RESCALED ENTROPY OF CELLULAR AUTOMATA

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Abstract. For a d-dimensional cellular automaton with $d \geq 1$ we introduce a rescaled entropy which estimates the growth rate of the entropy at small scales by generalizing previous approaches [1, 9]. We also define a notion of Lyapunov exponent and proves a Ruelle inequality as already established for $d = 1$ in [20, 18]. Finally we generalize the entropy formula for 1-dimensional permutative cellular automata [21] to the rescaled entropy in higher dimensions. This last result extends recent works [19] of Shinoda and Tsukamoto dealing with the metric mean dimensions of two-dimensional symbolic dynamics.

1. Introduction

In this paper we estimate the dynamical complexity of multidimensional cellular automata. In the following the main results will be stated in a more general setting, but let us focus in this introduction on the following algebraic cellular automaton on $(\mathbb{F}_p)^{\mathbb{Z}^d}$ with p prime given for some finite family $(a_i)_{i\in I}$ in $\mathbb{F}_p^*$, $I \subset \mathbb{Z}^d$, by

$$\forall (x_j)_j \in (\mathbb{F}_p)^{\mathbb{Z}^d},\ f((x_j)_j) = \left(\sum_{i\in I} a_i x_{i+j}\right)_j .$$

Let $I' = I \cup \{0\}$. For $d = 1$ the topological entropy of f is finite and equal to $\mathrm{diam}(I') \log p$ where $\mathrm{diam}(I')$ denotes the diameter of I' for the usual distance on $\mathbb{R}$ [21]. However in higher dimensions the topological entropy of f is always infinite unless $I = \{0\}$ [15, 10]. Moreover the topological entropy of the $\mathbb{N}\times\mathbb{Z}^d$-action given by f and the shift vanishes. It was expected that the topological entropy of any cellular automaton for $d > 1$ was either zero or infinity, but T. Meyerovitch built a two-dimensional counterexample [13].

In this paper we investigate the growth rate of $\left(h_{top}(f, \mathsf{P}_{J_n})\right)_n$ for nondecreasing sequences (J_n) of convex subsets of $\mathbb{R}^d$ where $(\mathsf{P}_{J_n})_n$ denotes the clopen partitions into $J_n \cap \mathbb{Z}^d$-coordinates. This sequence appears to increase as the perimeter $p(J_n)$ of J_n. We define the rescaled entropy $h^d_{top}(f)$ of f as $\limsup_{J_n} \frac{h_{top}(f,\mathsf{P}_{J_n})}{p(J_n)}$. In [9] another renormalization is used, whereas in [1] the authors only investigate the case of squares $J_n = [-n, n]^2,\ n \in \mathbb{N}$. For $d = 1$ we get $h^1_{top}(f) = \frac{h_{top}(f)}{2}$. We generalize the entropy formula for algebraic cellular automata as follows :

Theorem 1. *Let f be an algebraic cellular automaton on $(\mathbb{F}_p)^{\mathbb{Z}^d}$ as above, then*

$$h^d_{top}(f) = R_{I'} \log p,$$

where $R_{I'}$ denotes the radius of the smallest bounding sphere containing I'.

In fact we establish such a formula for any permutative cellular automaton (see Section 7). In [19] the authors compute, inter alia, the metric mean dimension of the horizontal shift in $\mathbb{Z}^2$ for some standard distances. These dimensions may be interpreted as the rescaled entropy

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Date: June 2018.
2010 *Mathematics Subject Classification.* 37B15, 37A35, 52C07.

with respect to some particular sequence of convex sets $(J_n)_n$. In particular we extend these results in higher dimensions for general permutative cellular automata.

We also consider a measure theoretical analogous quantity of the rescaled entropy. In dimension one, a notion of Lyapunov exponent has been defined in [18]. Then Tisseur [20] proved in this case a Ruelle inequality relating this exponent with the Kolmogorov-Sinai entropy. In this paper we also introduce a notion of Lyapunov exponent in higher dimensions, which bounds from above the rescaled entropy of measures.

The paper is organized as follows. In Section 2 we state some measure geometrical properties of convex sets in $\mathbb{R}^d$. We estimate the cardinality of integer points in the morphological boundary of large convex sets in Section 3. We recall the dynamical background of cellular automata in Section 4 and we introduce then a Lyapunov exponent for multidimensional cellular automata. In Section 5 we define and study the topological and measure theoretical rescaled entropy. We prove the Ruelle type inequality in Section 6. Section 7 is devoted to the proof of the entropy formula for permutative cellular automata. Finally we discuss in the last section a generalization of the rescaled entropy for any endomorphism of a $\mathbb{Z}^d$-action.

2. Background on convex geometry

2.1. **Convex bodies, domains and polytopes.** For a fixed positive integer d we endow the vector space $\mathbb{R}^d$ with its usual Euclidean structure. The associated scalar product (resp. norm) is simply denoted by $\cdot$ (resp. $\| \|$) and we let $\mathbb{S}^{d-1}$ be the unit sphere. For a subset F of $\mathbb{R}^d$ we let $\overline{F}$, $\mathrm{Int}(F)$ and ∂F be respectively its closure, interior set and boundary. We let $\sharp F$ be the number of integer points in F, i.e. $\sharp F = |F \cap \mathbb{Z}^d|$. We also denote by $V(F)$ the d-Lebesgue measure of F (also called the volume of F) when the set F is Borel.

The extremal set of a convex set J is denoted by $\mathrm{ex}(J)$ and the convex hull of $F \subset \mathbb{R}^d$ by $\mathrm{cv}(F)$. A convex body is a compact convex set of $\mathbb{R}^d$. A convex body containing the origin $0 \in \mathbb{R}^d$ in its interior set is said to be a **convex domain**. The set of convex bodies endowed with the Hausdorff topology is a locally compact metrizable space. In the following we denote by $\mathcal{D}$ the set of convex domains endowed with the Hausdorff topology. A **convex polytope** (resp. k-polytope with $k \leq d$) in $\mathbb{R}^d$ is a convex body given by the convex hull of a finite set (resp. with topological dimension equal to k). When this finite set lies inside the lattice $\mathbb{Z}^d$, the convex polytope is said **integral**. We let $\mathcal{F}(P)$ be the set of faces of a convex d-polytope P.

A convex domain J has Lipshitz boundary and finite perimeter $p(J)$. We let $\mathcal{D}^1$ be the subset of $\mathcal{D}$ given by convex domains with unit perimeter. We denote by $\widetilde{J} = p(J)^{-\frac{1}{d-1}} \in \mathcal{D}^1$ the normalization of a convex domain J. For convex domains the perimeter in the distributional sense of De Giorgi coincides with the $(d-1)$-Hausdorff measure $\mathcal{H}_{d-1}$ of the boundary. For $J \in \mathcal{D}$ we let $\partial' J$ be the subset of points $x \in \partial J$, where the tangent space $T_x J$ is well defined. The set $\partial' J$ has full $\mathcal{H}_{d-1}$-measure in ∂J. We let $N^J(x) \in \mathbb{S}^{d-1}$ be the unit J-external normal vector at $x \in \partial' J$. For any $x \in \partial' J$ we let $T_x^+ J$ (resp. $T_x^- J$) be the open external (resp. closed internal) semi-space with boundary $T_x J$. With these notations we have $J = \bigcap_{x \in \partial' J} T_x^- J$. For $\epsilon \in \mathbb{R}$ we denote by $T_x^{\pm} J(\epsilon)$ the semi-planes $T_x^{\pm} J(\epsilon) = T_x^{\pm} J + \epsilon N^J(x)$. When J is a convex d-polytope and $F \in \mathcal{F}(J)$, we write T_F to denote the tangent affine space supporting F, $T_F^{\pm}$ for the associated semi-spaces and N^F for the unit external normal to F.

The **support function** of a convex body I is the real continuous function h_I on $\mathbb{S}^{d-1}$:

$$\forall x \in \mathbb{S}^{d-1},\ h_I(x) = \max_{u \in I} u \cdot x.$$

The support function completely characterizes the convex body I. The **area measure** σ_J of a convex domain J is the Borel measure on $\mathbb{S}^{d-1}$ given by $N_*^J \mathcal{H}_{d-1}$:

$$\forall B \text{ Borel of } \mathbb{S}^{d-1},\ \sigma_J(B) = \mathcal{H}_{d-1}\left((N^J)^{-1} B\right).$$

If a sequence $(J_n)_n$ in $\mathcal{D}$ is converging to $J_\infty \in \mathcal{D}$ (for the Hausdorff topology), then σ_{J_n} is converging weakly to σ_{J_∞}, in particular the perimeter of J_n goes to the perimeter of J_∞ (see Proposition 10.2 in [7]). Consequently, $\mathcal{D}^1$ is a closed subset of $\mathcal{D}$.

2.2. **Convex exhaustions.** An exhaustion is a sequence $\mathcal{J} = (J_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^d$ satisfying $\bigcup_n J_n = \mathbb{R}^d$. In this paper we consider exhaustions $\mathcal{J} = (J_n)_{n\in\mathbb{N}}$ of convex domains with $p(J_n) \xrightarrow{n} +\infty$, such that the sets $\widetilde{J_n} = p\left(J_n\right)^{-\frac{1}{d-1}} J_n \in \mathcal{D}^1$ are converging to a limit $J_\infty \in \mathcal{D}$ in the Hausdorff topology. Then the limit J_∞ has unit perimeter. The sequences $\mathcal{J} = (J_n)_n$ satisfying the above properties are said to be **convex exhaustions**. For $O \in \mathcal{D}^1$ we denote by $\mathcal{E}(O)$ the set of convex exhaustions $\mathcal{J} = (J_n)_n$ with $J_\infty = O$. Moreover for $O \in \mathcal{D}$ we let $\mathcal{J}_O \in \mathcal{E}(\widetilde{O})$ be the convex exhaustion given by $\mathcal{J}_O := (nO)_n$.

The inner radius $r(E)$ of a subset E of $\mathbb{R}^d$ is the largest $a \geq 0$ such that E contains a Euclidean ball of radius a. For two subsets E and F of $\mathbb{R}^d$ we let $E\Delta F$ be the symmetric difference of E and F given by $E\Delta F := (E \setminus F) \cup (F \setminus E)$.

Lemma 1. *Let $O \in \mathcal{D}$ and $\mathcal{J} = (J_n)_n \in \mathcal{E}(O)$. Then any sequence of convex bodies $\mathcal{K} = (K_n)_n$ with $r\left(K_n \Delta J_n\right) = o\left(p(J_n)^{\frac{1}{d-1}}\right)$ belongs to $\mathcal{E}(O)$ and $p(K_n) \sim^n p(J_n)$.*

Proof. We claim that $p\left(J_n\right)^{-\frac{1}{d-1}} K_n$ is converging to J_∞ in the Hausdorff topology. Then by taking the perimeter in this limit we get $\lim_n \frac{p(K_n)}{p(J_n)} = p(J_\infty) = 1$ and therefore $\widetilde{K_n} = p\left(K_n\right)^{-\frac{1}{d-1}} K_n$ also goes to $J_\infty = O$. Let us prove now the claim. Fix an Euclidean ball B with $J_\infty \subset \operatorname{Int} B$. It is enough to show that $p\left(J_n\right)^{-\frac{1}{d-1}} K_n \cap B$ is converging to J_∞. Indeed as K_n is convex, this will imply that $p\left(J_n\right)^{-\frac{1}{d-1}} K_n$ is contained in B for n large enough (if not $p\left(J_n\right)^{-\frac{1}{d-1}} K_n \cap \partial B$ is non empty for infinitely many n and therefore we should have $J_\infty \cap \partial B \neq \emptyset$). By extracting a subsequence we may assume $p\left(J_n\right)^{-\frac{1}{d-1}} K_n \cap B$ is converging to a convex body K_∞ and we need to prove $K_\infty = J_\infty$. We argue by contradiction. As J_∞ is a convex domain, we have either $\operatorname{Int}(J_\infty) \setminus K_\infty \neq \emptyset$ or $\operatorname{Int}(K_\infty) \setminus J_\infty \neq \emptyset$. But for x in one of these sets, there is $s > 0$ such that the balls $p(J_n)^{\frac{1}{d-1}} B(x, s)$ are contained in $K_n \Delta J_n$, therefore $r\left(K_n \Delta J_n\right) \geq s p(J_n)^{\frac{1}{d-1}}$, for n large enough. □

Remark 2. *If $\sharp K_n \Delta J_n = o\left(p(J_n)^{\frac{d}{d-1}}\right)$ then the condition on the inner radius in Lemma 1 holds and therefore $\mathcal{K}$ belongs to $\mathcal{E}(O)$.*

2.3. **Internal and external morphological boundary.** We recall some terminology of mathematical morphology used in image processing. For two subsets I and J of $\mathbb{R}^d$, the **dilation** (also known as the Minkowski sum) $J \oplus I$ and the **erosion** $J \ominus I$ of J by I are defined as follows

$$J \oplus I = \{i + j \mid i \in I \text{ and } j \in J\},$$
$$J \ominus I = \{j \in \mathbb{R}^d \mid \forall i \in I,\, i + j \in J\}.$$

When the origin 0 belongs to I then we have $J \subset J \oplus I$ and $J \ominus I \subset J$. When J is a convex body then $J \ominus I$ is a convex body. Assume now that I is also a convex body. The dilation $J \oplus I$ is then also a convex body with $\operatorname{ex}(J \oplus I) \subset \operatorname{ex}(I) \oplus \operatorname{ex}(J)$. In particular, when I and J are moreover convex polytopes, then so is $J \oplus I$. We have $J \ominus I = \bigcap_{x \in \partial' J} T_x^- J\left(h_I(-N^J(x))\right)$ (also $J \oplus I \subset \bigcap_{x \in \partial' J} T_x^- J\left(h_I(N^J(x))\right)$, but this last inclusion may be strict). When J is a convex polytope, the above intersection is finite, thus $J \ominus I$ is also a convex polytope. The convex bodies given by the erosion $J \ominus I$ and the dilation $J \oplus I$ are also known as the inner and outer parallel bodies of J relative to I. We recall that $h_{J\oplus I} = h_J + h_I$. In particular when $I = \{i\}$ is a singleton, we get $h_{J+i}(x) = h_J(x) + i \cdot x$ for all $x \in \mathbb{S}^{d-1}$. In general we only have $h_{J\ominus I} \leq h_J - h_I$.

The **internal and external (morphological) boundaries** of J relative to I denoted respectively by $\partial_I^- J$ and $\partial_I^+ J$ are given by

$$\partial_I^+ J = (I \oplus J) \setminus J,$$
$$\partial_I^- J = J \setminus (J \ominus I).$$

Clearly we have $\partial_I^{\pm} J = \partial_{I'}^{\pm} J$ with $I' = I \cup \{0\}$. When J is a convex domain then we have $\partial_I^- J = \partial_{\mathrm{cv}(I)}^- J$ and $\partial_I^+ J \subset \partial_{\mathrm{cv}(I)}^+ J$. In the following the set I will be fixed so that we omit the index I in the above definitions when there is no confusion.

Finally we observe that $r\left(J_n \Delta (J_n \oplus I)\right), r\left(J_n \Delta (J_n \ominus I)\right) \leq \mathrm{diam}(I')$. Therefore it follows from Lemma 1, that if $(J_n)_n$ is a convex exhaustion and I a convex body then $(J_n \ominus I)_n$ and $(J_n \oplus I)_n$ define convex exhaustions with the same limit as $(J_n)_n$.

3. Counting integer points in morphological boundary of large convex sets

For a large convex domain J and a fixed integral polytope I we estimate the cardinality of the integer points in the morphological boundaries of J relative to I.

3.1. **First relative quermass integral.** Let O be a convex domain and let I be a convex body. For $\rho \in \mathbb{R}$ we let

$$O_\rho = \begin{cases} O \oplus \rho I \text{ when } \rho \geq 0, \\ O \ominus \rho I \text{ when } \rho < 0. \end{cases}$$

Proposition 3.

$$\lim_{\rho \to 0} \frac{V(O_\rho) - V(O)}{\rho} = \int_{\mathbb{S}^{d-1}} h_I \, d\sigma_O.$$

For $\rho > 0$ the formula follows from Minkowski's formula on mixed volume (see Theorem 6.5 and Corollary 10.1 in [7]). For $\rho < 0$ we refer to [12] (see also Lemma 2 in [4] for the 2-dimensional case).

In the following we denote by $V_I(O)$ the integral $\int_{\mathbb{S}^{d-1}} h_I \, d\sigma_O$. The product $d \cdot V_I(O)$ is known as the **first I-relative quermass integral** of O. For convex bodies $I \subset H$ and $k \in \mathbb{N}$, we have $V_I(O) \leq V_H(O)$ and $V_{kI}(O) = kV_I(O)$ for any convex domain O. The support function h_I being continuous, the first I-relative quermass integral of O is continuous with respect to the Hausdorff topology, i.e. if $(O_n)_n$ is a sequence of convex domains converging to a convex domain O_∞ in the Hausdorff topology, then we have

$$V_I(O_n) \xrightarrow{n \to +\infty} V_I(O_\infty).$$

We deduce now from Proposition 3 an estimate on the volume of the morphological boundary for large convex sets.

Corollary 4. *Let I be a convex body containing 0 and let $O \in \mathcal{D}$. Then*

$$V\left(\partial_I^{\pm} nO\right) \sim n^{d-1} \int_{\mathbb{S}^{d-1}} h_I \, d\sigma_O.$$

Proof. We only consider the case of the external boundary as one may argue similarly for the internal boundary. For all $n > 0$ we have

$$\begin{aligned} V\left(\partial_I^+ nO\right) &= V\left(nO \oplus I\right) - V\left(nO\right), \\ &= n^d \left(V(O \oplus n^{-1} I) - V(O)\right). \end{aligned}$$

According to Proposition 3 we conclude that

$$V\left(\partial_I^+ nO\right) \sim n^{d-1} \int_{\mathbb{S}^{d-1}} h_I \, d\sigma_O.$$

□

3.2. **Counting integer points in large convex sets.** After Gauss circle problem, counting lattice points in convex sets has been extensively investigated. Let $\mathsf{C} = [0,1]^d$. Clearly for any Borel subset K of $\mathbb{R}^d$ we have always

$$\sharp K \leq V(K \oplus \mathsf{C}). \tag{3.1}$$

In the other hand, Bokowski, Hadwiger and Wills have proved the following general (sharp) inequality for any convex domain O [2] :

$$V(O) - \frac{p(O)}{2} \leq \sharp O. \tag{3.2}$$

There exist precise asymptotic estimates of $\sharp xO$ for large $x > 0$ for convex smooth domains O having positive curvature, in particular we have in this case $\sharp xO = V(xO) + o(x^{d-1})$ [8].

3.3. **Estimate of $\sharp\partial_I^{\pm} nO$ for $O \in \mathcal{D}$.** For a real sequence $(a_n)_n$ and two numbers l and $c > 0$ we write $a_n \sim l \pm c$ when the accumulation points of $(a_n)_n$ lie in $[l-c, l+c]$.

Lemma 2. *There exists a constant c depending only on d such that we have for any convex domain $O \in \mathcal{D}$ and any convex body I of $\mathbb{R}^d$ with $0 \in I$:*

$$\frac{\sharp\partial_I^{\pm} nO}{n^{d-1}} \sim V_I(O) \pm c.$$

Proof. We only argue for $\partial_I^{+} nO$, the other case being similar. We have $\sharp\partial_I^{+} nO = \sharp nO \oplus I - \sharp nO$, and then by combining Equation (3.1) and (3.2) we get :

$$V(nO \oplus I) - \frac{p(nO \oplus I)}{2} - V(nO + \mathsf{C}) \quad \leq \sharp\partial_I^{+} nO \quad \leq V(nO \oplus I \oplus \mathsf{C}) - V(nO) + \frac{p(nO)}{2},$$

After dividing by n^{d-1}, the right (resp. left) hand side term is going to $\int_{\mathbb{S}^{d-1}} (h_I - h_{\mathsf{C}} - 1/2)\, d\sigma_O$ (resp. $\int_{\mathbb{S}^{d-1}} (h_I + h_{\mathsf{C}} + 1/2)\, d\sigma_O$) according to Corollary 4. □

3.4. **Upperbound of $\sharp\partial^{-} J_n$ for general convex exhaustions.** For a subset E of $\mathbb{R}^d$ and for $r > 0$ we let $E(r) := \{x \in E, d(x, \partial E) \leq r\}$ with d being the Euclidean distance. With the previous notations we may also write $E(r) = \partial_{B_r}^{-} E$ where B_r denotes the Euclidean ball centered at 0 with radius r.

Lemma 3. *For any convex domain J in $\mathbb{R}^d$, we have*

$$V(J(r)) \leq rp(J).$$

Proof. We first assume that J is a convex d-polytope. Let $x \in J(r)$. There is $F \in \mathcal{F}(\mathsf{J})$ with $\|x - x_F\| \leq d(x, F) = d(x, \partial J) \leq r$, where x_F denotes the orthogonal projection of x onto T_F. Observe that x_F belongs to F : if not the segment line $[x, x_F]$ would have a non empty intersection with ∂J and the intersection point $y \in \partial J$ would satisfy $\|x - y\| < \|x - x_F\| \leq d(x, \partial J)$. Therefore $J(r) \subset \bigcup_{F \in \mathcal{F}(J)} R_F(r)$ with $R_F(r) := \{x - tN^F(x),\ x \in F \text{ and } t \in [0, r]\}$. Finally we get

$$\begin{aligned} V(J(r)) &\leq \sum_{F \in \mathcal{F}(J)} V(R_F(r)), \\ &\leq rp(J). \end{aligned}$$

For a general convex domain, there is a nondecreasing sequence $(J_p)_p$ of convex d-polytopes contained in J converging to J in the Hausdorff topology. Then the characteristic function of $J_p(r)$ is converging pointwisely to the characteristic function of $J(r)$, in particular $V(J_p(r)) \xrightarrow{p} V(J(r))$. Moreover $p(J_p)$ goes to $p(J)$, so that the desired inequality is obtained by taking the limit in the inequalities for the convex d-polytopes J_p. □

Proposition 5. *For any convex exhaustion $(J_n)_n$ in $\mathbb{R}^d$, we have*

$$\limsup_n \frac{\sharp\partial_I^- J_n}{p(J_n)} \leq \mathrm{diam}(I') + \sqrt{d}.$$

Proof. As already observed, we have $\sharp\partial^- J_n \leq V(\partial^- J_n \oplus \mathsf{C})$ with $\mathsf{C} = [0,1]^d$. Let $(J'_n)_n$ be the sequence given by $J'_n = J_n \oplus \mathsf{C}$ for all n. By Lemma 1 this sequence is a convex exhaustion with $p(J'_n) \sim^n p(J_n)$. Moreover $\partial^- J_n \oplus \mathsf{C}$ is contained in $J'_n(c)$ with $c = \mathrm{diam}(I') + \mathrm{diam}(\mathsf{C})$. Therefore we conclude according to Lemma 3 :

$$\begin{aligned}\sharp\partial^- J_n &\leq V\left(J'_n(c)\right), \\ &\leq cp(J'_n), \\ &\lesssim^n cp(J_n).\end{aligned}$$

□

Remark 6. *We conjecture that $\lim_n \frac{\sharp\partial_I^- J_n}{p(J_n)} = V_I(J_\infty)$ holds for any convex exhaustion $(J_n)_n$ in $\mathbb{R}^d$. We manage to show it only in dimension 2, but we prefer to omit the proof as such finer estimates are useless in the dynamical applications given in the present paper.*

3.5. **Supremum of $O \mapsto V_I(O)$.** In this section we investigate the supremum of V_I on $\mathcal{D}^1$ for a given convex polytope I of $\mathbb{R}^d$. We recall that there is a unique sphere S_I containing I with minimal radius, usually called the **smallest bounding sphere** of I. We let R_I and x_I be respectively the radius and the center of S_I. There are at least two distinct points in $S_I \cap I$, whenever I is not reduced to a singleton, and $S_I \cap I \subset \mathrm{ex}(I)$. Moreover we have the following alternative :

- either there is a finite subset of $S_I \cap I$ generating an inscribable polytope T with $\mathrm{Int}(T) \ni x_I$ (in particular the interior set of I is non empty),
- or there is a hyperplane H containing x_I such that I lies in an associated semispace and $S_I \cap H$ is the smallest bounding sphere of $I \cap H$.

The smallest bounding sphere S_I (or I itself) will be said **nondegenerated** (resp. **degenerated**) and an associated polytope T (resp. hyperplane H) is said **generating**. For an inscribable polytope T in $\mathbb{R}^d$ we may define its dual T' as the polytope given by the intersection of the inner semispaces tangent to the circumsphere of T at the vertices of T. In the following T' always denotes the dual polytope of a generating polytope T with respect to I.

When S_I is degenerated, there is a sequence of affine spaces $H = H_1 \supset H_2 \supset \cdots H_l \ni x_I$ such that $I \cap H_l$ is nondegenerated in H_l and for all $1 \leq i < l$ the convex polytope $I \cap H_i$ is degenerated in H_i with H_{i+1} as an associated generating hyperplane (H_i is a $d-i$ dimensional affine space). We denote by L a generating polytope of $I \cap H_l$ in H_l and by L' its dual polytope in H_l. Let U be an isometry of $\mathbb{R}^d$ mapping H_i for $i = 1, \cdots, l$ to $\{0_i\} \times \mathbb{R}^{d-i}$ (where 0_i denotes the origin of $\mathbb{R}^i$) with $U(x_I) = 0$. Then for $R > 0$ we let $T'_R := U^{-1}\left([-R,R]^l \times U(L')\right)$. The faces F of T'_R satisfy

(1) either $F = U^{-1}\left([-R,R]^l \times U(\mathsf{F})\right)$ for some face F of L',
(2) or $F = U^{-1}\left([-R,R]^{l-1} \times \{\pm R\}_i \times U(L')\right)$ for $i = 1, \cdots, l$ (where $\{\pm R\}_i$ coresponds to the i^{th} coordinate of the product).

For $i = 1, 2$ we let $\mathcal{F}_i(T'_R)$ be the subset of $\mathcal{F}(T'_R)$ given by the faces of the i^{th} category.

Observe that when x_I coincides with the origin then T' or T'_R, $R > 0$ are convex domains.

Proposition 7.

$$\sup_{O \in \mathcal{D}^1} V_I(O) = R_I.$$

The supremum of V_I is achieved if and only if S_I is nondegenerated. The supremum is then achieved at $\widetilde{T'}$ with T' being the dual polytope of a generating polytope T.

Proof. For any $v \in \mathbb{R}^d$ we have

$$\begin{aligned} V_{I+v}(O) &= \int h_{I+v}\, d\sigma_O, \\ &= \int h_I\, d\sigma_O + \int_{\mathbb{S}^{d-1}} v \cdot u \, d\sigma_O(u), \\ &= \int h_I\, d\sigma_O + \int_{\partial O} v \cdot N^O \, d\mathcal{H}_{d-1}. \end{aligned}$$

By the divergence formula we have $\int_{\partial O} v \cdot N^O \, d\mathcal{H}_{d-1} = 0$ for any $v \in \mathbb{R}^d$ and $O \in \mathcal{D}^1$. Therefore we may assume $x_I = 0$. With the above notations we have $\max_{u \in I} u \cdot v \leq R_I$ for all $v \in \mathbb{R}^d$ with $\|v\| = 1$ with equality iff v belongs to $R_I^{-1} I$. Therefore $V_I(O) \leq R_I$ for any $O \in \mathcal{D}^1$. Moreover if the equality occurs then for x in a subset E of ∂O with full $\mathcal{H}_{d-1}$-measure, $h_I\left(N^O(x)\right) = \max_{u \in I} u \cdot x = R_I$ and therefore the normal unit vector $N^O(x)$ belongs to $R_I^{-1} I$. But as O is a convex domain, we may find $d+1$ points $x_1, \cdots, x_{d+1}$ in E in such a way the origin belongs to the interior of the simplex $T = R_I \,\mathrm{cv}\left(N^O(x_1), \cdots, N^O(x_{d+1})\right)$. Thus S_I is nondegenerated and the polytope T is a generating polytope with respect to I. Moreover we have with the above notations

$$\int h_I \, d\sigma_{T'} = R_I p(T').$$

Therefore $\widetilde{T'}$ achieves the supremum of V_I on $\mathcal{D}^1$. We consider now the degenerated case. With the above notations, we have $h_I(N^F) = R_I$ for any $F \in \mathcal{F}_1(T'_R)$ (recall we assume $x_I = 0$ without loss of generality). Moreover $\mathcal{H}_{d-1}\left(\bigcup_{F \in \mathcal{F}_2(T'_R)} F\right) = o(p(T'_R))$ when R goes to infinity. Therefore the renormalization $\widetilde{T'_R} \in \mathcal{D}^1$ of T'_R satisfies

$$V_I(\widetilde{T'_R}) \xrightarrow{R \to +\infty} R_I.$$

□

4. Cellular automata

4.1. **Definitions.** We consider a finite set $\mathcal{A}$. We endow the set $\mathcal{A}$ with the discrete topology and $X_d = \mathcal{A}^{\mathbb{Z}^d}$ with the product topology. We consider the $\mathbb{Z}^d$-shift σ on $\mathcal{A}^{\mathbb{Z}^d}$ defined for $l \in \mathbb{Z}^d$ and $u = (u_k)_k \in X_d$ by $\sigma^l(u) = (u_{k+l})_k$. Any closed subset X of X_d invariant under the action of σ is called a $\mathbb{Z}^d$**-subshift**. We fix such a subshift X in the remaining of the paper.

For a bounded subset J of $\mathbb{R}^d$ we consider the partition P_J into $J \cap \mathbb{Z}^d$**-cylinders**, i.e. the element P_J^x of P_J containing $x = (x_i)_{i \in \mathbb{Z}^d} \in X$ is given by $\mathsf{P}_J^x := \{y = (y_i)_{i \in \mathbb{Z}^d} \in X, \ \forall i \in J \cap \mathbb{Z}^d \ y_i = x_i\}$. In other terms we may define P_J as the joined partition $\bigvee_{j \in J \cap \mathbb{Z}^d} \sigma^{-j} \mathsf{P}_0$ with P_0 being the zero-coordinate partition.

A **cellular automaton** (CA for short) defined on a $\mathbb{Z}^d$-subshift X is a continuous map $f : X \to X$ which commutes with the shift action σ. By a famous theorem of Hedlund [16] the cellular automaton f is given by a local rule, i.e. there exists a finite subset I of $\mathbb{Z}^d$ and a map $F : \mathcal{A}^I \to \mathcal{A}$ such that

$$\forall j \in \mathbb{Z}^d \ \ (fx)_j = F\left((x_{j+i})_{i \in I}\right).$$

The (smallest) subset I is called the **domain** of the CA. Recall $I' = I \cup \{0\}$ and let $\mathbb{I}$ be the convex hull of I'.

4.2. **Lyapunov exponents for higher dimensional cellular automata.** Lyapunov exponent of one-dimensional cellular automata have been defined in [18, 20]. We develop a similar theory in higher dimensions. Let f be a CA on a $\mathbb{Z}^d$-subshift X with domain I.

Given a convex body J of $\mathbb{R}^d$ and $x \in X$, we let

$$\mathcal{E}_f(x,J) := \{K \text{ convex body, } f\mathsf{P}_J^x \subset \mathsf{P}_K^{fx}\}$$

A priori the family $\mathcal{E}_f(x,J)$ does not admit a greatest element for the inclusion. Observe also that the convex body $J \ominus I$ belongs to $\mathcal{E}_f(x,J)$, in particular this family is not empty. Then we let for all x :

$$\mathrm{gr}_J f(x) := \min\{\sharp\, J \setminus K,\ K \in \mathcal{E}_f(x,J)\}.$$

The family $\mathcal{E}_f(x,J)$ and the function $\mathrm{gr}_J f(x)$ are constant on each atom A of P_J, thus we let $\mathcal{E}_f(A,J)$ and $\mathrm{gr}_J f(A)$ be these quantities. We denote by $\mathcal{D}_f(x,J)$ the subfamily of $\mathcal{E}_f(x,J)$ consisting in K with $\sharp\, J \setminus K = \mathrm{gr}_J f(x)$. For K in $\mathcal{D}_f(x,J)$ the intersection $K \cap J$ defines a convex body, which belongs also to $\mathcal{D}_f(x,J)$.

For a convex exhaustion $\mathcal{J} = (J_n)_n$, we define **the growth** $\mathrm{gr}_{\mathcal{J}} f$ with respect to $\mathcal{J}$ as the following real functions on X :

$$\mathrm{gr}_{\mathcal{J}} f := \limsup_n \frac{\mathrm{gr}_{J_n} f}{p(J_n)}.$$

Finally we let for a convex domain $O \in \mathcal{D}^1$:

$$\mathrm{gr}_O f = \sup_{\mathcal{J} \in \mathcal{E}(O)} \mathrm{gr}_{\mathcal{J}} f.$$

Lemma 4. *The sequence of functions $\left(\mathrm{gr}_O f^k\right)_k$ is a subadditive cocycle, i.e.*

$$\forall k,l \in \mathbb{N}\ \forall x \in X,\ \mathrm{gr}_O f^{k+l}(x) \le \mathrm{gr}_O f^l(f^k x) + \mathrm{gr}_O f^k(x).$$

Proof. Fix $x \in X$ and $k,l \in \mathbb{N}$. Let $\mathcal{J} = (J_n)_n \in \mathcal{E}(O)$. We consider a sequence $\mathcal{K} := (K_n)_n$ of convex bodies in $\prod_n \mathcal{D}_{f^k}(x,J_n)$ with $K_n \subset J_n$ for all n. Let I_k be the domain of f^k. The convex body $J_n \ominus I_k$ belongs to $\mathcal{E}_{f^k}(x,J_n)$ for all n. By Proposition 5, we have $\sharp\, J_n \setminus K_n \le \sharp \partial^-_{I_k} J_n = O\left(p(J_n)\right)$. It follows from Lemma 1 and Remark 2 that $\mathcal{K}$ is a convex exhaustion in $\mathcal{E}(O)$ with $p(K_n) \sim^n p(J_n)$. We also let $\mathcal{L} = (L_n)_n \in \prod_n \mathcal{D}_{f^l}(f^k x, K_n)$ with $L_n \subset K_n$ for all n. Similarly the sequence $\mathcal{L}$ belongs to $\mathcal{E}(O)$ with $p(L_n) \sim^n p(J_n)$. Then we have for all positive integers n :

$$\begin{aligned} f^{k+l}\mathsf{P}^x_{J_n} &= f^l(f^k \mathsf{P}^x_{J_n}), \\ &\subset f^l\left(\mathsf{P}^{f^k x}_{K_n}\right), \\ &\subset \mathsf{P}^{f^{k+l}x}_{L_n}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \mathrm{gr}_{J_n} f^{k+l}(x) &\le \sharp\, J_n \setminus L_n, \\ &\le \sharp\, J_n \setminus K_n + \sharp\, K_n \setminus L_n, \end{aligned}$$

then

$$\begin{aligned}
\mathrm{gr}_{\mathcal{J}} f^{k+l}(x) &= \limsup_n \frac{\mathrm{gr}_{J_n} f}{p(J_n)}, \\
&\leq \limsup_n \frac{\mathrm{gr}_{K_n} f}{p(J_n)} + \limsup_n \frac{\mathrm{gr}_{L_n} f}{p(J_n)}, \\
&\leq \limsup_n \frac{\mathrm{gr}_{K_n} f}{p(K_n)} + \limsup_n \frac{\mathrm{gr}_{L_n} f}{p(L_n)}, \\
&\leq \mathrm{gr}_{\mathcal{K}} f^k(x) + \mathrm{gr}_{\mathcal{L}} f^l(f^k x).
\end{aligned}$$

As the sequences $\mathcal{K}$ and $\mathcal{L}$ lie in $\mathcal{E}(O)$ we conclude that

$$\mathrm{gr}_O f^{k+l}(x) \leq \mathrm{gr}_O f^k(x) + \mathrm{gr}_O f^l(f^k x).$$

□

The nonnegative function $\mathrm{gr}_O f$ satisfies $\mathrm{gr}_O f \leq \sup_{\mathcal{J}\in\mathcal{E}(O)} \limsup_n \frac{\sharp\partial_I^- J_n}{p(J_n)}$ and this last term is finite according to Proposition 5. Therefore the subadditive ergodic theorem applies : for any $\mu \in \mathcal{M}(X, f)$ the sequence $\left(\frac{1}{n}\mathrm{gr}_O f^n(x)\right)_k$ converge almost everywhere to a f-invariant function χ_O with $\int \chi_O \, d\mu = \lim / \inf_n \frac{1}{n} \int \mathrm{gr}_O f^n \, d\mu$. We call the function χ_O **the Lyapunov exponent of f with respect to O**.

Remark 8. *The exponent χ_O for $O \in \mathcal{D}$ plays somehow the role of the sum of the positive Lyapunov exponents in smooth dynamical systems.*

5. Rescaled entropy of cellular automata

5.1. **Definition.** We let $\mathcal{M}(f)$ (resp. $\mathcal{M}(f,\sigma)$) be the set of invariant Borel probability measures on X which are f-invariant (resp. f- and σ-invariant). For a finite clopen partition P of X we let $H_{top}(\mathsf{P}) = \log \sharp\mathsf{P}$ and $H_\mu(\mathsf{P}) = -\sum_{A\in\mathsf{P}} \mu(A)\log\mu(A)$ with $\mu \in \mathcal{M}(f)$. In the following the symbol $*$ denotes either $* = top$ or $* = \mu \in \mathcal{M}(f)$. We let $h_*(f,\mathsf{P})$ be the entropy with respect to the clopen partition P :

$$h_*(f,\mathsf{P}) := \lim_n \frac{1}{n} H_* \left(\bigvee_{k=0}^{n-1} f^{-k}\mathsf{P} \right).$$

For two partitions P, Q of X, we say P is finer than Q and we write $\mathsf{P} > \mathsf{Q}$, when any atom of P is contained in an atom of Q. The functions $H_*(\cdot)$ and $h_*(f,\cdot)$ are nondecreasing with respect to this order.

The rescaled entropy with respect to a convex exhaustion $\mathcal{J} = (J_n)_n$ is defined as follows

$$h_*^d(f,\mathcal{J}) = \limsup_n \frac{h_*(f,\mathsf{P}_{J_n})}{p(J_n)}.$$

In [9] the authors defines a similar notion for the rescaled topological entropy with the renormalization factor $\sharp\partial_I^- J_n$ (which depends on the domain I of f) rather than $p(J_n)$.

Remark 9. *For $d = 2$, when $J = \bigcup_{i\in I} J_i$ is a finite disjoint union of Jordan domains J_i with Lipshitz boundary, we have*

$$\begin{aligned}
\frac{h_{top}(f,\mathsf{P}_J)}{p(J)} &\leq \frac{\sum_{i\in I} h_{top}(f,\mathsf{P}_{J_i})}{\sum_{i\in I} p(J_i)}, \\
&\leq \sup_{i\in I} \frac{h_{top}(f,\mathsf{P}_{J_i})}{p(J_i)}.
\end{aligned}$$

Moreover for each i, we have $p(J_i) \geq p(\mathrm{cv}(J_i))$ and $\mathsf{P}_{\mathrm{cv}(J_i)}$ is finer than P_{J_i}. Therefore

$$\begin{aligned}\frac{h_{top}(f, \mathsf{P}_J)}{p(J)} &\leq \frac{\sum_{i\in I} h_{top}(f, \mathsf{P}_{J_i})}{\sum_{i\in I} p(J_i)}, \\ &\leq \sup_{i\in I} \frac{h_{top}(f, \mathsf{P}_{\mathrm{cv}(J_i)})}{p(\mathrm{cv}(J_i))}.\end{aligned}$$

This inequality justifies that we focus on convex bodies J of $\mathbb{R}^d$.

We let also for any $O \in \mathcal{D}^1$

$$h_*^d(f, O) = \sup_{\mathcal{J}\in\mathcal{E}(O)} h_*^d(f, \mathcal{J})$$

and

$$h_*^d(f) = \sup_{\mathcal{J}} h_*^d(f, \mathcal{J}),$$

where the last supremum holds over all convex exhaustions $\mathcal{J}$. For $d = 1$ we have $p(J) = 2$ for any convex subset J. Therefore up to a factor 2 we recover the usual definition of entropy, $2h_*^1(f) = h_*(f)$.

Remark 10. *As the CA f commutes with the shift action σ we have for all $k \in \mathbb{Z}^d$ and any subset J of $\mathbb{Z}^d$ $h_{top}(f, \mathsf{P}_{J+k}) = h_{top}(f, \sigma^{-k}\mathsf{P}_J) = h_{top}(f, \mathsf{P}_J)$ and the same holds for the measure theoretical entropy with respect to measures in $\mathcal{M}(f, \sigma)$. Let us call generalized convex domain any convex body with a non empty interior set. Replacing convex domains by generalized convex domains, we may define generalized convex exhaustions $\mathcal{J}$ and the associated rescaled entropies. Then it follows from the aforementioned invariance by translation of the entropy, that $h_{top}^d(O) = h_{top}^d(O+\alpha)$ for all $\alpha \in \mathbb{R}^d$ and all generalized convex domain O with unit perimeter. Indeed for any $(J_n)_n \in \mathcal{E}(O)$ (resp. $\mathcal{E}(\mathcal{O}+\alpha)$) there is a sequence of integers $(k_n)_n$ with $(J_n + k_n)_n \in \mathcal{E}(\mathcal{O}+\alpha)$ (resp. $(J_n)_n \in \mathcal{E}(O)$).*

In a seminal work [14], Milnor investigated the d-dimensional topological entropy of a compact set O in $\mathbb{R} \times \mathbb{R}^d$ with respect to the $\mathbb{N} \times \mathbb{Z}^d$-action generated by a CA f and the $\mathbb{Z}^d$-shift σ. When $O = \{0\} \times O'$ for some $O' \in \mathcal{D}$, this d-dimensional entropy $\eta_d(O)$ may be written as follows :

$$\eta_d(O) = \sup_{m\in\mathbb{N}} \left(\limsup_n \frac{1}{n^d} H_{top} \left(\bigvee_{k=0}^{m-1} f^{-k}\mathsf{P}_{nO'} \right) \right),$$

whereas another renormalization is used here in the definition of the rescaled entropy with respect to O' :

$$h_{top}^d(f, \mathcal{J}_{O'}) = \limsup_n \left(\lim_m \frac{1}{mn^{d-1}} H_{top} \left(\bigvee_{k=0}^{m-1} f^{-k}\mathsf{P}_{nO'} \right) \right).$$

These quantities have different behaviour, e.g. $\eta_d(O)$ is proportional to the d-Lebesgue measure $V(O')$ of O' (Theorem 2 in [14]), but we will see in the proof of Theorem 1 in Section 7 that when the smallest bounding sphere of the domain I of the algebraic CA f is degenerated then $0 < h_{top}^d(f) = \lim_{R\to+\infty} h_{top}^d(f, \mathcal{J}_{\widetilde{T'_R}})$, but $V(\widetilde{T'_R}) \xrightarrow{R\to+\infty} 0$ (with $T'_R \in \mathcal{D}$ as defined in Subsection 3.5).

5.2. **Link with the metric mean dimension in dimension two.** In a compact metric space (X, d), the ball of radius $\epsilon \geq 0$ centered at $x \in X$ will be denoted by $B_{\mathsf{d}}(x, \epsilon)$. For a continuous map $f : X \to X$ we denote by d_n the dynamical distance defined for all $n \in \mathbb{N}$ by

$$\forall x, y \in X,\ \mathsf{d}_n(x, y) = \max\{\mathsf{d}(f^k x, f^k y),\ 0 \leq k < n\}.$$

The metric mean dimension of f is defined as $\mathrm{mdim}(f,\mathsf{d}) = \limsup_{\epsilon\to 0}\frac{h_{top}(f,\epsilon)}{|\log\epsilon|}$ where $h_{top}(f,\epsilon)$ denotes the topological entropy at the scale $\epsilon>0$:

$$h_{top}(f,\epsilon) := \limsup_n \frac{1}{n}\log\min\{\sharp C,\ \bigcup_{x\in C} B_{\mathsf{d}_n}(x,\epsilon)=X\}.$$

The topologial mean dimension is conjectured to be the infimum of $\mathrm{mdim}(f,\mathsf{d})$ over all distances on X (this is known for systems with the marker property). We refer to [11] for alternative definitions and further properties of mean dimension. The topological mean dimension of a finite dimensional topological system is null. Here f is a CA on a $\mathbb{Z}^d$-subshift X. In particular it has zero topological mean dimension.

Fix $\alpha>1$. To any exhaustion $\mathcal{J}=(J_n)_n$ of $\mathbb{R}^d$, we may associate an ultrametric distance $\mathsf{d}_{\mathcal{J}}$ on X_d as follows :

$$\forall x=(x_k)_{k\in\mathbb{Z}^d} \text{ and } y=(y_k)_{k\in\mathbb{Z}^d},\quad \mathsf{d}_{\mathcal{J}}(x,y)=\alpha^{-\max\{n\in\mathbb{N},\ x_k=y_k\ \forall k\in J_n\}}.$$

Then for $n\in\mathbb{N}$ the ball $B_{\mathsf{d}_{\mathcal{J}}}(x,\alpha^{-n})$ with respect to $\mathsf{d}_{\mathcal{J}}$ coincides with the cylinder $\mathsf{P}^x_{J_n}$. Therefore we have for any $O\in\mathcal{D}$:

$$\begin{aligned} h^d_{top}(f,\mathcal{J}_O) &= \limsup_n \frac{h_{top}(f,\mathsf{P}_{nO})}{p(nO)},\\ &= \limsup_n \frac{h_{top}(f,\alpha^{-n})}{n^{d-1}p(O)},\\ &= \frac{(\log\alpha)^{d-1}}{p(O)}\limsup_{\epsilon\to 0}\frac{h_{top}(f,\epsilon)}{|\log\epsilon|^{d-1}} \end{aligned}$$

In particular in dimension two we get :

$$h^2_{top}(f,\mathcal{J}_O) = \frac{\log\alpha}{p(O)}\,\mathrm{mdim}(f,\mathsf{d}_{\mathcal{J}_O}).$$

For $d>2$ the mean dimension $\mathrm{mdim}(f,\mathsf{d}_{\mathcal{J}_O})$ is infinite whenever the rescaled entropy $h^d_{top}(f,\mathcal{J}_O)$ is positive. In [19] the authors compute explicitly the mean dimension of the particular CA given by the horizontal shift on a $\mathbb{Z}^2$-subshift with respect to some metrics of the form $d_{\mathcal{J}_O}$ with O being the unit ball of standard norms on $\mathbb{R}^d$.

Remark 11. *In* [19] *the authors also work with a measure theoretical quantity, called the measure distorsion rate dimension and show a variational principle with the metric mean dimension of* $\mathsf{d}_{\mathcal{J}_O}$. *Does this quantity coincides with* $\mu\mapsto h^2_\mu(f,\mathcal{J}_O)$ *?*

5.3. **Monotonicity and Power.** We investigate now basic properties of the rescaled entropy.

Lemma 5. *For any* $O\in\mathcal{D}$ *and any* $\alpha>0$*, we have*

$$h^d_*(f,\mathcal{J}_O) = h^d_*(f,\mathcal{J}_{\alpha O}).$$

Proof. For $n\in\mathbb{N}$, we let $k_n=\lceil\frac{n}{\alpha}\rceil$, thus $nO\subset k_n\alpha O$ and $p(nO)\sim^n p(k_n\alpha O)$. Therefore

$$\begin{aligned} h^d_*(f,\mathcal{J}_O) &= \limsup_n \frac{h_*(f,\mathsf{P}_{nO})}{p(nO)},\\ &\leq \limsup_n \frac{h_*(f,\mathsf{P}_{k_n\alpha O})}{p(nO)},\\ &\leq \limsup_n \frac{h_*(f,\mathsf{P}_{k_n\alpha O})}{p(k_n\alpha O)},\\ &\leq h^d_*(f,\mathcal{J}_{\alpha O}). \end{aligned}$$

The other inequality is obtained by considering αO and α^{-1} in place of O and α. □

Lemma 6. *For any $O \in \mathcal{D}^1$ and $O' \in \mathcal{D}$ with $O \subset \mathrm{Int}(O')$, we have*

$$h_*^d(f, \mathcal{J}_O) \leq h_*^d(f, O) \leq p(O') h_*^d(f, \mathcal{J}_{O'}).$$

Proof. As $\mathcal{J}_O \in \mathcal{E}(O)$ the inequality $h_*^d(f, \mathcal{J}_O) \leq h_*^d(f, O)$ follows from the definitions. Let now $\mathcal{J} \in \mathcal{E}(O)$. For n large enough we have $\widetilde{J_n} \subset \mathrm{Int}(O')$, therefore $J_n \subset p(J_n)^{\frac{1}{d-1}} O'$. We conlude that

$$\begin{aligned} h_*^d(f, \mathcal{J}) &\leq \limsup_n \frac{p\left(p(J_n)^{\frac{1}{d-1}} O'\right)}{p(J_n)} h_*^d(f, \mathcal{J}_{O'}), \\ &\leq p(O') h_*^d(f, \mathcal{J}_{O'}). \end{aligned}$$

□

For $O \in \mathcal{D}^1$ the origin belongs to $\mathrm{Int}(O)$ so that $\alpha O \in \mathcal{D}$ and $O \subset \mathrm{Int}(\alpha O)$ for any $\alpha > 1$. Moreover we have $h_*^d(f, \mathcal{J}_{\alpha O}) = h_*^d(f, \mathcal{J}_O)$ by Lemma 5. Together with Lemma 6 we get immediately :

Corollary 12.

$$\forall O \in \mathcal{D}^1, \ h_*^d(f, O) = h_*^d(f, \mathcal{J}_O).$$

Corollary 13.

$$O \mapsto h_*^d(f, O) \textit{ is continuous on } \mathcal{D}^1.$$

Convex d-polytopes are dense in $\mathcal{D}$. Therefore we get with $\mathcal{P}$ being the collection of convex d-polytopes with the origin in their interior set :

Corollary 14.

$$\sup_{O \in \mathcal{D}^1} h_*^d(f, O) = \sup_{P \in \mathcal{P}} h_*^d(f, \mathcal{J}_P).$$

However we will see that the supremum is not always achieved. We prove now a formula for the rescaled entropy of a power.

Lemma 7.

$$\forall O \in \mathcal{D}^1 \ \forall k \in \mathbb{N}, \ h_*^d(f^k, O) = k h_*^d(f, O).$$

Proof. Let $O \in \mathcal{D}^1$ and $\mathcal{J} = (J_n)_n \in \mathcal{E}(O)$. Let $J_n^k = J_n \oplus \underbrace{I \oplus \cdots \oplus I}_{k \text{ times}}$ for all n. The sequence $\mathcal{J}^k = (J_n^k)_n$ belongs also to $\mathcal{E}(O)$. Moreover the partition $\mathsf{P}_{J_n^k}$ is finer than $\bigvee_{l=0}^{k-1} f^{-l} \mathsf{P}_{J_n}$. Therefore

$$h_*(f^k, \mathsf{P}_{J_n}) \leq k h_*(f, \mathsf{P}_{J_n}) = h_*\left(f^k, \bigvee_{l=0}^{k-1} f^{-l} \mathsf{P}_{J_n}\right) \leq h_*(f^k, \mathsf{P}_{J_n^k})$$

and we then obtain

$$h_*^d(f^k, \mathcal{J}) \leq k h_*^d(f, \mathcal{J}) \leq h_*^d(f^k, \mathcal{J}^k).$$

We conclude by taking the supremum in $\mathcal{J} \in \mathcal{E}(O)$. □

Remark 15. *Clearly we have $h_\mu^d(f) \leq h_{top}^d(f)$ for any $\mu \in \mathcal{M}(f)$ but we ignore if a general variational principle holds true.*

5.4. **A first upperbound for the rescaled entropy.** Let (X, f) be a cellular automaton with domain I. We relate the entropy of P_J with the entropy of $\mathsf{P}_{\partial^\pm J}$ and we prove an upperbound for the rescaled entropy $h^d_{top}(f, O)$ in term of the first $\mathbb{I}$-relative quermass integral of O with $\mathbb{I}$ being the convex hull of I'.

Lemma 8. *For any bounded subset J of $\mathbb{R}^d$, we have*

$$h_*(f, \mathsf{P}_J) = h_*(f, \mathsf{P}_{\partial^-_I J}) \text{ and } h_*(f, \mathsf{P}_J) \leq h_*(f, \mathsf{P}_{\partial^+_I J}).$$

Proof. The inequality $h_*(f, \mathsf{P}_J) \geq h_*(f, \mathsf{P}_{\partial^- J})$ follows directly from the inclusion $\partial^- J \subset J$. By definition of the domain I and the erosion $J \ominus I$, we have $P_J > f^{-1} P_{J \ominus I}$. Therefore we get $f^{-1}\mathsf{P}_J \vee \mathsf{P}_J = f^{-1}\mathsf{P}_{\partial^- J} \vee P_J$ and then by induction $\mathsf{P}_J \vee \bigvee_{l=0}^{k-1} f^{-l}\mathsf{P}_{\partial^- J} = \bigvee_{l=0}^{k-1} f^{-l}\mathsf{P}_J$ for all k. We conclude that :

$$\begin{aligned} h_*(f, \mathsf{P}_J) &= \lim_k \frac{1}{k} H_*(f, \bigvee_{l=0}^{k-1} f^{-l}\mathsf{P}_J), \\ &\leq \lim_k \frac{1}{k} \left(H_*\left(\mathsf{P}_J\right) + H_*\left(\bigvee_{l=0}^{k-1} f^{-l}\mathsf{P}_{\partial^- J} \right) \right), \\ &\leq h_*(f, \mathsf{P}_{\partial^- J}). \end{aligned}$$

We also have

$$\mathsf{P}_J \vee \mathsf{P}_{\partial^+ J} > \mathsf{P}_{J \oplus I} > f^{-1}\mathsf{P}_J.$$

Therefore we get now by induction on k

$$\mathsf{P}_J \vee \bigvee_{l=0}^{k-2} f^{-l}\mathsf{P}_{\partial^+ J} > \bigvee_{l=0}^{k-1} f^{-l}\mathsf{P}_J.$$

This implies $h_*(f, \mathsf{P}_{\partial^+_I J}) \leq h_*(f, \mathsf{P}_J)$.

□

Proposition 16. *For any $O \in \mathcal{D}^1$,*

$$h^d_{top}(f, O) \leq V_{\mathbb{I}}(O) \log |\mathcal{A}|.$$

Proof. Recall that

$$\begin{aligned} h^d_{top}(f, O) &= h^d_{top}(f, \mathcal{J}_O), \\ &= \limsup_n \frac{h_{top}(f, \mathsf{P}_{nO})}{p(nO)}. \end{aligned}$$

Then by applying Lemma 8 we obtain

$$\begin{aligned} h^d_{top}(f, O) &\leq \limsup_n \frac{h_{top}(f, \mathsf{P}_{\partial^\pm nO})}{p(nO)}, \\ &\leq \limsup_n \frac{\sharp \partial^\pm nO \log |\mathcal{A}|}{p(nO)}. \end{aligned}$$

For all $k \in \mathbb{N} \setminus \{0\}$ we let I_k be the domain of f^k and we denote by $\mathbb{I}_k$ the convex hull of $I'_k = I_k \cup \{0\}$. Clearly we have $I_k \subset \underbrace{I \oplus \cdots \oplus I}_{k \text{ times}}$, therefore $\mathbb{I}_k \subset k\mathbb{I}$. By Lemma 2, we get for some constant $c = c(d)$:

$$\begin{aligned} h^d_{top}(f^k, O) &\leq (V_{\mathbb{I}_k}(O) + c) \log |\mathcal{A}|, \\ &\leq (V_{k\mathbb{I}}(O) + c) \log |\mathcal{A}|, \\ &\leq (kV_{\mathbb{I}}(O) + c) \log |\mathcal{A}|. \end{aligned}$$

But by Lemma 11 we have $h^d_{top}(f^k, O) = kh^d_{top}(f, O)$, so that we finally conclude when k goes to infinity

$$h^d_{top}(f, O) \leq V_{\mathbb{I}}(O) \log |\mathcal{A}|.$$

□

6. Ruelle inequality

Recall (X, σ) denotes a $\mathbb{Z}^d$-subshift. The topological entropy of σ is defined for any Fölner sequence $\mathcal{L} = (L_n)_n$ (see e.g. [22]) as

$$h_{top}(\sigma) = \limsup_n \frac{H_{top}(\mathsf{P}_{L_n})}{|L_n|}.$$

Lemma 9. *For all $\epsilon > 0$ there exists $c > 0$ such that we have for any $K \subset J$ convex bodies:*

$$H_{top}(\mathsf{P}_{J \setminus K}) \leq (\sharp\, J \setminus K + cp(J \oplus \mathsf{C})) \cdot (h_{top}(\sigma) + \epsilon).$$

Proof. Let $\epsilon > 0$. As the sequence of cubes $\mathcal{C} = (C_n)_n$ defined by $C_n = [-n, n[^d \cap \mathbb{Z}^d$ is a Fölner sequence, there is a positive integer m such that $\frac{H_{top}(\mathsf{P}_{C_m})}{|C_m|} < h_{top}(\sigma) + \epsilon$. Then for some $c = c(m) > 0$ we may cover $\mathbb{Z}^d \cap (J \setminus K)$ by a family $\mathcal{F}$ at most $\frac{\sharp J \setminus K + cp(J \oplus \mathsf{C})}{|C_m|}$ disjoint translated copies of C_m. Indeed if R_m denotes a partition of $\mathbb{R}^d$ into translated copies of C_m, then any atom A of R_m with $\mathbb{Z}^d \cap A \cap (J \setminus K) \neq \emptyset$ either satisfies $\mathbb{Z}^d \cap A \subset J \setminus K$ or $\mathbb{Z}^d \cap A \cap \left(\partial^-_{C_m} J \cup \partial^-_{C_m} K\right) \neq \emptyset$. Clearly the number of A's in the first case is less than $\frac{\sharp J \setminus K}{|C_m|}$, whereas the numbers of atoms A satisfying the second condition is less than $\sharp \partial^-_{C_m} J + \sharp \partial^-_{C_m} K$. Arguing as in the proof of Proposition 5, this last term is less than $c\,(p(J \oplus \mathsf{C}) + p(K \oplus \mathsf{C}))$ for some constant c depending on m. As K is contained in J we have $p(J \oplus \mathsf{C}) \leq p(K \oplus \mathsf{C})$.

Therefore

$$\begin{aligned} H_{top}(\mathsf{P}_{J \setminus K}) &\leq (\sharp\, J \setminus K + 2cp(J \oplus \mathsf{C})) \frac{H_{top}\left(\mathsf{P}_{C_m}\right)}{|C_m|}, \\ &\leq (\sharp\, J \setminus K + 2cp(J \oplus \mathsf{C})) \cdot (h_{top}(\sigma) + \epsilon). \end{aligned}$$

□

We refine now the inequality obtained in Proposition 16 at the level of invariant measures. We recall that χ_O denotes the Lyapunov exponent of f with respect to O as defined at the end of Section 4.

Lemma 10.

$$\forall \mu \in \mathcal{M}(f),\ h_\mu(f, O) \leq h_{top}(\sigma) \int \chi_O \, d\mu.$$

Proof. For any convex domain J and any $\mu \in \mathcal{M}(f)$ we have

$$\begin{aligned} h_\mu(f, \mathsf{P}_J) &\leq H_\mu(f^{-1}\mathsf{P}_J | \mathsf{P}_J), \\ &\leq \sum_{A \in \mathsf{P}_J} \mu(A) H_{\mu_A}(f^{-1}\mathsf{P}_J). \end{aligned}$$

Fix $\epsilon > 0$ and let c be as in Lemma 9. Then if $(K_A)_{A\in \mathsf{P}_J}$ is a family of convex bodies in $\prod_{A\in\mathsf{P}_J}\mathcal{E}_f(A,J)$ with $K_A\subset J$ for all A we obtain

$$\begin{aligned} h_\mu(f,\mathsf{P}_J) &\leq \sum_{A\in\mathsf{P}_J}\mu(A)H_{\mu_A}(f^{-1}\mathsf{P}_{J\setminus K_A}),\\ &\leq \sum_{A\in\mathsf{P}_J}\mu(A)H_{top}(\mathsf{P}_{J\setminus K_A}),\\ &\leq \sum_{A\in\mathsf{P}_J}\mu(A)\left(\sharp\, J\setminus K_A + cp(J\oplus\mathsf{C})\right)\cdot(h_{top}(\sigma)+\epsilon). \end{aligned}$$

By choosing K_A with $\sharp\, J\setminus K_A$ minimal we obtain

$$h_\mu(f,\mathsf{P}_J)\leq (h_{top}(\sigma)+\epsilon)\cdot\left(\int \mathrm{gr}_J f\, d\mu + cp(J\oplus\mathsf{C})\right).$$

Therefore we have for any convex exhaustion $\mathcal{J}=(J_n)_n$ (recall that $p(J_n\oplus\mathsf{C})\sim^n p(J_n)$) :

$$\begin{aligned} h^d_\mu(f,\mathcal{J}) &= \limsup_n \frac{h_\mu(f,\mathsf{P}_J)}{p(J_n)},\\ &\leq (h_{top}(\sigma)+\epsilon)\cdot\left(\limsup_n \int \frac{\mathrm{gr}_{J_n} f}{p(J_n)}\, d\mu + c\right). \end{aligned}$$

By Proposition 5 we have for all $x\in X$

$$\sup_{n\in\mathbb{N}}\frac{\mathrm{gr}_{J_n}f(x)}{p(J_n)}\leq \sup_{n\in\mathbb{N}}\frac{\sharp\partial^- J_n}{p(J_n)} < +\infty.$$

We may therefore apply Fatou's Lemma to the sequence of functions $\left(-\frac{\mathrm{gr}_{J_n}f}{p(J_n)}\right)_n$:

$$\limsup_n \int\frac{\mathrm{gr}_{J_n}f}{p(J_n)}\, d\mu \leq \int \limsup_n \frac{\mathrm{gr}_{J_n}f}{p(J_n)}\, d\mu,$$

then

$$h^d_\mu(f,\mathcal{J})\leq (h_{top}(\sigma)+\epsilon)\left(\int \mathrm{gr}_{\mathcal{J}} f\, d\mu + c\right).$$

By taking the supremum over $\mathcal{J}\in\mathcal{E}(O)$ we get

$$h^d_\mu(f,O)\leq (h_{top}(\sigma)+\epsilon)\left(\int \mathrm{gr}_O f\, d\mu + c\right).$$

By Lemma 7 we have $\frac{h^d_\mu(f^k,O)}{k} = h^d_\mu(f,O)$ for any k. Apply the above inequality to f^k :

$$h^d_\mu(f,O)\leq (h_{top}(\sigma)+\epsilon)\left(\int \frac{\mathrm{gr}_O f^k}{k}\, d\mu + \frac{c}{k}\right).$$

When k goes to infinity and then ϵ goes to zero, we conclude $h^d_\mu(f,O)\leq h_{top}(\sigma)\int\chi_O\, d\mu$. □

7. Entropy formula for permutative CA

The cellular automaton f is said **permutative** at $i\in\mathbb{Z}^d$ if for all pattern P on $I\setminus\{i\}$ and for all $a\in\mathcal{A}$ there is $b\in\mathcal{A}$ such that the pattern P^i_b on $I\cup\{i\}$ given by the completion of P at i by b satisfies $F(P^i_b)=a$, in particular i belongs to the domain I of f. The CA is said permutative when it is permutative at the nonzero extreme points of the convex hull $\mathbb{I}$ of $I'=I\cup\{0\}$ (these points lie in I). The algebraic CA as described in the introduction are permutative.

Proposition 17. *The topological rescaled entropy of a permutative CA f on X_d is given by*

$$h^d_{top}(f) = R_{I'} \log |\mathcal{A}|.$$

The sets I' and $\mathbb{I}$ have the same smallest bounding sphere, thus $R_{I'} = R_{\mathbb{I}}$. Theorem 1, stated in the introduction, follows from Proposition 17.

Question. For a permutative CA, the uniform measure $\lambda^{\mathbb{Z}^d}$ with λ being the uniform measure on $\mathcal{A}$ is known to be invariant [23]. Does the uniform measure maximize the rescaled entropy?

Recall that for any $k \in \mathbb{N} \setminus \{0\}$ we denote by I_k the domain of f^k and $\mathbb{I}_k$ the convex hull of $I'_k = I_k \cup \{0\}$. In the following we also let $C(P, L) = \{(x_i)_{i\in\mathbb{Z}^d} \in X,\ x_j = p_j \, \forall j \in L\}$ be the cylinder associated to the pattern $P = (p_j)_{j\in L} \in \mathcal{A}^L$ on $L \subset \mathbb{Z}^d$. We also write $C(P)$ for this cylinder when there is no confusion on L.

Lemma 11. *For any permutative CA f and any $k \in \mathbb{N} \setminus \{0\}$, the CA f^k is also permutative and*

$$\mathbb{I}_k = k\mathbb{I}.$$

Proof. As already observed, the inclusion $\mathbb{I}_k \subset k\mathbb{I}$ holds for any CA (not necessarily permutative). We will show $k\,\mathrm{ex}(\mathbb{I}) \subset I'_k$, which implies together with $\mathbb{I}_k \subset k\mathbb{I}$ the equality $\mathbb{I}_k = k\mathbb{I}$. Let $i \in \mathrm{ex}(\mathbb{I}) \setminus \{0\} \subset I$. For a fixed k we prove by induction on k that f^k is permutative at ki, in particular $ki \in I'_k$. Let P be a pattern on $I_k \setminus \{ki\}$ and let $a \in \mathcal{A}$. Since we have $I_k \subset I_{k-1} \oplus I$, we may complete P by a pattern Q on $(I_{k-1} \oplus I) \setminus \{ki\}$. By induction hypothesis, $(k-1)i$ lies in $\mathrm{ex}(\mathbb{I}_{k-1})$ and i lies in $\mathrm{ex}(\mathbb{I})$, therefore ki does not belong to $I_{k-1} \oplus (I \setminus \{i\})$, so that we have $I_{k-1} \oplus (I \setminus \{i\}) \subset (I_{k-1} \oplus I) \setminus \{ki\}$. Therefore there is a pattern R on $I \setminus \{i\}$ such that $f^{k-1}C\left(Q, (I_{k-1} \oplus I) \setminus \{ki\}\right)$ is contained in the cylinder $C(R, I \setminus \{i\})$. As f is permutative at i there is $b \in \mathcal{A}$ with $F(R^i_b) = a$ or in other terms $f\left(C(R^i_b, I)\right) \subset C\left(a, \{0\}\right)$. Since f^{k-1} is permutative at $(k-1)i$, we may find $c \in \mathcal{A}$ with $f^{k-1}\left(C(Q^{ki}_c, I_{k-1} \oplus I)\right) \subset C\left(b, \{i\}\right)$. Therefore we get

$$f^k\left(C(Q^{ki}_c, I_{k-1} \oplus I)\right) \subset f\left(C(R^i_b, I)\right) \subset C\left(a, \{0\}\right).$$

But I_k is the domain of f^k and P is the restriction of Q to $I_k \setminus \{ki\}$, so that we also have $f^k\left(C(P^{ki}_c, I_k)\right) \subset C\left(a, \{0\}\right)$, i.e. f^k is permutative at ki. □

For a convex d-polytope J and a face F of J we consider the subset of $\partial^-_{\mathbb{I}} J$ given by $\partial^-_{\mathbb{I}} F := \partial^-_{\mathbb{I}} J \cap T^+_F J(-h_{\mathbb{I}}(N^F))$. The sets $\partial^-_{\mathbb{I}} F$ for $F \in \mathcal{F}(J)$ are covering $\partial^-_{\mathbb{I}} J$ but do not define a partition in general. For any $F \in \mathcal{F}(J)$ we let $u^F \in \mathrm{ex}(\mathbb{I}) \subset I'$ with $u^F \cdot N^F = h_{\mathbb{I}}(N^F)$ and we also let d_F be the the Euclidean distance to T_F. Then for $j \in \mathbb{Z}^d \cap \partial^-_{\mathbb{I}} J$ we let F_j be a face of J such that $d_{F_j}(j+u^{F_j}) = -d_{F_j}(j) + u^{F_j} \cdot N^{F_j}$ is maximal among faces F with $j \in \partial^-_{\mathbb{I}} F$. We consider then a total order $\prec$ on $\mathbb{Z}^d \cap \partial^-_{\mathbb{I}} J$ such that $i \prec j$ if $d_{F_i}(i + u^{F_i}) < d_{F_j}(j + u^{F_j})$. We also let $\mathcal{F}_{\mathbb{I}}(J)$ be the subset of $\mathcal{F}(J)$ given by faces F for which u_F is uniquely defined. We denote by $\partial^{\perp}_{\mathbb{I}} J$ the subset of $\partial^-_{\mathbb{I}} J$ given by

$$\partial^{\perp}_{\mathbb{I}} J := \bigcup_{F \in \mathcal{F}_{\mathbb{I}}(J)} \partial^-_{\mathbb{I}} F.$$

Lemma 12. *With the above notations, let $j \in \mathbb{Z}^d \cap \partial^{\perp}_{\mathbb{I}} J$. Then*

$$\forall k \in \mathbb{N},\ j + ku^{F_j} \notin \{j',\, j' \prec j\} \oplus k\mathbb{I}.$$

Proof. We argue by contradiction : there are $j' \prec j$ and $u \in \mathbb{I}$ with $j + ku^{F_j} = j' + ku$. Observe that

$$d_{F_j}(j + ku^{F_j}) = d_{F_j}(j + u^{F_j}) + (k-1)u^{F_j} \cdot N^{F_j},$$
$$d_{F_j}(j' + ku) = d_{F_j}(j' + u) + (k-1)u \cdot N^{F_j}.$$

We will show that the equality between these two distances implies $u = u^{F_j}$, therefore $j = j'$. Indeed we have

$$\begin{aligned} d_{F_j}(j'+u) &\leq \sup_{v\in \mathrm{ex}(\mathbb{I})} d_{F_j}(j'+v), & u\cdot N^{F_j} &\leq \sup_{v\in \mathrm{ex}(\mathbb{I})} v\cdot N^{F_j}, \\ &\leq d_{F_{j'}}(j'+u^{F_{j'}}), & &\leq h_{\mathbb{I}}(N^{F_j}), \\ d_{F_j}(j'+u) &\leq d_{F_j}(j+u^{F_j}) & u\cdot N^{F_j} &\leq u^{F_j}\cdot N^{F_j}, \end{aligned}$$

therefore $u \cdot N^{F_j} = u^{F_j} \cdot N^{F_j}$, and finally $u = u^{F_j}$ as j belongs to $\mathbb{Z}^d \cap \partial_{\mathbb{I}}^{\perp} J$. □

For a partition P of X and a positive integer k, we write P^k to denote the iterated partition $\bigvee_{l=0}^{k-1} f^{-l}\mathsf{P}$ in order to simplify the notations.

Lemma 13. *Let J be a convex d-polytope and let k, n be positive integers. For any $A^k \in \mathsf{P}_J^k$ and any pattern P on $\mathbb{Z}^d \cap \partial_{\mathbb{I}}^{\perp} J$, there is $w \in A^k$ such that $f^k w$ belongs to $C(P, \mathbb{Z}^d \cap \partial_{\mathbb{I}}^{\perp} J)$.*

Proof. For any $j \in \partial_{\mathbb{I}}^{\perp} J$ we let P_j be the restriction of $P = (p_l)_{l\in\partial^{\perp} J}$ to $\{j',\, j' \prec j\}$. We show now by induction on $j \in \mathbb{Z}^d \cap \partial^{\perp} J$ that there is $w \in A^k$ with $f^k w \in C(P_j)$. By Lemma 11 the CA f^k is permutative at ku^{F_j} so that we may change the $(j + ku^{F_j})^{\text{th}}$-coordinate of w to get $w' \in X$ with $(f^k w')_j = p_j$. Moreover the j'-coordinates of $f^k w$ for $j' \prec j$ only depends on the coordinates of w on $\{j',\, j' \prec j\} \oplus k\mathbb{I}$ so that by Lemma 12 we still have $f^k w' \in C(P_j, \{j',\, j' \prec j\})$, thus $f^k w' \in C(P_{j''})$ with j'' being the successor of j for $\prec$ in $\mathbb{Z}^d \cap \partial^{\perp} J$. □

Lemma 14. *Let T' and T'_R, $R > 0$ be the polytopes associated to $\mathbb{I}$ as defined in Subsection 3.5. We have*

$$\mathcal{F}(T') = \mathcal{F}_{\mathbb{I}}(T')$$

and

$$\forall R > 0,\ \mathcal{F}_1(T'_R) \subset \mathcal{F}_{\mathbb{I}}(T'_R).$$

Proof. Let $F \in \mathcal{F}(T')$ or $F \in \mathcal{F}_1(T'_R)$. Such a face F is tangent to $S_{I'}$ at some $u \in \mathrm{ex}(\mathbb{I})$ with $u \cdot N^F = h_{\mathbb{I}}(N^F)$. Then any v with $v \cdot N^F = h_{\mathbb{I}}(N^F)$ belongs to T_F. But $T_F \cap \mathbb{I} \subset T_F \cap S_{I'} = \{u\}$, therefore we have necessarily $u_F = u$.

□

We are now in a position to prove Proposition 17.

Proof of Proposition 17. The inequality $h^d_{top}(f) \leq R_{I'} \log |\mathcal{A}|$ follows immediately from Proposition 16 and Proposition 7. By Lemma 13 we have for any convex d-polytope O and any positive integer n

$$\forall A^k \in \mathsf{P}^k_{nO},\ \sharp\{A^{k+1} \in \mathsf{P}^{k+1}_{nO},\ A^{k+1} \subset A^k\} \geq \sharp\partial^{\perp} nO.$$

Consequently we have

$$\begin{aligned} h_{top}(f, \mathsf{P}_{nO}) &\geq \sharp\partial^{\perp} nO \log |\mathcal{A}|, \\ h^d_{top}(f, \mathcal{J}_O) &\geq \limsup_n \frac{\sharp\partial^{\perp} nO}{n^{d-1} p(O)} \log |\mathcal{A}|. \end{aligned}$$

We first assume that $S_{\mathbb{I}} = S_{I'}$ is nondegenerated. Let T' be the dual polytope of a generating polytope T. Note that T' is a convex body with nonempty interior containing 0 (but the origin does not lie necessarily in its interior set). By Lemma 14 we have $\mathcal{F}(T') =$

$\mathcal{F}_{\mathbb{I}}(T')$, therefore $\mathcal{F}(nT') = \mathcal{F}_{\mathbb{I}}(nT')$ and $\partial^{\perp} nT' = \partial^{-} nT'$ for all n. Applying then Lemma 2 we get for some constant $c = c(d)$:

$$\begin{aligned} h_{top}^{d}(f, \mathcal{J}_{T'}) &\geq \limsup_{n} \frac{\sharp \partial^{-} nT'}{n^{d-1} p(T')} \log |\mathcal{A}|, \\ &\geq \frac{V_{\mathbb{I}}(T')}{p(T')} \log |\mathcal{A}| - c. \end{aligned}$$

Then it follows from Proposition 7 that :

$$h_{top}^{d}(f, \mathcal{J}_{T'}) \geq R_{\mathbb{I}} \log |\mathcal{A}| - c.$$

For any positive integer k the above equality also holds for f^{k} and $\mathbb{I}_{k}$ in place of f and $\mathbb{I}$. Moreover we have $\mathbb{I}_{k} = k\mathbb{I}$ according to Lemma 11, so that we get together with the power formula of Lemma 7 and $\widetilde{T'} = p(T')^{-\frac{1}{d-1}} T' \in \mathcal{D}^{1}$:

$$\begin{aligned} h_{top}^{d}(f, \widetilde{T'}) &= \frac{h_{top}^{d}(f^{k}, \widetilde{T'})}{k}, \\ &\geq \frac{R_{\mathbb{I}_{k}}}{k} \log |\mathcal{A}| - \frac{c}{k}, \\ &\geq \frac{R_{k\mathbb{I}}}{k} \log |\mathcal{A}| - \frac{c}{k}, \\ &\geq R_{\mathbb{I}} \log |\mathcal{A}| - \frac{c}{k}, \\ h_{top}^{d}(f, T') &\geq R_{I'} \log |\mathcal{A}|. \end{aligned}$$

This conclude the proof in the nondegenerated case.

We deal now with the degenerated case. By Lemma 14 we have for all $R > 0$ with the notations of Subsection 3.5 :

$$h_{top}^{d}(f, \mathcal{J}_{T'_{R}}) \geq \limsup_{n} \frac{\sharp \partial^{-} nT'_{R} - \sum_{F \in \mathcal{F}_{2}(T'_{R})} \sharp \partial^{-} nF}{p(nT'_{R})} \log |\mathcal{A}|.$$

But for $F \in \mathcal{F}_{2}(T'_{R})$ we have

$$\begin{aligned} \sharp \partial^{-} nF &\leq V(\partial^{-} nF \oplus \mathsf{C}), \\ &= n^{d-1} \operatorname{diam}(\mathbb{I}) O(R^{l-1}) \end{aligned}$$

Since $\lim_{R \to \infty} \frac{p(T'_{R})}{R^{l}} = \mathcal{H}_{d-l}(L') > 0$ and $|\mathcal{F}_{2}(T'_{R})| = 2l$, we get

$$\limsup_{n} \frac{\sum_{F \in \mathcal{F}_{2}(T'_{R})} \sharp \partial^{-} nF}{p(nT'_{R})} = \operatorname{diam}(\mathbb{I}) O(R^{-1}).$$

Together with Proposition 2 we get for some constant $c = c(d)$:

$$h_{top}^{d}(f, \mathcal{J}_{T'_{R}}) \geq \big(V_{\mathbb{I}}(T'_{R}) - c - \operatorname{diam}(\mathbb{I}) O(R^{-1})\big) \log |\mathcal{A}|.$$

We conclude as in the degenerated case by using the power rule. Fix $\epsilon > 0$ and let $k > c\epsilon^{-1}$. We obtain finally

$$\begin{aligned} h^d_{top}(f, \widetilde{T'_R}) &= \frac{h^d_{top}(f^k, \widetilde{T'_R})}{k}, \\ &\geq \left(\frac{V_{\mathbb{I}_k}(T'_R)}{kp(T'_R)} - \epsilon - \frac{\operatorname{diam}(\mathbb{I}_k)}{k} O(R^{-1}) \right) \log |\mathcal{A}|, \\ &\geq \left(\frac{V_{\mathbb{I}}(T'_R)}{p(T'_R)} - \epsilon - \operatorname{diam}(\mathbb{I}) O(R^{-1}) \right) \log |\mathcal{A}|, \\ &\xrightarrow{R \to +\infty} (R_{I'} - \epsilon) \log |\mathcal{A}|. \end{aligned}$$

□

8. Rescaled topological entropy for endomorphisms of $\mathbb{Z}^d$-actions

Let X be a compact metric space endowed with a $\mathbb{Z}^d$-action τ. A discrete system ($\mathbb{N}$-action) $f : X \to X$ is called an endomorphism of (X, τ) when f commutes with the $\mathbb{Z}^d$-action τ. We may define the rescaled topological entropy for any endomorphism f of a $\mathbb{Z}^d$-action (X, τ) as follows. For an open (finite) cover $\mathcal{U}$ of X and any convex exhaustion $\mathcal{J} = (J_n)_n$ we first let

$$\begin{aligned} h^\tau_{top}(f, \mathcal{U}, \mathcal{J}) &= \limsup_n \frac{h_{top}(f, \bigvee_{k \in J_n \cap \mathbb{Z}^d} \tau^{-k}\mathcal{U})}{p(J_n)}, \\ h^\tau_{top}(f, \mathcal{J}) &= \sup_{\mathcal{U}} h^\tau_{top}(f, \mathcal{U}, \mathcal{J}). \end{aligned}$$

Then for any $O \in \mathcal{D}^1$

$$h^\tau_{top}(f, O) = \sup_{\mathcal{J} \in \mathcal{E}(O)} h^\tau_{top}(f, \mathcal{J})$$

and

$$h^\tau_{top}(f) = \sup_{\mathcal{U}, \mathcal{J}} h^\tau_{top}(f, \mathcal{U}, \mathcal{J}) \quad \left(= \sup_{\mathcal{J}} h^\tau_{top}(f, \mathcal{J}) = \sup_{O \in \mathcal{D}^1} h^\tau_{top}(f, O) \right).$$

Lemma 15. *The rescaled entropies $h^\tau_{top}(f)$, $h^\tau_{top}(f, O)$ and $h^\tau_{top}(f, \mathcal{J})$ are invariant under conjugacy for the $\mathbb{N}$-action of f and the $\mathbb{Z}^d$-action of τ.*

Proof. Clearly it is enough to consider $h^\tau_{top}(f, \mathcal{J})$ for some convex exhaustion $\mathcal{J} = (J_n)_n$. Let $\psi : X \to Y$ be an homeomorphism. We check that $h^\tau_{top}(f, \mathcal{J}) = h^{\tau'}_{top}(g, \mathcal{J})$ with $g = \psi \circ f \circ \psi^{-1}$ being the endomorphism of the $\mathbb{Z}^d$-action τ' on Y given by $\tau' = \psi \circ \tau \circ \psi^{-1}$. For any open cover $\mathcal{U}$ of X we have with $\mathcal{V} = \psi(\mathcal{U})$:

$$\begin{aligned} h^\tau_{top}(f, \mathcal{U}, \mathcal{J}) &= \limsup_n \frac{h_{top}(f, \bigvee_{k \in J_n \cap \mathbb{Z}^d} \tau^{-k}\mathcal{U})}{p(J_n)}, \\ &= \limsup_n \frac{h_{top}(g, \bigvee_{k \in J_n \cap \mathbb{Z}^d} \tau'^{-k}\mathcal{V})}{p(J_n)}, \\ &= h^{\tau'}_{top}(g, \mathcal{V}, \mathcal{J}). \end{aligned}$$

The map $\mathcal{U} \mapsto \psi(\mathcal{U})$ is a bijection between open covers of X and Y. Therefore we get $h^\tau_{top}(f, \mathcal{J}) = h^{\tau'}_{top}(g, \mathcal{J})$.

□

Remark 18. *(i) If Y is a a compact subset of X invariant under f and τ, then the restriction f_Y of f to Y satisfies $h^\tau_{top}(f_Y, \mathcal{J}) \leq h^\tau_{top}(f, \mathcal{J})$ for any convex exhaustion $\mathcal{J}$.*

(ii) By following straightforwardly the proofs in Section 5.3 we get again $h^{\tau}_{top}(f,O) = h^{\tau}_{top}(f,\mathcal{J}_O)$ *and* $h^{\tau}_{top}(f^k,O) = kh^{\tau}_{top}(f,O)$ *for any* $k\in\mathbb{N}$ *and any* $O\in\mathcal{D}^1$.

Let $\tau_1,\cdots,\tau_d$ be the commuting transformations on X generating the $\mathbb{Z}^d$-action τ, i.e. $\tau^k=\tau_1^{k_1}\circ\cdots\circ\tau_d^{k_d}$ for any integer d-tuple $k=(k_1,\cdots,k_d)$. For an integer matrix $A=(a_{ij})_{i,j}\in M_d(\mathbb{Z})$ with non-zero determinant, we let τ_A be the $\mathbb{Z}^d$-action generated by $\tau^{l_1},\cdots,\tau^{l_d}$ with $l_1,\cdots,l_d$ being the columns of A. Then $\tau_A^k=\tau^{Ak}$ for any integer d-tuple k. Let $\mathbb{B}^d$ be the unit Euclidean ball of $\mathbb{R}^d$.

Lemma 16. *With the previous notations, we have for any* $O\in\mathcal{D}^1$ *:*

$$h^{\tau_A}_{top}(f,O)=\det(A)h^{\tau}_{top}(f,\widetilde{AO})\int h_{A^{-1}\mathbb{B}^d}\,d\sigma_O.$$

Proof. Firstly we observe that $p\,(AJ)=\det(A)\int h_{A^{-1}\mathbb{B}^d}\,d\sigma_J$ for any convex domain J. Indeed, it follows from Proposition 3 that :

$$\begin{aligned}
p\,(AJ) &= \lim_{\rho\to 0}\frac{V(AJ\oplus\rho\mathbb{B}^d)-V(AJ)}{\rho},\\
&= \lim_{\rho\to 0}\frac{V(A(J\oplus\rho A^{-1}\mathbb{B}^d)-V(AJ)}{\rho},\\
&= \det(A)\lim_{\rho\to 0}\frac{V(J\oplus\rho A^{-1}\mathbb{B}^d)-V(J)}{\rho},\\
&= \det(A)\int h_{A^{-1}\mathbb{B}^d}\,d\sigma_J.
\end{aligned}$$

For any subset J of $\mathbb{R}^d$ and $x\in J$ there is $y\in(J\oplus\mathsf{C})\cap\mathbb{Z}^d$ with $\|x-y\|\leq\sqrt{d}$. In particular we have $AJ\cap\mathbb{Z}^d\subset B_J:=\{-\lceil\sqrt{d}|||A|||\rceil,\cdots,-\lceil\sqrt{d}|||A|||\rceil\}\oplus A\left((J\oplus\mathsf{C})\cap\mathbb{Z}^d\right)$.

Let $\mathcal{U}$ be an open cover of X and put $\mathcal{U}_A=\bigvee_{|k|\leq\lceil\sqrt{d}|||A|||\rceil}\tau^{-k}\mathcal{U}$. Let $\mathcal{J}\in\mathcal{E}(O)$. We recall that $\mathcal{J}\oplus\mathsf{C}:=(J_n\oplus\mathsf{C})_n$ defines a convex exhaustion in $\mathcal{E}(O)$ with $p(J_n\oplus\mathsf{C})\sim^n p(J_n)$. Then we have :

$$\begin{aligned}
h^{\tau_A}_{top}(f,\mathcal{U}_A,\mathcal{J}\oplus\mathsf{C}) &= \limsup_n\frac{h_{top}(f,\bigvee_{k\in(J_n\oplus\mathsf{C})\cap\mathbb{Z}^d}\tau_A^{-k}\mathcal{U}_A)}{p(J_n\oplus\mathsf{C})},\\
&= \limsup_n\frac{h_{top}(f,\bigvee_{k\in A((J_n\oplus\mathsf{C})\cap\mathbb{Z}^d)}\tau^{-k}\mathcal{U}_A)}{p(J_n)},\\
&= \limsup_n\frac{h_{top}(f,\bigvee_{k\in B_{J_n}}\tau^{-k}\mathcal{U})}{p(J_n)},\\
&\geq \limsup_n\frac{h_{top}(f,\bigvee_{k\in AJ_n\cap\mathbb{Z}^d}\tau^{-k}\mathcal{U})}{p(J_n)},\\
&\geq \det(A)\limsup_n\left(\frac{h_{top}(f,\bigvee_{k\in AJ_n\cap\mathbb{Z}^d}\tau^{-k}\mathcal{U})}{p(AJ_n)}\int h_{A^{-1}\mathbb{B}^d}\,d\sigma_{\widetilde{J_n}}\right),\\
&\geq \det(A)h^{\tau}_{top}(f,\mathcal{U},A\mathcal{J})\int h_{A^{-1}\mathbb{B}^d}\,d\sigma_O.
\end{aligned}$$

As the map $\mathcal{J}=(J_n)_n\mapsto A\mathcal{J}=(AJ_n)_n$ is a bijection from $\mathcal{E}(O)$ to $\mathcal{E}(\widetilde{AO})$, we get by taking the supremum over $\mathcal{U}$ and $\mathcal{J}\in\mathcal{E}(O)$:

$$h^{\tau_A}_{top}(f,O)\geq\det(A)h^{\tau}_{top}(f,\widetilde{AO})\int h_{A^{-1}\mathbb{B}^d}\,d\sigma_O.$$

In the same way the other inequality is obtained (more easily) by observing that $AJ\cap\mathbb{Z}^d\supset A(J\cap\mathbb{Z}^d)$ for any subset J. □

For $A = k\,\mathrm{Id}$ with $k \in \mathbb{N}$ we get $h^{\tau_A}_{top}(f,O) = k^{d-1}h^{\tau}_{top}(f,\widetilde{AO})$ and therefore $h^{\tau_A}_{top}(f) = k^{d-1}h^{\tau}_{top}(f)$. In particular the rescaled entropy may be not invariant under topological conjugacy of the $\mathbb{N}$-action of the endomorphism f when the conjugacy does not preserve the $\mathbb{Z}^d$-action.

The $\mathbb{Z}^d$-action (X,τ) is said expansive when there is an open cover $\mathcal{U}$ such that the cover $\bigcap_{k\in\mathbb{Z}^d}\tau^{-k}\mathcal{U}$ is the partition into singletons. Such an open cover $\mathcal{U}$ is called a τ-generator.

Lemma 17. *Assume (X,τ) is expansive and let $\mathcal{U}$ be a τ-generator. Then for any $O \in \mathcal{D}^1$*

$$h^{\tau}_{top}(f,O) = \sup_{\mathcal{J}\in\mathcal{E}(O)} h^{\tau}_{top}(f,\mathcal{U},\mathcal{J}).$$

Proof. Let $\mathcal{V}$ be an open cover of X. There is a bounded subset I of $\mathbb{Z}^d$ such that the open cover $\bigvee_{k\in I}\tau^{-k}\mathcal{U}$ is finer that $\mathcal{V}$. Let $\mathcal{J} = (J_n)_n \in \mathcal{E}(O)$ for $O \in \mathcal{D}^1$. Then we get :

$$\begin{aligned}
h^{\tau}_{top}(f,\mathcal{U},\mathcal{J}\oplus I) &= \limsup_n \frac{h_{top}(f,\bigvee_{k\in(J_n\oplus I)\cap\mathbb{Z}^d}\tau^{-k}\mathcal{U})}{p(J_n\oplus I)},\\
&= \limsup_n \frac{h_{top}\left(f,\bigvee_{k\in J_n\cap\mathbb{Z}^d}\tau^{-k}(\bigvee_{l\in I}\tau^{-l}\mathcal{U})\right)}{p(J_n)},\\
&\geq \limsup_n \frac{h_{top}(f,\bigvee_{k\in J_n\cap\mathbb{Z}^d}\tau^{-k}\mathcal{V})}{p(J_n)},\\
&\geq h^{\tau}_{top}(f,\mathcal{V},\mathcal{J}).
\end{aligned}$$

By taking the supremum over convex exhaustions $\mathcal{J} \in \mathcal{E}(O)$ and open covers $\mathcal{V}$ of X, we get $\sup_{\mathcal{J}\in\mathcal{E}(O)} h^{\tau}_{top}(f,\mathcal{U},\mathcal{J}) \geq h^{\tau}_{top}(f,O)$. This concludes the proof of the lemma as the other inequality follows straightforwardly from the definition of $h^{\tau}_{top}(f,O)$. □

For a CA we recover the definition of rescaled entropy of Section 5 by considering the generator given by the zero-coordinate partition.

An algebraic $\mathbb{Z}^d$-action τ is a $\mathbb{Z}^d$-action by automorphisms of a compact abelian group X. By Pontryagin duality, there is a one-to-one correspondence between algebraic $\mathbb{Z}^d$-actions and modules M over the ring $R_d = \mathbb{Z}[u_1^{\pm1},\cdots,u_d^{\pm1}]$. The $\mathbb{Z}^d$-shift on $X_p = (\mathbb{F}_p)^{\mathbb{Z}^d}$ (resp. $X_\infty = (\mathbb{R}/\mathbb{Z})^{\mathbb{Z}^d}$) is associated to the module $M = \widehat{X_p} = R_d\,/ < p >$ with p a rational prime (resp. $M = \widehat{X_\infty} = R_d$). Then algebraic endomorphisms of these $\mathbb{Z}^d$-actions, i.e. group homomorphisms $f : X \to X$ commmuting with the $\mathbb{Z}^d$-action, are given by algebraic CA. As a consequence of Theorem 1 we get :

Corollary 19. *Let $f \neq \pm\mathrm{Id}, 0$ be an algebraic CA on X_∞. Then we have*

$$h^{d}_{top}(f) = +\infty.$$

Proof. For some finite family $(a_i)_{i\in I}$ in $\mathbb{Z}^*$ we have :

$$\forall (x_j)_j \in (\mathbb{R}/\mathbb{Z})^{\mathbb{Z}^d},\ f((x_j)_j) = \left(\sum_{i\in I} a_i x_{i+j}\right)_j.$$

We first consider the case $I \neq \{0\}$. Then for some arbitrarily large rational prime p, the domain of the algebraic CA f_p on $(\mathbb{F}_p)^{\mathbb{Z}^d}$ associated to the family $(\overline{a_i})_{i\in I}$ in $\mathbb{F}_p$ is also non trivial and therefore $h^d_{top}(f_p) \geq \frac{\log p}{2}$. But (X_p, f_p) is conjugated for the $\mathbb{N}$- and $\mathbb{Z}^d$-actions to the subsystem (Y, f_Y) of (X_∞, f) with $Y = \left(\frac{1}{p}\mathbb{Z}/\mathbb{Z}\right)^{\mathbb{Z}^d} \subset X_\infty$. By Lemma 15 and Remark 18 (i) we conclude $h^d_{top}(f) = +\infty$.

Finally assume $I = \{0\}$ and $a_0 \neq \pm 1$. Let f_{a_0} be the $\times a_0$ circle map. We consider an open cover $\mathcal{U}$ of $\mathbb{R}/\mathbb{Z}$ with $h_{top}(f_{a_0}, \mathcal{U}) \simeq h_{top}(f_{a_0}) = \log |a_0|$. Let $\mathcal{V} = \mathcal{U} \times (\mathbb{R}/\mathbb{Z})^{\mathbb{Z}^d \setminus \{0\}}$ be the induced zero-coordinate cover of X_∞. Then we have for any convex exhaustion $\mathcal{J} = (J_n)_n$:

$$\begin{aligned} h_{top}(f, \bigvee_{k \in J_n \cap \mathbb{Z}^d} \sigma^{-k}\mathcal{V}) &\simeq \sharp J_n h_{top}(f_{a_0}), \\ &\simeq \sharp J_n \log |a_0|, \\ h^d_{top}(f, \mathcal{V}, \mathcal{J}) &= \limsup_n \frac{h_{top}(f, \bigvee_{k \in J_n \cap \mathbb{Z}^d} \sigma^{-k}\mathcal{V})}{p(J_n)}, \\ &= \log |a_0| \limsup_n \frac{\sharp J_n}{p(J_n)} = +\infty. \end{aligned}$$

Note that we clearly have $h^d_{top}(f) = 0$ for $a_0 \in \{\pm 1\}$ and $I = \emptyset$ $(f \equiv 0)$. □

Question. Does the formula of the rescaled entropy for algebraic CA obtained in Theorem 1 generalize to algebraic endomorphisms of other $\mathbb{Z}^d$-actions (associated to modules $M \neq R_d, R_d / < p >$) ?

Remark 20. *We only deal in this last section with the generalization of the rescaled topological entropy, but one may also define similarly a measure theoretical rescaled entropy for general endomorphisms of $\mathbb{Z}^d$-actions.*

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