(I \cup \{0\})$. For permutative CA, we have $X_\infty = X$, therefore Proposition 4.3 is useless to produce such examples.

5.2. **Maximal mean dimension for near strongly permutative CA.** Let $\mathcal{C}(X)$ be the set of continuous maps from a compact metrizable space X to itself endowed with the topology of uniform convergence. A family $\mathcal{F}$ in $\mathcal{C}(X)$ is said to be *m-expansive* if one of the equivalent conditions is satisfied:

- there exists an open cover α of X with $\mathrm{mdim}(X, T, \alpha) = \mathrm{mdim}(X, T)$ for every $T \in \mathcal{F}$. Such a cover α is called a *generator* of $\mathcal{F}$,
- there exist $\epsilon > 0$ and a compatible metric d with $\mathrm{mdim}(X, T, d, \epsilon) = \mathrm{mdim}(X, T)$ for every $T \in \mathcal{F}$,
- for all compatible metric d there exists $\epsilon > 0$ with $\mathrm{mdim}(X, T, d, \epsilon) = \mathrm{mdim}(X, T)$ for every $T \in \mathcal{F}$.

Lemma 5.5. *Let $\mathcal{F}$ be a m-expansive family. Then $T \mapsto \mathrm{mdim}(X, T)$ is upper semi-continuous on the closure of $\mathcal{F}$.*

Proof. Let α be a generator of $\mathcal{F}$. We will prove $T \mapsto \mathrm{mdim}(X, T, \alpha)$ is upper semi-continuous. The conclusion then follows : if $T_n \xrightarrow{n} T$ with $T_n \in \mathcal{F}$ for all n, then we get

$$\begin{aligned}
\mathrm{mdim}(X, T) &\geq \mathrm{mdim}(X, T, \alpha), \\
&\geq \limsup_{n\to\infty} \mathrm{mdim}(X, T_n, \alpha), \\
&\geq \limsup_{n\to\infty} \mathrm{mdim}(X, T_n).
\end{aligned}$$

To check the upper semi-continuity of $T \mapsto \mathrm{mdim}(X, T, \alpha)$, it is enough to see $f_n : T \mapsto D(\bigvee_{k=0}^{n-1} T^{-k}\alpha)$ is upper semi-continuous for any n. Indeed we have $\mathrm{mdim}(X, T) = \inf_{n\in\mathbb{N}} \frac{f_n(T)}{n}$ by sub-additivity of the sequence $(f_n(T))_{n\in\mathbb{N}}$ and the infimum of a sequence of upper semi-continuous functions is itself upper semicontinuous. By [Coo15, Proposition 1.6.5], we have $D(\bigvee_{k=0}^{n-1} T^{-k}\alpha) = \min_{\gamma} \mathrm{ord}(\gamma)$, where the minimum holds over all closed covers γ finer than $\bigvee_{k=0}^{n-1} T^{-k}\alpha$. Let γ_n be such a cover realizing the minimum. It follows that γ_n is finer than $\bigvee_{k=0}^{n-1} T_m^{-k}\alpha$ for m large enough, so that $D(\bigvee_{k=0}^{n-1} T_m^{-k}\alpha) \leq \mathrm{ord}(\gamma_n) = D(\bigvee_{k=0}^{n-1} T^{-k}\alpha)$ for m large enough. It implies that f_n is upper semicontinuous function for every $n \in \mathbb{N}$. □

When a m-expansive family consists of a single map T, we say that the map T is *m-expansive.*

Lemma 5.6 ([Tsu19], Lemma 3.1 and Theorem 2.5)**.** *The shift map σ on $X^{\mathbb{N}}$ is m-expansive. Moreover there is a generator α of the form $\alpha = \mathcal{U} \times X^{\mathbb{N}^*}$ for some open cover $\mathcal{U}$ of X.*

Remark 5.7. *This concept of m-expansiveness is inspired by the notion of h-expansiveness relative to the entropy. We may build examples of non m-expansive systems with any given mean dimension as follows. Consider a sequence* $(X_n, T_n)_{n\in\mathbb{N}}$ *of topological systems such that* $\mathrm{mdim}(X_n, T_n)$ *is strictly increasing (either bounded or divergent to infinity). Then the one point compactification* (X, T) *of the disjoint union of* (X_n, T_n) *by a* T*-fixed point at the infinity is not m-expansive.*

For $N \in \mathbb{N}$ we denote by $\mathcal{F}_N(X)$ (resp. $\mathcal{G}_N(X)$) the family of strongly permutative CA (resp. all CA) with domain I contained in $[-N, N]$.

Lemma 5.8. *With the above notations, the family* $\mathcal{F}_N(X)$ *is m-expansive.*

Proof. Let J be an interval of integers contained in $[-N, N]$. By Lemma 5.6 there is a generator $\alpha = \mathcal{U} \times (X^{J^*})^{\mathbb{N}^*}$ of the shift map on $(X^{J^*})^{\mathbb{N}}$ for some cover $\mathcal{U}$ of X^{J^*}. Let $\beta = \mathcal{U} \times X^{\mathbb{Z}^*}$ and $\gamma = X \times \alpha$ be the induced covers of $X^{\mathbb{Z}}$ and $X \times (X^{J^*})^{\mathbb{N}}$ respectively. Let F be a strongly permutative CA with J being the convex hull of its domain I. We claim that $\beta = \beta_J$ is a generator of F. Indeed the conjugacy ϕ sends β to γ, therefore $\mathrm{mdim}(F, \beta) = \mathrm{mdim}(g, \gamma)$. But g being a skew-product overs σ, we have by Lemma 3.9 $\mathrm{mdim}(g, \gamma) \geq \mathrm{mdim}(\sigma, \alpha) = \mathrm{mdim}(F)$. Finally we observe that $\bigvee_J \beta_J$, where the joining holds over all intervals of integers $J \subset [-N, N]$, is a generator of $\mathcal{F}_N(X)$. □

Question 5.9. *Is the larger family* $\mathcal{G}_N(X)$ *also m-expansive?*

A CA on $X^{\mathbb{Z}}$ is said *near strongly permutative* when it belongs to the closure of $\mathcal{F}_N(X)$ for some N. For example if g and h are surjective non-injective monotone interval maps, then the CA with local rule $f : [0, 1]^2 \to [0, 1]$, defined by $f(x_0, x_1) = g(x_0) + h(x_1)$, is near strongly permutative but not strongly permutative.

Corollary 5.10. *Let* F *be a near strongly permutative CA. Then*

$$\mathrm{mdim}(F) = \mathrm{stabdim}(X) \cdot \mathrm{diam}(I \cup \{0\}).$$

Proof. It follows from Lemma 5.8, Lemma 5.5 and Theorem 5.3. □

5.3. **Infinite topological entropy.**

Lemma 5.11. *Assume* X *is infinite. Then any permutative CA on* $X^{\mathbb{Z}}$ *has infinite topological entropy.*

Proof. The factor map ϕ of Subsection 4.2 is surjective, therefore $h_{top}(F) \geq h_{top}(g)$. But g is an extension of the unilateral full shift on $\left(X^{J^*}\right)^{\mathbb{N}}$ which has infinite entropy, therefore $h_{top}(F) = +\infty$. □

However a cellular automaton with infinite set of states and surjective local rule may have finite, even zero, topological entropy.

Example 5.12. *Let* $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. *Let* $I = \{-1, 0, 1\}$. *Define a surjective continuous map* $f : X^I \to X$ *by* $f(x_{-1,}, x_0, x_1) = \max\{x_{-1,}, x_0, x_1\}$. *Notice that* $F^n(x) \xrightarrow{n} a^{\mathbb{Z}}$ *with* $a = \max_{k\in\mathbb{Z}} x_k$ *for every* $x \in X^{\mathbb{Z}}$. *Therefore the topological entropy of* $(X^{\mathbb{Z}}, T_f)$ *is zero.*

6. Unit CA

In this section, we consider one-dimensional CA with domain $I = \{1\}$ and continuous rule $f : X \circlearrowleft$, i.e.

$$F((x_k)_{k\in\mathbb{Z}}) = (f(x_{k+1}))_{k\in\mathbb{Z}}. \tag{6·1}$$

We call such CA a *unit CA*.

6.1. **Non-wandering set of unit CA's.** For unit CA's one may wonder if $NW(F) \subset NW(f)^{\mathbb{Z}}$. This is false in general. For example if one considers a North-South invertible dynamic on the circle $\mathbb{S}^1$, $NW(f)$ is reduced to the two poles and therefore $h_{top}(F) \leq \log 2$, in particular $\mathrm{mdim}(F) = 0$. But f being invertible, we have $\mathrm{mdim}(F) = 1$ by Theorem 5.3. When the North-South dynamic f is moreover assumed to be smooth, the non-wandering set is the whole set $(\mathbb{S}^1)^{\mathbb{Z}}$. Indeed if μ is any probability measure on $\mathbb{S}^1$, one easily checks that $\prod_{k\in\mathbb{Z}} f^{-k}\mu$ is invariant by F, in particular its support is contained in the non-wandering set. If μ is the Lebesgue measure, $\prod f^{-k}\mu$ has full support in $(\mathbb{S}^1)^{\mathbb{Z}}$ for a diffeomorphism f.

For a unit CA T_f, we have $X_\infty = \bigcap_n f^n(X)$. Moreover the restriction of F to $X_\infty^{\mathbb{Z}}$ is permutative. In particular it has infinite topological entropy if and only if $\sharp X_\infty = +\infty$ by Lemma 5.11. If X_∞ is finite then there are finitely many periodic orbits attracting all the points of X. When X is connected so is X_∞. Therefore in this case $\sharp X_\infty < \infty$ is equivalent to (X, f) has an attracted fixed point with full basin.

6.2. **Upper bound on the mean dimension.**

Proposition 6.1. *Let T_f be a unit CA with local rule $f : X \circlearrowleft$. Then we have*

$$\mathrm{mdim}(X^{\mathbb{Z}}, T_f) \leq \mathrm{stabdim}(\widetilde{X_f}).$$

Proof. We first prove $\mathrm{mdim}(T_f) \leq \dim(\widetilde{X_f})$. It is enough to consider the unilateral action T_f^+ on $X^{\mathbb{N}}$. To simplify the notations we write here T_f for T_f^+. Let $\mathcal{U}$ be an open cover of X and $\mathcal{V} = \mathcal{V}(\mathcal{U}, M)$ be the induced cover of $X^{\mathbb{N}}$ given by open sets of the form $\{(x_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}} : x_l \in U_l \text{ for } 0 \leq l \leq M\}$ with some $U_0, \cdots, U_M \in \mathcal{U}$. To conclude it is enough to prove

$$\lim_{n\to\infty} \frac{D(\bigvee_{k=0}^{n-1} T_f^{-k}\mathcal{V})}{n} \leq \dim(\widetilde{X_f}) \tag{6·2}$$

for open covers $\mathcal{V}$ of the previous form, because they may have arbitrarily small diameter.

We identify $\widetilde{X_f}$ as a closed subspace of $X^{\mathbb{N}}$. We consider an open cover $\widetilde{\mathcal{V}}$ of $\widetilde{X_f}$ finer than $\mathcal{V} \cap \widetilde{X_f}$ with $\mathrm{ord}(\widetilde{\mathcal{V}}) \leq \dim(\widetilde{X_f})$. For a cover $\mathcal{A}$, we let $\mathrm{cl}(\mathcal{A})$ be the associated closed cover $\mathrm{cl}(\mathcal{A}) := \{\overline{A} : A \in \mathcal{A}\}$. By Corollary 1.6.4 in [Coo15] we may assume $\mathrm{ord}(\mathrm{cl}(\widetilde{\mathcal{V}})) = \mathrm{ord}\left(\widetilde{\mathcal{V}}\right)$ without loss of generality. Any $W \in \widetilde{\mathcal{V}}$ may be written as $W = O_W \cap \widetilde{X_f}$ for some open subset O_W of $X^{\mathbb{N}}$ in such a way that the family $\mathcal{W} = \{O_W : W \in \widetilde{\mathcal{V}}\}$ is finer than $\mathcal{V}$ and satisfies $\mathrm{ord}(\mathrm{cl}(\mathcal{W})) \leq \mathrm{ord}(\widetilde{\mathcal{V}})$. Indeed if this last condition could not be satisfied, there would be a point $x \in \widetilde{X_f}$ with $\sum_{W\in\mathcal{V}} 1_{\overline{W}}(x) - 1 > \mathrm{ord}(\widetilde{\mathcal{V}}) = \mathrm{ord}(\mathrm{cl}(\widetilde{\mathcal{V}}))$. Then by letting N large enough, we may ensure that :

- $\mathrm{ord}(\mathrm{cl}(\mathcal{W}_N)) \leq \mathrm{ord}(\mathrm{cl}(\mathcal{W})) \leq \mathrm{ord}(\widetilde{\mathcal{V}})$ where $\mathcal{W}_N$ is the family of open subsets of X^N consisting in the N-first coordinates projections $\pi_N(O_W)$, $W \in \widetilde{\mathcal{V}}$;
- $\mathcal{W}$ is covering the compact set $X_N := \{(x_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}} : f(x_{k+1}) = x_k \text{ for } k = 0, \cdots, N-1\}$ since we have $\widetilde{X_f} = \bigcap_{n\in\mathbb{N}} X_n$.

Fix such a $N > M$. Let $\mathcal{X}_n$ be the cover of $X^{\mathbb{N}}$ given by the sets

$$F_E := \{(x_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}} : (f^l x_l, \cdots, f^{l-N+1} x_l) \in E_l,\ N \leq l < n\}$$

for $E_N, \cdots, E_{n-1} \in \mathcal{W}_N$. Clearly $\mathrm{ord}(\mathcal{X}_n) \leq (n-N)\mathrm{ord}(\mathcal{W}_N) \leq (n-N)\dim(\widetilde{X_f})$. We check now $\mathcal{X}_n$ is finer than $\bigvee_{k=0}^{n-M} T_f^{-k}\mathcal{V}$, which will imply $\mathrm{mdim}(T_f) \leq \dim(\widetilde{X_f})$. Let E_l, $N \leq l < n$, be in $\mathcal{W}_N$ and let F_E be the corresponding element of $\mathcal{X}_n$. We will show that F_E is contained in some element of $\bigvee_{k=0}^{n-M} T_f^{-k}\mathcal{V}$, i.e. for any $0 \leq k < n-M$ and $0 \leq l \leq M$ there is $U \in \mathcal{U}$ such that $f^k x_{k+l}$ lies in U for all $x \in F_E$. Note that $(f^i x_{k+l})_{i=k+l,\cdots,k+l-N+1}$ lies in some $E_{k+l} \in \mathcal{W}_N$. Since $\mathcal{W} \succ \mathcal{V}$, there is $U \in \mathcal{U}$ such that $f^k x_{k+l}$ lies in U for all $x \in F_E$.

Let us show now $\mathrm{mdim}(T_f) \leq \mathrm{stabdim}(\widetilde{X_f})$. For $n \in \mathbb{N}^*$ we consider the direct n-product $T_f^{\times n} := \underbrace{T_f \times \cdots \times T_f}_{n \text{ times}}$. This product is conjugated to $T_{f^{\times n}}$ with $f^{\times n} : X^n \circlearrowleft$, $(x_1, \cdots, x_n) \mapsto$

$(f(x_1), \cdots, f(x_n))$. Moreover the natural extension of $f^{\times n}$ is conjugated to the direct n-product of $(\widetilde{X_f}, \widetilde{T_f})$, in particular $\dim(\widetilde{X_{f^{\times n}}}) = \dim(\widetilde{X_f}^n)$. Therefore we get

$$\begin{aligned}\operatorname{mdim}(T_f) &\leq \lim_n \frac{\operatorname{mdim}(T_f^{\times n})}{n}, \\ &\leq \lim_n \frac{\operatorname{mdim}(T_{f^{\times n}})}{n}, \\ &\leq \lim_n \frac{\dim(\widetilde{X_{f^{\times n}}})}{n}, \\ &\leq \lim_n \frac{\dim(\widetilde{X_f}^n)}{n} = \operatorname{stabdim}(\widetilde{X_f}).\end{aligned}$$

This completes the proof. □

6.3. **Natural Extension.** The natural extension of a unit CA is again a unit CA. More precisely we have :

Proposition 6.2. *Let $f : X \circlearrowleft$ be a topological dynamical system and let T_f be the associated unit cellular automaton. Then the natural extension $(\widetilde{X^{\mathbb{Z}}_{T_f}}, \widetilde{T_f})$ is topologically conjugated to $(\widetilde{X_f}^{\mathbb{Z}}, T_{\widetilde{f}})$.*

Proof. Here we denote $Y_{\mathbb{Z}} = Y^{\mathbb{Z}}$ for any set Y. The natural extension $\widetilde{X_{T_f}}$ is given by

$$\widetilde{X_{\mathbb{Z}}} := \widetilde{X^{\mathbb{Z}}_{T_f}} = \{(x^l)_{l\in\mathbb{N}} \in (X_{\mathbb{Z}})^{\mathbb{N}},\ \forall l\ T_f(x^{l+1}) = x^l\}.$$

With $x^l = (x^l_k)_k$, the equality $T_f(x^{l+1}) = x^l$ may be rewritten as $f(x^{l+1}_{k+1}) = x^l_k$ for all $k \in \mathbb{Z}$. On the other hand the natural extension $\widetilde{X_f}$ of (X, f) is defined as

$$\widetilde{X} = \widetilde{X_f} := \{(x^l)_{l\in\mathbb{N}} \in X^{\mathbb{N}},\ \forall l\ f(x^{l+1}) = x^l\}.$$

We consider the map

$$\begin{aligned}\pi : &\widetilde{X}_{\mathbb{Z}} \to \widetilde{X_{\mathbb{Z}}}, \\ &(\tilde{x}_k)_k \mapsto (X^l)_l\end{aligned}$$

with $\tilde{x}_k = (x^l_k)_l \in \widetilde{X}$ and $X^l = (x^l_{k-l})_k \in X_{\mathbb{Z}}$. This maps takes value in $\widetilde{X_{\mathbb{Z}}}$ because

$$(T_f(X^{l+1}))_k = f(x^{l+1}_{k-l}) = x^l_{k-l} \text{ and consequently } T_f(X^{l+1}) = X^l.$$

One shows also easily that π is bijective and continuous. For $k \in \mathbb{Z}$ we let $x^{-1}_k = f(x^0_k)$. Let us check now π is a conjugacy :

$$\begin{aligned}\pi \circ T_{\widetilde{f}}((\tilde{x}_k)_k) &= \pi\left((\widetilde{f}(\tilde{x}_{k+1})_k\right), \\ &= \pi\left((x^{l-1}_{k+1})_{l,k}\right), \\ &= (x^{l-1}_{k-l+1})_{l,k}.\end{aligned}$$

and

$$\begin{aligned}\widetilde{T_f} \circ \pi((\tilde{x}_k)_k) &= \widetilde{T_f}\left((x^l_{k-l})_{l,k}\right), \\ &= (x^{l-1}_{k-l+1})_{l,k}.\end{aligned}$$

This completes the proof. □

Remark 6.3. *The natural extension of a general is a generalized subshift of finite type. The proof is presented in Appendix B.*

6.4. **Mean dimension of unit CA.** The system $\left(\left(\widetilde{X_f}\right)^{\mathbb{Z}}, T_{\widetilde{f}}\right)$ is topologically conjugated to the shift on $\widetilde{X_f}$. From Proposition 6.2, Proposition 6.1 and Proposition 4.1 we derive the following formula for the mean dimension of unit CA's.

Theorem 6.4. *Let $F = T_f$ be a unit CA associated to a topological system (X, f). Then we have*

$$\mathrm{mdim}(X^{\mathbb{Z}}, T_f) = \mathrm{mdim}(\widetilde{X^{\mathbb{Z}}_{T_f}}, \widetilde{T_f}) = \mathrm{stabdim}(\widetilde{X_f}).$$

By considering the unit CA associated to the topological systems given by Proposition 3.1 we get :

Corollary 6.5. *For any integers $0 \leq k \leq n$, there exists a permutative unit cellular automaton F on $X^{\mathbb{Z}}$ with $\mathrm{mdim}(F) = k$ and $\mathrm{stabdim}(X) = n$. For $k \geq 1$ we may assume X connected.*

Remark 6.6. *When f is a near homeomorphism on X, i.e. the uniform limit of homeomorphisms of X, then T_f is a near strongly permutative CA and therefore $\mathrm{mdim}(T_f) = \mathrm{stabdim}(X) = \mathrm{stabdim}(\widetilde{X_f})$. This last inequality follows in this case from the stronger fact that X and $\widetilde{X_f}$ are homeomorphic for near homeomorphisms* [Bro60] .

7. Algebraic CA

In this section, we investigate the cellular automaton with algebraic structure. Firstly, we show that one-dimensional algebraic permutative CA has maximal mean dimension. Secondly, we prove that higher dimensional algebraic permutative CA has infinite mean dimension. Finally, building on an example of Meyerovitch [Mey08], we give an example of multidimensional algebraic surjective CA with positive finite mean dimension.

A cellular automaton F on $X^{\mathbb{Z}^d}$ with local rule $f : X^I \to X$ for $I \subset \mathbb{Z}^d$ is said to be *algebraic* if X is a compact metrizable abelian group and f is a continuous group homomorphism.

7.1. **One-dimensional algebraic permutative CA.** An *algebraic* system (X, T) is a topological system such that X is a compact metrizable abelian group and T is a continuous group homomorphism on X. A topological extension $\psi : (Y, S) \to (X, T)$ between two algebraic systems (Y, S) and (X, T) is said algebraic when $\psi : Y \to X$ is a homomorphism of group. We recall in the next two Lemmas the algebraic structure of the natural extension of an algebraic dynamical system. The easy proofs are left to the reader.

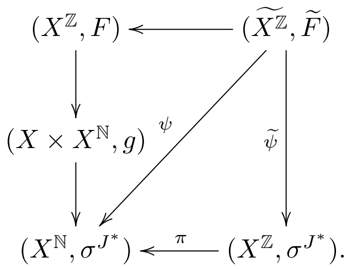
Lemma 7.1. *Let (X, T) be an algebraic dynamical system. Then the natural extension $(\widetilde{X_T}, \widetilde{T})$ is also algebraic.*

Lemma 7.2. *Let (Y, S) and (X, T) be algebraic dynamical systems with (Y, S) being invertible. Assume $\psi : (Y, S) \to (X, T)$ is an algebraic extension. Then the induced topological extension $\widetilde{\psi} : (Y, S) \to (\widetilde{X_T}, \widetilde{T})$ is also algebraic.*

Now we deduce the formula of mean dimension of algebraic permutative CA.

Lemma 7.3. *Let F be an algebraic permutative cellular automaton on $X^{\mathbb{Z}}$. Then* $\mathrm{mdim}(X^{\mathbb{Z}}, F) = \mathrm{stabdim}(X) \cdot \mathrm{diam}(I \cup \{0\})$.

Proof. By Proposition 4.1, it is sufficient to show $\mathrm{mdim}(X^{\mathbb{Z}}, F) \geq \mathrm{stabdim}(X) \cdot \sharp J^*$, where $J^* = J \setminus \{0\}$ and J is the collection of integers in the convex hull of I. Let $(\widetilde{X^{\mathbb{Z}}}, \widetilde{F})$ be the natural extension of $(X^{\mathbb{Z}}, F)$. Since F is permutative and the natural extension of $(X^{\mathbb{N}}, \sigma^{J^*})$ is $(X^{\mathbb{Z}}, \sigma^{J^*})$, by Subsection 4.2 we have the following commutative graph:



By Lemma 7.1 and Lemma 7.2, the invertible system $(\widetilde{X_F^{\mathbb{Z}}}, \widetilde{F})$ is algebraic and $\widetilde{\psi} : (\widetilde{X_F^{\mathbb{Z}}}, \widetilde{F}) \to (X^{\mathbb{Z}}, \sigma^{J^*})$ is an algebraic extension. By [LL18, Corollary 6.1], we have $\mathrm{mdim}(\widetilde{X^{\mathbb{Z}}}, \widetilde{F}) \geq \mathrm{mdim}(X^{\mathbb{Z}}, \sigma^{J^*}) = \mathrm{stabdim}(X) \cdot \sharp J^*$. Meanwhile, by Proposition 3.5, we obtain

$$\mathrm{mdim}(X^{\mathbb{Z}}, F) \geq \mathrm{mdim}(\widetilde{X_F^{\mathbb{Z}}}, \widetilde{F}) \geq \mathrm{stabdim}(X) \cdot \sharp(J^*).$$

□

Example 7.4. *Let $f : \mathbb{T}^{\{0,1\}} \to \mathbb{T}$ defined by $(x, y) \mapsto 2x + 3y$. The associated cellular automaton T_f on $\mathbb{T}^{\mathbb{Z}}$ is an algebraic permutative CA. By Lemma 7.3, we have* $\mathrm{mdim}(\mathbb{T}^{\mathbb{Z}}, T_f) = 1$.

7.2. **Higher dimensional algebraic permutative CA.** Let $d > 1$. Following the notations used in [Bur] we let $\mathbb{I}$ be the convex hull of the domain I of a CA on $X^{\mathbb{Z}^d}$. The *support function* of $\mathbb{I}$ is the function $h_{\mathbb{I}} : \mathbb{S}^{d-1} \to \mathbb{R}$, which maps $u \in \mathbb{S}^{d-1}$ to $\max_{i \in \mathbb{I}} i \cdot u$ with $\cdot$ be the usual scalar product on $\mathbb{R}^d$. For a convex d-polytope J in $\mathbb{R}^d$, a face F of J and $\epsilon \in \mathbb{R}$ we denote by N^F the exterior normal vector to F and by $T_F^+ J(\epsilon)$ the closed semi-space normal to N^F satisfying $[F + \epsilon' N^F \subset T_F^+ J(\epsilon)] \Leftrightarrow \epsilon' \geq \epsilon$.

The $\mathbb{I}$-morphological boundary of J is the subset

$$\partial_{\mathbb{I}}^- J := \{j \in J,\, i + j \in J \text{ for all } i \in \mathbb{I}\}.$$

We consider the subset of $\partial_{\mathbb{I}}^- J$ given by $\partial_{\mathbb{I}}^- F := \partial_{\mathbb{I}}^- J \cap T_F^+ J(-h_{\mathbb{I}}(N^F))$. The sets $\partial_{\mathbb{I}}^- F$ over faces F of J are covering $\partial_{\mathbb{I}}^- J$ but do not define a partition in general. For any face F of J we let $u^F \in ex(\mathbb{I}) \subset I'$ with $u^F \cdot N^F = h_{\mathbb{I}}(N^F)$. We also let $\mathcal{F}_{\mathbb{I}}(J)$ be the set of faces F for which u_F is uniquely defined. We denote by $\partial_{\mathbb{I}}^\perp J$ the subset of $\partial_{\mathbb{I}}^- J$ given by

$$\partial_{\mathbb{I}}^\perp J := \bigcup_{F \in \mathcal{F}_{\mathbb{I}}(J)} \partial_{\mathbb{I}}^- F.$$

Let $T_f : X^{\mathbb{Z}^d} \circlearrowleft$, $d > 1$, be a higher-dimensional CA associated to a local rule $f : X^I \to X$ with $I \subset \mathbb{Z}^d$. We say that T_f is *permutative* when for any extreme point $j \neq 0^d$ of $\mathbb{I}$ and for any $x^j = (x_i)_{i \in I \setminus \{j\}} \in X^{I \setminus \{j\}}$, the map $f_{x^j} : X \circlearrowleft$, $x_j \mapsto f((x_i)_{i \in I})$, is surjective.

For a subset E of $\mathbb{R}^d$, we let $\underline{E}$ the set of integers in E. By Lemma 13 in [Bur], the semi-conjugacy $g = g_{\underline{J}, \underline{J \setminus \partial_{\mathbb{I}}^\perp J}}$ is surjective. Moreover for any domain $I \neq \{0\}$ we may choose J such that $\underline{\partial_{\mathbb{I}}^\perp J}$ has arbitrarily large cardinality (see Section 7 in [Bur]). Recall now that $g_{\underline{J}, \underline{J \setminus \partial_{\mathbb{I}}^\perp J}}$ semi-conjugates F with the a skew-product over the full shift on $X^{\underline{\partial_{\mathbb{I}}^\perp J}}$. Arguing as in Lemma 7.3 we conclude that :

Lemma 7.5. *Let F be an algebraic, permutative, non trivial (i.e. with $I \neq \{0\}$) cellular automaton on $X^{\mathbb{Z}^d}$ with $d > 1$. Then $\mathrm{mdim}(X^{\mathbb{Z}^d}, F)$ is infinite.*

7.3. **Examples of higher dimensional CA's with finite nonzero mean dimension.** A $\mathbb{Z}^d$-*tiling system* is a pair (S, R) where S is a finite set of "square tiles" and $R \subset S^E$ is some adjacency rules with $E = \{0, \pm e_1, \pm e_2, \ldots, \pm e_d\}$, which determine when a tile $s_1 \in S$ is allowed to be placed next to a tile $s_2 \in S$ (and in which directions). A configuration $x \in S^F$ for some $F \subset \mathbb{Z}^d$ is called *valid* at $n \in F$ if the neighbours of the cell at n obey the adjacency rules, i.e. $x_{F+n} \in R$. Clearly, the set of infinite valid configurations in $S^{\mathbb{Z}^d}$ forms a subshift of finite type.

A set of *directed tiles* S with direction d is defined by a tiling system with a forward direction $d(s) \in \{\pm e_1, \dots, \pm e_d\}$ associated to each tile $s \in S$. Given a configuration $x \in S^{\mathbb{Z}^d}$, a *path* defined by x is a sequence $p_0, p_1, p_2, \dots$ with p_0 given and $p_{n+1} \in \mathbb{Z}^d, n \in \mathbb{N}$, obtained by traversing the forward directions of x, that is to say, $p_{n+1} = p_n + d(x_{p_n})$. Given $x \in S^{\mathbb{Z}^d}$, a path $\{p_n\}_{n\in\mathbb{N}}$ is *valid* if x is valid at every p_n. A set of directed tiles S is called an *acyclic* set of tiles if any valid path in $x \in S^{\mathbb{Z}^d}$ is not a loop.

Let S be a set of directed set of tiles. Let $X := (S \times \mathbb{T})^{\mathbb{Z}^d}$. Define $T : X \to X$ by

$$T(x, y)_n = \begin{cases} (x_n, y_n + y_{n+d(x_n)}) & \text{if } x \text{ is valid at } n, \\ (x_n, y_n) & \text{otherwise.} \end{cases}$$

Notice that if S is an acyclic set, then T is surjective. Now suppose S is an acyclic set. Let $\omega \in S^{\mathbb{Z}^d}$ and

$$X_\omega := \{(s, y) \in X : s = \omega\}$$

which is a closed T-invariant subset of X. Now define a directed graph $G_\omega = (V_\omega, E_\omega)$ with the vertex $V_\omega = \mathbb{Z}^d$ and the edges

$$E_\omega = \{(n, n + d(\omega_n)) : n \in \mathbb{Z}^d, \omega \text{ is valide at } n\}.$$

A subset of vertex $K \subset V_\omega$ is said to be *connected* in the graph G_ω if for any $a, b \in K$ there exists $c \in K$ such that there are directed paths in K from a to c and from b to c. A *connected component* of G_ω is a connected set in G_ω which is maximal with respect to inclusion. A connected component K of G_ω is called *forward-infinite* if there exists a forward-infinite directed graph in G_ω starting at some/any cell of K. Note that an infinite connected component is not always forward-infinite, in other words, it may be backward-infinite. Since K is acyclic, the set of vertex V_ω is a disjoint union of connected components of G_ω.

By definition of T, it is clear that T acts independently on each connected component of G_ω. Suppose G_ω has m forward-infinite connected component, denoted respectively by $K_1, \dots, K_m$. Let $K_0 = V_\omega \setminus \sqcup_{i=1}^m K_i$ which is the union of vertex having a forward-finite path. Let (X_i, T_i) be the system corresponding to the action of T on K_i. Clearly, $(X_\omega, T) = \prod_{i=0}^m (X_i, T_i)$.

Lemma 7.6. *We have* $\mathrm{mdim}(X_0, T_0) = 0$ *and* $\mathrm{mdim}(X_i, T_i) = 1$ *for every* $1 \le i \le m$*. Moreover,* $1 \le \mathrm{mdim}(X_\omega, T) \le m$ *whenever* $m \ge 1$*.*

Proof. Since any valid path in X_0 is forward-finite, the system (X_0, T_0) is isomorphic to an inverse limit of finite-dimensional systems: the k-th system in this sequence consists of the cells in K_0 with forward-path in G_ω of length at most k, which has the topological dimension at most k[3]. Since a finite-dimensional system has zero mean dimension, it follows by [Shi21, Proposition 5.8] that $\mathrm{mdim}(X_0, T_0) = 0$.

It remains to show $\mathrm{mdim}(X_i, T_i) = 1$ for every $1 \le i \le m$. Let $K = K_i$ be a forward-infinite connected component in G_ω. We define inductively a sequence $J_n \subset K$ as follows. Let $J_0 = \{p_0, p_1, p_2, \dots\} \subset \mathbb{Z}^d$ be a forward-infinite path in K. If $J_n = K$, then let $J_{n+1} = J_n$; otherwise pick a cell $a \in K \setminus J_n$ whose successor in J_n and let $J_{n+1} = J_n \cup \{a\}$. Then we have

$$J_0 \subset J_1 \subset \cdots \subset J_n \subset \cdots \subset K \text{ and thus } K = \bigcup_{n\ge 0} J_n.$$

Notice that T acts independently on each J_n. It is clear that the action of T on J_0 is isomorphic to the algebraic CA on $\mathbb{T}^{\mathbb{N}}$ of the form $(x_n)_{n\in\mathbb{N}} \mapsto (x_n + x_{n+1})_{n\in\mathbb{N}}$. It follows by Lemma 7.3 that the action of T on J_0 has the mean dimension 1. Since the action of T on J_n is a skew-product extension of that on J_{n-1}, the action T on K is an inverse limit of these systems. By [Shi21, Proposition 5.8], we have $\mathrm{mdim}(X_i, T_i) \le 1$. On the other hand, since the action T on K is a skew-product extension of that on J_0, by Lemma 3.9, we have $\mathrm{mdim}(X_i, T_i) \ge 1$. Thus we have $\mathrm{mdim}(X_i, T_i) = 1$.

By mean dimension of product systems, we conclude that $1 \le \mathrm{mdim}(X_\omega, T) \le m$ whenever $m \ge 1$. □

[3]Because it is a countable union of spaces having dimension at most k.

For each $\omega \in S^{\mathbb{Z}^d}$, define $I(\omega)$ to be the number of forward-infinite connected components of G_ω. For a directed set of tiles S, let $I(S) = \sup_{\omega \in S^{\mathbb{Z}^d}} I(\omega)$.

Proposition 7.7. *For any $d \geq 1$ there exist a surjective (algebraic) $\mathbb{Z}^d$-CA with positive, finite mean dimension.*

Proof. Let $d \geq 1$. By [Mey08, Section 4], there exists a directed set of tiles S which is an acyclic set and has $0 < I(S) < +\infty$. It follows from Lemma 7.6 and $\mathrm{mdim}(X,T) = \sup_{\omega \in S^{\mathbb{Z}^d}} \mathrm{mdim}(X_\omega, T)$ that

$$1 \leq \mathrm{mdim}(X,T) \leq I(S) < +\infty.$$

This completes the proof. □

Finally, we remark that by Lemma 7.5 the CA presented in Proposition 7.7 for $d > 1$ is not permutative.

8. Higher-dimensional CA having a spaceship

Let X be a compact metric space. For $v \in \mathbb{Z}^d$, let $\sigma_v : X^{\mathbb{Z}^d} \to X^{\mathbb{Z}^d}$ be the shift $(x_u)_{u \in \mathbb{Z}^d} \mapsto (x_{u+v})_{u \in \mathbb{Z}^d}$. Let Y (resp. x_*) be a distinguished subset (resp. point) of X. Let T be another compact metric space with $\mathrm{stabdim}(T) > 0$ and $(h_t : X \circlearrowleft)_{t \in T}$ a family of functions on X, such that

- the map $t \mapsto h_t(x)$ is continuous for $x \in X$,
- the map $t \mapsto h_t(x)$ is injective for $x \in Y$,
- $h_t(x_*) = x_*$ for all t.

To simplify the notations we also let $h_t(x) = t \cdot x$ and $h_t \circ h_{t'}(x) = tt' \cdot x$.

Let F be a CA on $X^{\mathbb{Z}^d}$ associated to a local rule $f : X^I \to X$ for a finite subset I of $\mathbb{Z}^d$ satisfying $f(t \cdot y_i,\ i \in I) = t \cdot f(y_i,\ i \in I)$ for all $(y_i)_{i \in I} \in (Y \cup \{x_*\})^I$ and for all $t \in T$.

The *support* of an element $x = (x_u)_{u \in \mathbb{Z}^d} \in X^{\mathbb{Z}^d}$ is the set

$$\mathrm{supp}(x) = \{u \in \mathbb{Z}^d : x_u \neq x_*\} \subset \mathbb{Z}^d.$$

An element $x \in (Y \cup \{x_*\})^{\mathbb{Z}^d}$ with finite non empty support is called a *spaceship* when $F^p x = a \cdot \sigma_v(x) := (a \cdot x_{u+v})_{u \in \mathbb{Z}^d}$ for some $p \in \mathbb{N}^*$, $v \in \mathbb{Z}^d \setminus \{0\}$ and $a \in T$ satisfying $a \cdot Y \subset Y$.

Proposition 8.1. *The mean dimension of a CA having a spaceship is infinite.*

Proof. Let F be a CA having a spaceship. By assumption, let $x \in (Y \cup \{x_*\})^{\mathbb{Z}^d}$ be a spaceship of period p with displacement vector $v \neq 0$, i.e. $F^p x = a \cdot \sigma_v(x)$ for some $a \in T$ with $a \cdot Y \subset Y$. By considering some iterate of F we may assume without loss of generality $p = 1$.

We can pick a vector $u \in \mathbb{Z}^d$ not proportional to v and a positive integer m such that the sets $\mathrm{supp}(x) - imv + ju - \mathbb{I}$ for $i, j \in \mathbb{Z}$ are pairwise disjoint (recall $\mathbb{I}$ denotes the convex hull of $I \cup \{0\}$). Let n be a positive integer. The direct product of n copies of the right shift $(T^{\mathbb{Z}}, \sigma^{-1})$ is denoted by $((T^{\mathbb{Z}})^n, \sigma^{-\otimes n})$ which has mean dimension equal to $n \cdot \mathrm{stabdim}(T) > 0$. Define $\varphi : (T^{\mathbb{Z}})^n \to X^{\mathbb{Z}^d}$ for all $\alpha = (\alpha^1, \cdots, \alpha^n) \in (T^{\mathbb{Z}})^n$ by

$$\varphi(\alpha)_k = \alpha_i^j a^{im} \cdot x_{k'} \text{ for } k = k' - imv + ju \text{ with } k' \in \mathrm{supp}(x),\ i \in \mathbb{Z},\ 1 \leq j \leq n$$

$$\text{and } \varphi(\alpha)_k = x_* \text{ for others } k \in \mathbb{Z}^d.$$

It is not hard to see that the map φ is continuous. As $t \mapsto t \cdot y$ is injective for $y \in Y$, the map φ is one-to-one. Thus the map φ is a homeomorphism from $(T^{\mathbb{Z}})^n$ to its image $\Omega := \varphi\left((T^{\mathbb{Z}})^n\right)$.

Claim: The dynamical system $((T^{\mathbb{Z}})^n, \sigma^{-\otimes n})$ is topologically conjugate to (Ω, F^m).

Suppose our claim holds. Then $\mathrm{mdim}(\Omega, F^m) = \mathrm{mdim}((T^{\mathbb{Z}})^n, \sigma^{-\otimes n}) = n \cdot \mathrm{stabdim}(T)$. Thus

$$\mathrm{mdim}(X^{\mathbb{Z}^d}, F) = \frac{\mathrm{mdim}(X^{\mathbb{Z}^d}, F^m)}{m} \geq \frac{\mathrm{mdim}(\Omega, F^m)}{m} = \frac{n \cdot \mathrm{stabdim}(T)}{m}.$$

This will complete the proof as it holds for all positive integers n. It remains to prove our claim. In fact, it is sufficient to show $F \circ \varphi = \varphi \circ \sigma^{\otimes n}$.

For $k \in \mathbb{Z}^d$ the k-coordinate of $F(\varphi(\alpha))$ depends only on the l-coordinates of $\varphi(\alpha)$ for $l \in k + \mathbb{I}$. Fix k. There is at most one pair $(i, j) \in \mathbb{Z} \times \{1, \cdots, n\}$ with $(\mathrm{supp}(x) - imv + ju) \cap (k + \mathbb{I}) \neq \emptyset$, because the sets $\mathrm{supp}(x) - i'mv + j'u - \mathbb{I}$ for $i', j' \in \mathbb{Z}$ are pairwise disjoint. As $t \cdot x_* = x_*$ for all $t \in T$, the configuration $\varphi(\alpha)$ coincides with $\alpha_i^j a^i \cdot \sigma_{imv-ju:}x$ on $k + \mathbb{I}$. Therefore

$$\begin{aligned} F(\varphi(\alpha))_k &= f(\alpha_i^j a^{im} \cdot x_{q+k+imv-ju},\ q \in I), \\ &= \alpha_i^j a^{im} \cdot f(x_{q+k+imv-ju}),\ q \in I), \\ &= \alpha_i^j a^{im} \cdot F(x)_{k+imv-ju}, \\ &= \alpha_i^j a^{im+1} \cdot x_{k+(im+1)v-ju}. \end{aligned}$$

Iterating again F we get finally

$$\begin{aligned} F^m(\varphi(\alpha))_k &= \alpha_i^j a^{(i+1)m} \cdot x_{k+(i+1)mv-ju}, \\ &= \phi(\sigma^{-\otimes n}\alpha)_k. \end{aligned}$$

Therefore $F^m(\varphi(\alpha) = \phi(\sigma^{-\otimes n}\alpha)$ for all $\alpha = (\alpha^1, \cdots, \alpha^n) \in (T^{\mathbb{Z}})^n$. This completes the proof of our claim. □

We illustrate Proposition 8.1 with a continuous state version of the celebrated Conway's game of life. We first recall the local rule of this famous CA on $\{0,1\}^{\mathbb{Z}^2}$. Let $I = [-1,1]^2 \cap \mathbb{Z}^2$ and $I^* = I \setminus \{0\}$. The local rule of the game of life is the map $f : \{0,1\}^I \to \{0,1\}$ such that $f(x_i, i \in I) = 1$ if and only if either $x_0 = 1$ and $\sharp\{i \in I^* :\ x_i = 1\} \in \{2,3\}$ or $x_0 = 0$ and $\sharp\{i \in I^* :\ x_i = 1\} = 3$. It is well known that this discrete CA has a spaceship, meaning here that there a finitely supported configuration $x \in \{0,1\}^{\mathbb{Z}^2}$ (i.e. with finitely many non-zero coordinates) satisfying $F(x) = \sigma_v(x)$ for some $v \in \mathbb{Z}^2 \setminus \{0\}$.

We describe below a continuous version of the game of life on $[0,1]^{\mathbb{Z}^2}$ which contains the standard discrete version as a subsystem. We let f be the local rule $f : [0,1]^I \to [0,1]$ defined as follows :

- if $x_0 > 0$ and $\sharp\{i \in I^* :\ x_i > 0\} \in \{2,3\}$ we let $f(x_i) = \frac{\sum_{i \in I^*} x_i}{\sharp\{i \in I^* :\ x_i > 0\}}$,
- if $x_0 = 0$ and $\sharp\{i \in I^* :\ x_i > 0\} = 3$ we let $f(x_i) = \frac{\sum_{i \in I^*} x_i}{3}$,
- in the remaining case, we let $f(x_i,\ i \in I) = 0$.

Corollary 8.2. *The continuous state version of the game of life has infinite mean dimension.*

Proof. Apply Proposition 8.1 with $X = [0,1]$, $Y =]0,1]$, $x_* = 0$, $T = [0,1]$, $a = 1$ and $h_t(x) = tx$ for all t, x. □

Appendix A. Zero metric mean dimension

Lindenstrauss and Weiss [LW00] showed that the metric mean dimension is always an upper bound of the topological mean dimension. For general dynamical systems, it is widely open whether there exists a metric d (compatible with the topology) such that the metric mean dimension in terms of d equals the mean dimension. This would be a dynamical version of Pontrjagin-Schnirelmann's theorem. In this appendix, we show that if a dynamical system (X, T) is an inverse limit of dynamical systems of finite topological entropy, then there exists a metric d on X with $\mathrm{mdim}_M(X, T, d) = 0$.

Proposition A.1. *Let (X, T) be the inverse limit of a sequence of dynamical systems of finite topological entropy. Then there exists a metric d such that* $\mathrm{mdim}(X, T) = \mathrm{mdim}_M(X, T, d) = 0$.

Proof. Let $(X, T) = \varprojlim_k (X_k, T_k)$ with (X_k, T_k) of finite topological entropy. Pick a metric ρ^k on X_k for each k and define a metric $\hat{\rho}^k(x_k, x_k') := \max_{0 \le j \le k} \rho^j(x_j, x_j')$ on X_k for each $k \ge 0$. Then for each $0 \le j < k$ the bonding map $x_k \mapsto x_j$ from $(X_k, \hat{\rho}^k)$ to $(X_j, \hat{\rho}^j)$ is 1-Lipschitz. Moreover it follows from $h_{top}(X_k, T_k) < +\infty$ that $\mathrm{mdim}_M(X_k, T_k, \hat{\rho}^k) = 0$ for each k. By Lemma 7.7 in [LL18] there is a distance d on X satisfying

$$\mathrm{mdim}_M(X, T, d) \leq \liminf_k \mathrm{mdim}_M(X_k, T_k, \hat{\rho}^k) = 0.$$

□

Corollary A.2. *If (X, T) is finite dimensional, has at most countably many ergodic measures or has the small boundary property, then there exists a metric d such that $\mathrm{mdim}(X, T) = \mathrm{mdim}_M(X, T, d) = 0$.*

Proof. Due to [SW91], [Lin95] and [Lin99], if (X, T) is finite dimensional, has at most countably many ergodic measures or has the small boundary property, then (X, T) is the inverse limit of a sequence of dynamical systems with finite topological entropy, then the result follows by Proposition A.1. □

Appendix B. The natural extension of general cellular automata

In Section 6, we show that the natural extension of a unit CA is a full shift. In this section, we prove that for a general CA, its natural extension is a subshift of finite type. Recall that a subshift (Y, σ) with $Y \subset X^{\mathbb{Z}}$ is said of *finite type* when there is a closed subset L of $X \times X$ with

$$Y = \{(x_n)_n \in X^{\mathbb{Z}},\ (x_n, x_{n+1}) \in L \text{ for all } n \in \mathbb{Z}\}.$$

Proposition B.1. *Let T_f be a cellular automaton on $X^{\mathbb{Z}}$ with local rule $f : X^I \to X$ for $I \subset \mathbb{Z}^d$. The natural extension $\widetilde{T_f}$ is topologically conjugated to a subshift of finite type.*

Proof. Here we denote $Y_{\mathbb{Z}} = Y^{\mathbb{Z}}$ for any set Y. Let J be the integers in the convex hull of $I \cup \{0\}$. Let $J_- = \min\{J\}$ and $J_+ = \max\{J\}$. Let nJ be the collection of integers in $[nJ_-, nJ_+]$. The natural extension $\widetilde{X_{\mathbb{Z}}}$ is given by

$$\widetilde{X_{\mathbb{Z}}} = \{x^l \in X_{\mathbb{Z}},\ T_f(x^{l+1}) = x^l\} \subset (X_{\mathbb{Z}})^{\mathbb{N}}.$$

With $x^l = (x^l_k)_k$, the equality $T_f(x^{l+1}) = x^l$ may be rewritten as $f((x^{l+1}_k)_{J_-+j \leq k \leq J_++j}) = x^l_j$ for all j.

We let

$$\widetilde{X} := \{(x^l)_l \in \prod_{l\in\mathbb{N}} X^{lJ} : x^l = (x^l_k)_{k\in lJ}, f((x^{l+1}_k)_{J_-+j\leq k\leq J_++j}) = x^l_j\}.$$

Consider the subset L of $\widetilde{X} \times \widetilde{X}$ given by

$$L := \{(x, y) \in \widetilde{X} \times \widetilde{X} : x^l_k = y^{l+1}_k, \forall l \geq 0, \forall\, lJ_- \leq k \leq lJ_+\}.$$

Then we define a subshift in $\widetilde{X}_{\mathbb{Z}}$ by

$$Y := \{x \in \widetilde{X}_{\mathbb{Z}} : (x_n, x_{n+1}) \in L\}.$$

The map

$$\pi : Y \to \widetilde{X_{\mathbb{Z}}},$$
$$(y_k)_{k\in\mathbb{Z}} \mapsto (x^l_n)_{n\in\mathbb{Z}, l\in\mathbb{N}}$$

with $y_k = ([y_k]^l_n)_{l\in\mathbb{N}, n\in lJ} \in \widetilde{X}$ and $x^l_n = [y_k]^{l+k}_n$ for some/any $n \in (l+k)J$. It is clear that π is a topological conjugacy.

□

Remark B.2. *In general, the space $\widetilde{X}$ may be an infinite-dimensional space. Since we know few about the mean dimension of a subshift of finite type (even the full shift) over an infinite-dimensional space, the natural extension $\widetilde{T_f}$ in Proposition B.1 does not provide information on the mean dimension of $(X^{\mathbb{Z}}, T_f)$.*

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