ZERO-DIMENSIONAL AND SYMBOLIC EXTENSIONS OF TOPOLOGICAL FLOWS

DAVID BURGUET AND RUXI SHI

Abstract. A zero-dimensional (resp. symbolic) flow is a suspension flow over a zero-dimensional system (resp. a subshift). We show that any topological flow admits a principal extension by a zero-dimensional flow. Following [Bur19] we deduce that any topological flow admits an extension by a symbolic flow if and only if its time-t map admits an extension by a subshift for any $t \neq 0$. Moreover the existence of such an extension is preserved under orbit equivalence for regular topological flows, but this property does not hold more true for singular flows. Finally we investigate symbolic extensions for singular suspension flows. In particular, the suspension flow over the full shift on $\{0,1\}^{\mathbb{Z}}$ with a roof function f vanishing at the zero sequence 0^∞ admits a principal symbolic extension or not depending on the smoothness of f at 0^∞.

1. Introduction

A symbolic extension of a discrete topological system (X,T) is a topological extension $\pi : (Y,S) \to (X,T)$ by a subshift (Y,S). Existence of symbolic extensions and their entropy are related with weak expansive entropy properties of the system [BD04]. Building on [BD04] the first author and T.Downarowicz also investigate uniform generators, which are symbolic extensions with a Borel embedding $\psi : (X,T) \to (Y,S)$ with $\pi \circ \psi = \mathrm{Id}_X$.

To study symbolic extensions for discrete systems, M. Boyle and T. Downarowicz [BD04] have developed a new entropy theory. The first step in their construction of symbolic extensions consists in building a zero-dimensional principal extension (see Section 2.2 for precise definitions). In [BD04] this is done by using the small boundary property for finite topological entropy systems with a minimal factor [Lin99]. In this way one even get a strongly isomorphic zero-dimensional extension. Later T. Downarowicz and D. Huczek gave a constructive proof of a zero-dimensional principal extension for any topological system (with finite topological entropy or not).

More recently the first author developed in [Bur19] a theory of uniform generators for topological regular (i.e. without fixed points) flows. In this context a symbolic extension (resp. uniform generator) is a topological extension by a regular suspension flow over a subshift (resp. with an embedding). To investigate existence of uniform generators he considered topological flows satisfying the so-called small flow boundary property[1]. Under this extra assumption, the flow admits a strongly isomorphic zero-dimensional extension. Here we investigate only symbolic extensions for topological flows and not uniform generators for these, so that to be reduced to the case of zero-dimensional suspension flows, we only need to build a principal zero-dimensional extension (this is achieved in Section 2). In this way we may also consider the symbolic extensions of a topological singular flow [2]. In Section 3 we build on [Bur19] to study symbolic extension for general topological flows. In particular we show that two regular topological flows which are orbit equivalent either both admit a (resp. principal) symbolic extension or not. This result was only proved in [Bur19] for regular flows with the small flow boundary property. We also give counterexamples for singular flows.

In the rest part (Section 4) we define singular suspension flows as suspension flows over a discrete system (X,T) with a roof function vanishing only at a fixed point of T. We investigate their symbolic extensions. In the case (X,T) is expansive, an associated singular suspension flow

[1]The small flow boundary property (for topological flows) is an analogy to the small boundary property (for discrete topological systems). This notion is introduced by the first author in [Bur19].

[2]An isomorphic extension of a singular flow is necessarily singular (so that if one wants to develop a theory of uniform generators one should consider singular suspension flows over a subshift).

with finite topological entropy does not admit in general a principal symbolic extension. This depends on the behaviour of the roof function near the singularity. For (X,T) being the two full shift we illustrate this phenomenon by computing the symbolic extension entropy function for a large class of roof functions. In fact we explicitly build a symbolic extension of the corresponding singular suspension flow.

2. Zero-dimensional principal extension

By a (discrete) topological system (X,T), we mean that X is a compact metrizable space and $T: X \to X$ is continuous. Moreover, the system is said to be *invertible* if T is a homeomorphism. A pair (X,Φ) is called a *topological semi-flow* (resp. *topological flow*) if X is a metrizable compact space and $\Phi = (\phi_t)_{t\geq 0}$ (resp. $\Phi = (\phi_t)_{t\in\mathbb{R}}$) are continuous maps from X to itself satisfying that $\phi_0(x) = x$ and $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$ for all $t, s \geq 0$ (resp. $t, s \in \mathbb{R}$). A point $x \in X$ is a fixed point of Φ when $\phi_t(x) = x$ for all $t \geq 0$. A flow without fixed point is called *regular*, otherwise the flow is said *singular*. We let $\mathcal{M}_T(X)$ or $\mathcal{M}_\Phi(X)$ (resp. $\mathcal{M}^e_T(X)$ or $\mathcal{M}^e_\Phi(X)$) be the compact set of Borel probability measures invariant (resp. ergodic) by the topological system or flow. We recall that the measure-theoretic entropy $h^\Phi(\mu)$ of $\mu \in \mathcal{M}_\Phi(X)$ is defined as the the entropy of its time-1 map, i.e. $h_\Phi(\mu) = h_{\phi_1}(\mu)$. Through this paper, we use the following notations: $\mathbb{R}_{\geq 0} = [0,\infty)$ and $\mathbb{R}_{>0} = (0,\infty)$.

2.1. **Suspension semi-flows.** Let (X,d) be a compact metric space and $T : X \to X$ a continuous map. Let $f : X \to \mathbb{R}_{>0}$ be a continuous map. Let $\overline{X^f}$ be the compact subset of $X \times \mathbb{R}_{\geq 0}$ defined by

$$\overline{X^f} := \{(x,t) : 0 \leq t \leq f(x), x \in X\}. \tag{2·1}$$

We consider the equivalence relation $\sim$ on $\overline{X^f}$ with $(x, f(x)) \sim (Tx, 0)$ for all $x \in X$ and we denote by X^f the quotient space $\overline{X^f}/\sim$ endowed with the quotient topology. By abuse of notations we also write (x,t) to denote the equivalence class of $(x,t) \in \overline{X^f}$. The *suspension* semi-flow over T under the *roof function* f, written by (X^f, T^f), is the semi-flow $(T^f_t)_{t\in\mathbb{R}}$ on the space X^f induced by the time translation T_t on $X \times \mathbb{R}_{\geq 0}$ defined by $T_t(x,s) = (x, t+s)$. If T is a homeomorphism, then (X^f, T^f) defines a flow. Such (semi-)flows are regular.

Bowen and Walters define a metric d_f compatible with the quotient topology on X^f as follows [BW72, Section 4]. For a point $(x,t) \in X^f$ we let $u_{(x,t)} = t/f(x)$. A pair of points $A = (x_A, t_A)$ and $B = (x_B, t_B)$ in X^f is said to be

- *horizontal* if $u_A = u_B$, then its length is $|AB| := (1-u_A)d(x_A, x_B) + u_A d(Tx_A, Tx_B)$.
- *vertical* if $x_A = x_B$, $x_B = Tx_A$ or $x_A = Tx_B$, then its length is $|AB| := |u_A - u_B|$, $1 - u_B + u_A$ or $1 - u_A + u_B$ respectively.

A sequence $A_1, \cdots, A_n$ of n points in X^f is said *admissible* when (A_i, A_{i+1}) is either vertical or horizontal for $i = 1, \cdots, n-1$ and the length of this sequence is defined as the sum of the length of its corresponding pairs. Then the distance d_f between two points A and B is defined as the infimum of the length of all admissible sequences $A_1, \cdots, A_n$ with $A_1 = A$ and $A_n = B$.

Denote by $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}$. Let (X,T) be a discrete topological system. Let $f : X \to \mathbb{R}_{>0}$ be a continuous function and (X^f, T^f) be the associated suspension semi-flow. For $\mu \in \mathcal{M}_T(X)$ the product measure $\mu \times \mathcal{L}$ induces a finite T^f-invariant measure on X^f, which defines a homeomorphism Θ between $\mathcal{M}_{T^f}(X^f)$ and $\mathcal{M}_T(X)$:

$$\begin{aligned} \Theta : \mathcal{M}_T(X) &\to \mathcal{M}_{T^f}(X^f) \\ \mu &\mapsto \frac{(\mu \times \mathcal{L})_{|X^f}}{\int f d\mu}. \end{aligned}$$

Due to Abramov [Abr59], the entropy of μ and $\Theta(\mu)$ are related by the following formula

$$h_{T^f}(\Theta(\mu)) = \frac{h_T(\mu)}{\int f d\mu}, \forall \mu \in \mathcal{M}_T(X). \tag{2·2}$$

2.2. **Extensions.** A suspension flow over a zero-dimensional invertible dynamical system will be called a *zero-dimensional suspension flow* and a topological extension by a zero-dimensional suspension flow is said to be a *zero-dimensional extension.* Similarly a suspension flow over a symbolic discrete topological dynamical system (a.k.a. *$\mathbb{Z}$-subshift*) will be called a *symbolic suspension flow* and a topological extension by a symbolic suspension flow is said to be a *symbolic extension.*

Let (X, Φ) and (Y, Ψ) be two topological semi-flows. Suppose that $\pi : Y \to X$ is a topological extension from (X, Φ) to (Y, Ψ). The topological extension is said to be

- *principal* when it preserves the entropy of invariant measures, i.e. $h(\mu) = h(\pi\mu)$ for all Ψ-invariant measure μ,
- *with an embedding* when there is a Borel embedding $\psi : (X, \Phi) \to (Y, \Psi)$ with $\pi \circ \psi = \mathrm{Id}_X$,
- *isomorphic* when the map induced by π on the sets of invariant Borel probability measures is bijective and $\pi : (Y, \Psi, \mu) \to (X, \Phi, \pi\mu)$ is a measurable isomorphism for any Ψ-invariant measure μ,
- *strongly isomorphic* when there is a set $E \subset X$ with $\mu(E) = 1$ for all $\mu \in \mathcal{M}_\Phi(X)$ such that the restriction of π to $\pi^{-1}E$ is one-to-one.

Clearly, we have the following implication:

$$\begin{array}{ccc} \text{strongly isomorphic} & \Longrightarrow & \text{isomorphic} \\ \Downarrow & & \Downarrow \\ \text{with an embedding} & & \text{principal.} \end{array}$$

2.3. **Construction.** In this section we build a zero-dimensional principal extension for any topological semi-flow.

Theorem 2.1. *Every topological semi-flow has a zero-dimensional principal extension. Moreover the roof function of this extension may be chosen constant equal to* 1*.*

To prove this theorem, we first show that every suspension semi-flow has a zero-dimensional principal extension (Proposition 2.3). Then, for a general topological flow we build a principal extension by a suspension flow.

2.3.1. *Zero-dimensional principal extension of suspension flow.* Let (Z, T) be a discrete topological system. Let $f : Z \to \mathbb{R}_{>0}$ be a continuous function. In this section, we construct a zero-dimensional principal extension of the suspension semi-flow (Z^f, T^f). Due to T. Downarowicz and D. Huczek [DH13], there exists an invertible dynamical system (X, S) which is a zero-dimensional principal extension of (Z, T). Denote by $\rho : X \to Z$ the factor map. We see that the map $g := f \circ \rho : X \to \mathbb{R}_{>0}$ is continuous. Define $\overline{\rho} : \overline{X^g} \to \overline{Z^f}$ by $\overline{\rho}(x, t) \mapsto (\rho(x), t)$ for all $(x, t) \in \overline{X^g}$. We have $\overline{\rho}(x, f(x)) = (\rho(x), f \circ \rho(x))$ for all $x \in X$, so that $\overline{\rho}$ induces a continuous map $\widehat{\rho} : X^g \to Z^f$. Moreover $\overline{\rho}$ commutes with the translation on the second coordinate, therefore $\widehat{\rho}$ is a continuous factor map.

Lemma 2.2. *$\widehat{\rho}$ is principal.*

Proof. Let $\mu \in \mathcal{M}_S(X)$. Since ρ is principal, we see that $h(\rho\mu) = h(\mu)$. It is clear that

$$\widehat{\rho}\Theta(\mu) = \frac{\widehat{\rho}(\mu \times \mathcal{L})_{|X^g}}{\int g \, d\mu} = \frac{(\rho\mu \times \mathcal{L})_{|Z^f}}{\int f \, d(\rho\mu)} = \Theta(\rho\mu).$$

Then by (2·2), we obtain that

$$h(\widehat{\rho}\Theta(\mu)) = h(\Theta(\rho\mu)) = \frac{h(\rho\mu)}{\int f \, d(\rho\mu)} = \frac{h(\mu)}{\int g \, d\mu} = h(\Theta(\mu)).$$

□

To sum up, we obtain the following proposition.

Proposition 2.3. *Every suspension semi-flow has a zero-dimensional principal extension.*

2.3.2. *General case.* We present the proof of Theorem 2.1 in the general case.

Proof of Theorem 2.1. Let (X, Φ) be a topological semi-flow. Let us denote by $\mathbb{1}$ the constant function on X equal to 1. Then the suspension semi-flow $(X^{\mathbb{1}}, (\phi_1)^{\mathbb{1}})$ over the time-1 map ϕ_1 under $\mathbb{1}$ defines an extension of (X, Φ) via the factor map $\pi : (x, t) \mapsto \phi_t(x)$ for $x \in X$ and $t \geq 0$. Notice that a $(\phi_1)^{\mathbb{1}}$-invariant measure on $X^{\mathbb{1}}$ has the form $\mu \times \mathcal{L}_{[0,1]}$ where μ is a ϕ_1-invariant measure and $\mathcal{L}_{[0,1]}$ is the Lebesgue measure on $[0, 1]$. Pick arbitrary $(\phi_1)^{\mathbb{1}}$-invariant measure $\mu \times \mathcal{L}_{[0,1]}$. It follows from Fubini's theorem that for all Borel subset B of $X^{\mathbb{1}}$:

$$\begin{aligned}
\pi(\mu \times \mathcal{L}_{[0,1]})(B) &= \mu \times \mathcal{L}_{[0,1]}(\pi^{-1}((B)), \\
&= \mu \times \mathcal{L}_{[0,1]}\left(\{(x,t) \in X^{\mathbb{1}},\ \phi_t(x) \in B\}\right), \\
&= \int_0^1 \mu(\phi_t^{-1}(B))\, dt, \\
&= \int_0^1 \phi_t\mu(B)\, dt.
\end{aligned}$$

For all $t \geq 0$, we observe that $h_{\phi_1}(\phi_t\mu) \geq h_{\phi_1}(\mu)$. Indeed, for $s \geq 0$ with $t + s \in \mathbb{N}$ we have $h_{\phi_1}(\phi_t\mu, \phi_s^{-1}P) = h_{\phi_1}(\mu, \phi_{s+t}^{-1}P) = h_{\phi_1}(\mu, \phi_1^{-(t+s)}P) = h_{\phi_1}(\mu, P)$ for any Borel finite partition P of X. Since the entropy function h_{ϕ_1} is harmonic, we get then

$$\begin{aligned}
h_\Phi(\pi(\mu \times \mathcal{L}_{[0,1]})) &= h_{\phi_1}(\pi(\mu \times \mathcal{L}_{[0,1]})), \\
&= \int_0^1 h_{\phi_1}(\phi_t\mu)\, dt, \\
&\geq h_{\phi_1}(\mu), \\
&\geq h_{(\phi_1)^{\mathbb{1}}}(\mu \times \mathcal{L}_{[0,1]}).
\end{aligned}$$

Therefore, we conclude that $(X^{\mathbb{1}}, (\phi_1)_{\mathbb{1}})$ is a principal extension of (X, Φ). By Proposition 2.3, the suspension flow $(X^{\mathbb{1}}, (\phi_1)^{\mathbb{1}})$ has a zero-dimensional principal extension. By composition we get a principal zero-dimensional extension of (X, Φ). This completes the proof. □

3. Applications to symbolic extensions

By the previous construction of a zero-dimensional principal extension, the results related to symbolic extensions obtained in [Bur19] for flows with the small flow boundary property may be straightforwardly extended to general topological flows. We first recall the framework of the entropy theory of M. Boyle and T. Downarowicz.

3.1. **Superenvelope of entropy structures.** Entropy structures are particular sequences of nonnegative real functions $\mathcal{H} = (h_k)_k$ defined on the set $\mathfrak{X}$ of invariant probability measures, $\mathfrak{X} = \mathcal{M}_T(X)$ or $= \mathcal{M}_\Phi(X)$ depending on the context, which are converging pointwisely to the entropy function h. For a zero-dimensional system, the entropy with respect to any sequence $(P_k)_k$ of clopen partitions with $\mathrm{diam}(P_k) \xrightarrow{k} 0$ defines an entropy structure. For a precise definition and examples we refer to [Dow11] for discrete systems and to [Bur19] for topological flows.

For a function $f : \mathfrak{X} \to \mathbb{R}$ we let $\tilde{f}$ be the *upper semi-continuous envelope* of f, i.e.

$$\forall \mu \in \mathfrak{X},\ \tilde{f}(\mu) = \limsup_{\nu \to \mu} f(\nu).$$

A *superenvelope* $E : \mathfrak{X} \to \mathbb{R}$ is an affine function such that for some (any) entropy structure $\mathcal{H} = (h_k)_k$

$$\lim_k \widetilde{(E - h_k)} = E - h.$$

3.2. **Characterization of symbolic extensions.** For a symbolic extension $\pi : (Y,S) \to (X,T)$ of a topological discrete system (X,T), we let h^π be the associate entropy function defined as

$$\forall \mu \in \mathcal{M}_T(X),\ h^\pi(\mu) = \sup_{\nu\in\mathcal{M}_S(Y),\,\pi\nu=\mu} h(\nu).$$

We may define in the same way the entropy function associated to the symbolic extension of a topological flow. Then the fundamental theorem in the theory of symbolic extension is the following characterization of the entropy function h^π in terms of superenvelopes.

Theorem 3.1 ([BD04], Theorem 5.5)**.** *For a discrete topological system (X,T), any function h^π is an affine superenvelope and conversely for any affine superenvelope E there is a symbolic extension π with $h^\pi = E$.*

The first author proved the corresponding statement for topological flows with the small flow boundary property (Theorem 3.6 in [Bur19]). The existence of a zero dimensional extension given by Theorem 2.1 implies the general following version :

Theorem 3.2. *For a topological flow (X,Φ), any function h^π is an affine superenvelope and conversely for any affine superenvelope E there is a symbolic extension π with $h^\pi = E$.*

Similarly, we relate the symbolic extensions of a general topological flow with the symbolic extensions of the discrete systems given by its time-t maps (see Lemma 3.19 in [Bur19]).

Theorem 3.3. *A topological flow admits a symbolic extension (resp. principal) if and only if ϕ_t admits a symbolic extension (resp. principal) for some (any) $t \neq 0$.*

We recall some terminology of the theory of symbolic extensions, which will be used in the next sections. Firstly the (topological) symbolic extension entropy $h_{sex}(T)$ (resp. $h_{sex}(\Phi)$) of a topological system (resp. flow) is the infimum of the topological entropy over all its symbolic extensions. The corresponding measure theoretic quantity is the real function h_{sex} defined on the set of invariant measures as the infimum of the functions h^π over all the symbolic extensions π.

3.3. **Invariance under orbit equivalence.** Two topological flows (X,Φ) and (Z,Ψ) are said to be *orbit equivalent* when there is a homeomorphism Γ from X onto Z mapping Φ-orbits to Ψ-orbits, preserving their orientation. In other words, the topological flows (X,Φ) and $(X,\widehat{\Phi})$ with $\widehat{\Phi} = (\widehat{\phi}_t)_t = (\Gamma^{-1}\circ\phi_t\circ\Gamma)_t$ has the same orbits with the same direction, i.e.

$$\{\phi_t(x) : t \in \mathbb{R}_{\geq 0}\} = \{\widehat{\phi}_t(x) : t \in \mathbb{R}_{\geq 0}\}, \forall x \in X.$$

Assume that (X,Φ) is regular. Then we can define the continuous map $\theta : X \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with the following properties (see [BS40]):

(i) $\phi_t(x) = \widehat{\phi}_{\theta(x,t)}(x)$,
(ii) $\theta(x, t+s) = \theta(x,s) + \theta(\phi_s(x), t)$,
(iii) $\theta(x,0) = 0$ and $\theta(x,t)$ is strictly increasing in t.

Zero and infinite topological entropy are invariants for orbit equivalent regular flows [Ohn80]. In [Bur19] the first author showed existence of symbolic extensions is preserved under orbit equivalence for topological regular flows with the small boundary property. Here by using the zero-dimensional principal extension built in Theorem 2.1 we remove this extra assumption. The proof differs completely from the one given in [Bur19] for flows with the small flow boundary property; it is not a direct consequence of the existence of principal zero dimensional extension as in the above Theorem 3.2 and Theorem 3.3.

Theorem 3.4. *The existence of symbolic extensions (resp. principal) is preserved by orbit equivalence for regular flows.*

Proof. Let Φ and $\widehat{\Phi}$ be two regular topological flows on a compact metric space X with the same orbits and let θ be defined as above. According to Theorem 2.1 the suspension flow $(X^{\mathbb{1}}, (\phi_1)^{\mathbb{1}})$ is a principal extension of (X,Φ) via the factor map $\pi : (x,t) \mapsto \phi_t x$. On the other hand, the function $g : x \mapsto \theta(x,1)$ is continuous and positive on X (due to (iii)). Then by the fact that

$\widehat{\phi}_{\theta(x,1)}(x) = \phi_1(x)$, the suspension flow $(X^g, (\phi_1)^g)$ is the extension of $(X, \widehat{\Phi})$ via the factor map $\widehat{\pi} : (x, t) \mapsto \widehat{\phi}_t x$:

$$\begin{array}{ccc} (X^{\mathbb{1}}, (\phi_1)^{\mathbb{1}}) & & (X^g, (\phi_1)^g) \\ \downarrow \pi & & \downarrow \widehat{\pi} \\ (X, \Phi) & & (X, \widehat{\Phi}) \end{array}$$

We claim that $(X^g, (\phi_1)^g)$ is a principal extension of $(X, \widehat{\Phi})$. To simplify the notations we denote by $\Psi = (\psi_t)_t$ the suspension flow $(\phi_1)^g$ on X^g. By Ledrappier-Walters formula [3] [LW77], it is enough to check $h_{\mathrm{top}}(\psi_1, \widehat{\pi}^{-1} z) = 0$ for all $z \in X$. Fix $z \in X$. Since the map $\widehat{\pi} : X^g \to X$ is given by $\widehat{\pi}(x, t) = \widehat{\phi}_t x$ for $x \in X$ and $0 \leq t < g(x)$, the set $\widehat{\pi}^{-1} z$ is contained in the set

$$\{(\widehat{\phi}_{-t} z, t) : 0 \leq t \leq \max g\}.$$

We recall that d_g denotes the Bowen-Walters metric on X^g. Let $\min g > \epsilon > 0$ and $N \in \mathbb{N}^*$. We will show the $h_{top}(\widehat{\pi}^{-1} z, c\epsilon) \leq \frac{\log 2}{N \min g}$ with $c = 1 + \frac{3 \max g}{(\min g)^2}$, where $h_{top}(\widehat{\pi}^{-1} z, c\epsilon)$ is the Bowen entropy of $\widehat{\pi}^{-1} z$ for d_g at the scale $c\epsilon$ with respect to ψ_1, i.e.

$$h_{top}(\widehat{\pi}^{-1} z, c\epsilon) = \limsup_n \frac{1}{n} \log \max\{\sharp E_n,\ E_n \text{ is } (c\epsilon, n) - \text{separated for } d_g \text{ w.r.t. } \psi_1\}.$$

This will conclude the proof as ϵ and N are chosen arbitrarily and $h_{top}(\widehat{\pi}^{-1} z) = \lim_{\epsilon \to 0} h_{top}(\widehat{\pi}^{-1} z, c\epsilon)$.

By uniform continuity of g and the flow map, there are $\eta, \delta > 0$ which can be assumed less than $\frac{\epsilon}{2}$ such that

$$|g(x) - g(y)| < \frac{\epsilon}{4N} \ \forall x, y \text{ with } d(x, y) < \eta$$

and

$$d\left(\phi_t z, \phi_s z\right) < \eta \ \forall t, s \in \mathbb{R} \text{ with } |t - s| < \delta.$$

We let $S_n g$ be the Birkhof sum

$$S_n g(x) = \sum_{k=0}^{n-1} g \circ \phi_k(x).$$

Observe that

$$S_{n+m} g = S_n g + S_m g \circ \phi_n.$$

We denote by $|J|$ the diameter of a subset J of $\mathbb{R}$. From the choice of δ the set $|S_n g(\phi_I z)| := \{S_n g(\phi_t z),\ t \in I\}$ satisfies $|S_n g(\phi_I z)| < \frac{\epsilon}{4}$ for any subset I with $|I| < \delta$ and any integer $0 \leq n \leq N$.

Claim 1. For any subset I of $\mathbb{R}$ with $|I| < \delta$, we can find a cover $\mathcal{F}_k = \mathcal{F}_k(I)$ of I with $\sharp \mathcal{F}_k \leq 2^k$ satisfying $|S_n g(\phi_J z)| < \frac{\epsilon}{2}$ for all $k \in \mathbb{N}$, all $n \leq kN$ and $J \in \mathcal{F}_k$.

Proof of Claim 1. Fix such an interval I with $|I| < \delta$. We will argue by induction on k. For $k = 1$ we let $\mathcal{F}_1 = \{I\}$. Assume $\mathcal{F}_k$ already built and take $J \in \mathcal{F}_k$. Then we may cover J by two subsets J^1 and J^2 in such a way the diameter of $S_{kN} g(\phi_{J^i} z)$ is less than $\frac{\epsilon}{4}$ for $i = 1, 2$. Since the diameter of $J^i + kN$ is less than δ, we have also $|S_j g(\phi_{J^i + kN} z)| < \frac{\epsilon}{4}$ for every $0 \leq j \leq N$ and therefore

$$|S_{kN+j} g(\phi_{J^i} z)| \leq |S_{kN} g(\phi_{J^i} z)| + |S_j g(\phi_{J^i + kN} z)| < \frac{\epsilon}{2}, \forall 0 \leq j \leq N.$$

We conclude by letting $\mathcal{F}_{k+1} = \{J^1, J^2\ :\ J \in \mathcal{F}_k\}$. □

[3] Let $\pi : (X, T) \to (Y, S)$ be an extension. Then for any S-invariant measure ν, we have that

$$\sup_{\mu : \pi\mu = \nu} h_\mu(T) = h_\nu(S) + \int_Y h_{\mathrm{top}}(\pi^{-1} y) d\nu(y).$$

We go back to the proof of $h_{top}(\widehat{\pi}^{-1}z, c\epsilon) \leq \frac{\log 2}{N \min g}$. The flow Φ and $\hat{\Phi}$ being orbit equivalent, there are finite families $\mathcal{I}$ and $\mathcal{K}$ of intervals with length respectively less than δ and $\epsilon/2$ such that $\{(\widehat{\phi}_{-t}z, t) : 0 \leq t < \max g\}$ is contained in $\bigcup_{I\in\mathcal{I},\, K\in\mathcal{K}} \phi_I z \times K$.

Let $m = [kN \min g]$ and let E_m be a maximal $(m, c\epsilon)$-separated set inside $\widehat{\pi}^{-1}z$ for the time-one map ψ_1 of the suspension flow on $(X^g, (\phi_1)^g)$.

Claim 2. Any set of the form $\phi_I z \times K$ with $|I| < \delta$ and $|K| < \epsilon/2$ contains at most 2^k points of E_m.

Suppose Claim 2 holds. Then we have

$$\sharp E_m \leq \sharp\mathcal{I}\sharp\mathcal{K} \cdot 2^k.$$

We conclude that

$$\begin{aligned} h_{top}(\widehat{\pi}^{-1}z, c\epsilon) &= \limsup_m \frac{1}{m} \log \sharp E_m \\ &\leq \limsup_k \frac{k \log 2 + \log \sharp\mathcal{I}\sharp\mathcal{K}}{kN \min g} = \frac{\log 2}{N \min g}. \end{aligned}$$

It remains to prove Claim 2.

Proof of Claim 2. Let $|I| < \delta$ and $|K| < \epsilon/2$. Let $x = (\phi_t z, s)$ and $x' = (\phi_{t'} z, s')$ in $\phi_J z \times K$ for some $J \in \mathcal{F}_k(I)$. We will show that x and x' are $(c\epsilon, n)$-closed. This would imply that there is at most one point of E_m in $\phi_J z \times K$ and Claim 2 then follows. By Claim 1, we have $|S_n g(\phi_J z)| < \frac{\epsilon}{2}$ for all $n \leq kN$. For $l \leq m$ the point $x_l = (\phi_t z, s+l)$ of $X \times \mathbb{R}$ has for representative in the quotient space X^g the point $(\phi_{n+t}z, s_l)$ where n is the largest integer with $s_l := s + l - S_n g(\phi_t z) \geq 0$. As $l \leq kN \min g$ we have $n \leq kN$. Similarly we may define similarly the integer n' and the real number s'_l associated to $x'_l := (\phi_{t'} z, s'+l)$. It follows from $|S_n g(\phi_J z)| < \frac{\epsilon}{2}$, $|K| < \frac{\epsilon}{2}$ and $\epsilon < \min g$ that we have $|n - n'| \leq 1$. Let $u_l = \frac{s_l}{g(\phi_{n+t}z)}$ and $u'_l = \frac{s'_l}{g(\phi_{n'+t'}z)}$ in $[0, 1]$. We consider the point x''_l in X^g defined as $x''_l = (\phi_{n+t'}z, u_l g(\phi_{n+t'}z)$. The pairs (x_l, x''_l) and (x''_l, x'_l) are respectively horizontal and vertical. As t, t' both lie in I we have $|t - t'| < \delta$ so that

$$|x_l x''_l| = u_l d(\phi_{n+t}(z), \phi_{n+t'}(z)) + (1 - u_l) d(\phi_{n+1+t}(z), \phi_{n+1+t'}(z)) \leq \epsilon.$$

Then the length $|x''_l x'_l|$ may be bounded from above depending on $|n - n'|$:

- either $n = n'$. Then $|g(\phi_{n+t}z) - g(\phi_{n+t'}z)| < \epsilon$ and

$$\begin{aligned} |s_l - s'_l| &\leq |s - s'| + |S_n g(\phi_t z) - S_n g(\phi_{t'} z)|, \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

  Therefore we get

$$\begin{aligned} |x''_l x'_l| &= |u_l - u'_l|, \\ &\leq \frac{|s_l - s'_l| g(\phi_{n+t'}z) + s_l |g(\phi_{n+t}z) - g(\phi_{n+t'}z)|}{g(\phi_{n+t'}z) g(\phi_{n+t}z)}, \\ &\leq 2\epsilon \frac{\max g}{(\min g)^2}, \end{aligned}$$

- or $|n-n'|=1$, say $n=n'+1$. Then $s_l<\epsilon$ and $|s_l+g(\phi_{n'+t'}z)-s'_l|<\epsilon$ so that with the previous notations we get

$$\begin{aligned}|x''_l x'_l| &= |u_l-u'_l+1|,\\ &= \left|\frac{s_l}{g(\phi_{n+t}z)}-\frac{s'_l-g(\phi_{n'+t'}z)}{g(\phi_{n'+t'}z)}\right|,\\ &\leq \left|\frac{s_l}{g(\phi_{n+t}z)}\right|+\left|\frac{s'_l-g(\phi_{n'+t'}(z))}{g(\phi_{n'+t'}z)}\right|,\\ &\leq \frac{\epsilon}{\min g}+\frac{2\epsilon}{\min g}\leq 3\epsilon\frac{\max g}{(\min g)^2}.\end{aligned}$$

Consequently we get in any case

$$\begin{aligned}d_g(x_l,x'_l) &\leq |x_l x''_l|+|x''_l x'_l|,\\ &\leq \epsilon+3\epsilon\frac{\max g}{(\min g)^2}=c\epsilon.\end{aligned}$$

□

Assume that (X,Φ) admits a symbolic extension. Then so does (X,Φ_1) by Theorem 3.3. It follows from that $(X^g,(\phi_1)^g)$ also has symbolic extension (resp. principal) by Lemma 3.18 in [Bur19]. Since $(X^g,(\phi_1)^g)$ is a principal extension of $(X,\widehat{\Phi})$, the flow $(X,\widehat{\Phi})$ admits a symbolic extension (resp. principal). □

4. Singular suspension flows

In Section 2.1, we have defined the suspension over a discrete invertible topological system (X,T) with a positive continuous roof function $f:X\to\mathbb{R}_{>0}$. Assume now (X,T) has a fixed point $*$ and $f:X\to\mathbb{R}_{\geq 0}$ only vanishes on the fixed point $*$. Then we may define again the topological quotient space X^f by (2·1). We assume that $\sum_{k\in\mathbb{N}}f(T^kx)$ and $\sum_{k\in\mathbb{N}}f(T^{-k}x)$ goes uniformly to infinity out of any neighborhood of $*$, i.e.

$$\forall U \text{ open with } *\in U,\ \forall M>0,\ \exists K\in\mathbb{N}, \text{ s.t.}$$

$$\forall x\in X\setminus U\qquad \sum_{k=0}^{K}f(T^kx),\ \sum_{k=0}^{K}f(T^{-k}x)>M. \tag{4·1}$$

Lemma 4.1. *Under the hypothesis (4·1) the time translation still induces a topological flow T^f on X^f. Moreover the Bowen-Walters metric d_f* [4] *is compatible with the quotient topology on X^f.*

The flow (X^f,T^f) is called a *singular suspension flow*. When (X,T) is a subshift we speak of *singular symbolic flow*. In the following we write $*^f=(*,0)\in X^f$ the singularity of the flow.

Proof. The properties $\sum_{k\in\mathbb{N}}f(T^kx)=+\infty$ and $\sum_{k\in\mathbb{N}}f(T^{-k}x)=+\infty$ clearly ensures the orbit of the induced flow T^f are defined on $\mathbb{R}$ at any $(x,t)\in X^f$ with $x\neq *$. Moreover T^f is continuous on this open set $X^f\setminus\{*^f\}$. We let $T^f_t(*^f)=*^f$ for all $t\in\mathbb{R}$. Let us show that for a fixed $s\in\mathbb{R}$, the time-s map T^f_s is continuous at $*^f$. Without loss of generality we may assume $s>0$. We argue then by contradiction : assume there is a neighborhood V of $*^f$ in X^f such that $T^f_s(A)$ does not lie in V for $A\in X^f$ arbitrarily close to $*^f$. For any $x\in X\setminus\{*\}$, we let $k(x)$ be the largest integer k with $\sum_{l=0}^{k}f(T^{-l}x)<s$. Let U be an open neighborhood of $*$ with $\{(x,t)\in X^f,\ x\in U\}\subset V$ and we write $T^f_s(A)=(x^s_A,t^s_A)$. Then $x^s_A\in X\setminus U$ for A arbitrarily close to $*^f$. It follows from (4·1) that $k(x^s_A)$ is bounded for A arbitrarily close to $*^f$ by some $K>0$. Then $T^{-k(x^s_A)}x^s_A$ does not belong to the open neighborhood $\bigcap_{k=0}^{K}T^{-k}U$ of $*$. But $A=(T^{-k(x^s_A)}x^s_A,t^s_A+s-\sum_{l=0}^{k(x^s_A)}f(T^{-l}x^s_A))$ is close to $*^f=(*,0)$. This is a contradiction.

[4] We use the convention $u_{(*,0)}=0$.

To prove the Bowen-Walters metric is compatible with the topology on X^f, in comparison with the regular case, one only needs to check that for two points $A=(x_A,t_A)$ and $B=(x_B,t_B)$ lying on the same orbit of the flow (in particular $x_B=T^k(x_A)$ for some integer k) with $d_f(A,B)$ small, the minimum $d(T^\epsilon x_A,x_B)$ for $\epsilon\in\{0,-1,1\}$ is also small. Indeed if A is close to $*^f$ it could happen that the length of a sequence $A=A_1,\cdots,A_n=B$ with A_iA_{i+1} vertical for $i=1,\cdots,n-1$ is small, but k so large that $x_B=T^k(x_A)$ and x_A are far from each other, in particular x_B would lie far form the fixed point $*$. But (4·1) prevent this case as it implies k is bounded. □

We let $\mathcal{M}_T^*(X)$ (resp. $\mathcal{M}_{T^f}^*(X^f)$) denote the convex subset (not closed) of $\mathcal{M}_T(X)$ (resp. $\mathcal{M}_{T^f}(X^f)$) given by measures distinct from δ_* (resp. with $*^f$ zero measure). We also let $\mathcal{M}_T^\dagger(X)=\mathcal{M}_T(X)\setminus\mathcal{M}_T^*(X)$ (resp. $\mathcal{M}_{T^f}^\dagger(X^f)=\mathcal{M}_{T^f}(X^f)\setminus\mathcal{M}_{T^f}^*(X^f)$). Notice that if $\mu\in\mathcal{M}_{T^f}^\dagger(X^f)$ then $\mu(*^f)>0$. Then the map

$$\Theta:\mathcal{M}_T^*(X)\to\mathcal{M}_{T^f}^*(X^f)$$
$$\mu\mapsto\frac{(\mu\times\mathcal{L})_{|X^f}}{\int f\,d\mu}.$$

is a homeomorphism (not affine in general). Sometimes, we write $\nu_\mu=\Theta(\mu)$ for $\mu\in\mathcal{M}_T^*(X)$. Moreover Θ^{-1} extends continuously to $\mathcal{M}_{T^f}(X^f)$ in such a way that $\Theta^{-1}(\xi)=\delta_*$ for any $\xi\in\mathcal{M}_{T^f}^\dagger(X^f)$.

It follows from Abramov formula and the variational principle for the entropy that

$$h_{top}(T^f)=\sup_{\mu\in\mathcal{M}_T^*(X)}h(\Theta(\mu))=\sup_{\mu\in\mathcal{M}_T^*(X)}\frac{h(\mu)}{\int f\,d\mu}.$$

4.1. **Entropy structure of zero dimensional singular suspension flow.** For a partition P of X we let $A^*=A_P^*$ be the atom of P containing $*$, then $X^*=X\setminus A^*$ and P^* the partition of X^* induced by P. Fix also $\delta=\delta(P)=1/p$ with $p\in\mathbb{N}^*$ such that $f(x)>\frac{1}{p}$ for all $x\in X^*$.

Lemma 4.2. *For all $\mu\in\mathcal{M}_T(X)$ with $\mu(X^*)\neq 0$*

$$\frac{h(\nu_\mu,\phi_\delta,P_\delta)}{\delta}\geq\mu(X^*)\frac{h(\mu_*,T_*,P^*)}{\int f\,d\mu},\tag{4·2}$$

where P_δ is the partition of X^f given by $P_\delta:=\{X^f\setminus(X^*\times[0,\delta[),\ B\times[0,\delta[\ :\ B\in P^*\}$, T_* is the first return map in X^* w.r.t. T and $\mu_*\in\mathcal{M}_{T^*}(X^*)$ the measure induced by μ on X^*.

We follow the lines of the proof of Lemma 3.8 in [Bur19].

Proof. Let X_δ^* be the subset of X^f given by $X_\delta^*=X^*\times[0,\delta[/\sim$. For the partition P^* of X^* we denote by $P_\delta^*=\{B\times[0,\delta[,\ B\in P^*\}$ the partition induced by P on X_δ^*. We also let R_δ be the partition of X_δ^* with respect to the first return time in X_δ^* w.r.t. ϕ_δ and we let ϕ_δ^* the corresponding first return map. By applying Lemma 3.7 in [Bur19] to $\nu_\mu=\Theta(\mu)$, ϕ_δ and P_δ^* we get

$$\begin{aligned} h\left(\nu_\mu^\delta,\phi_\delta^*,P_\delta^*\vee R_\delta\right) &= \frac{h\left(\nu_\mu,\phi_\delta,P_\delta\right)}{\nu_\mu(B_\delta)},\\ &= \frac{\int f\,d\mu}{\delta\mu(A_\delta)}h\left(\nu_\mu,\phi_\delta,P_\delta\right).\end{aligned}\tag{4·3}$$

But the partition $\bigvee_{k=0}^{n-1}(\phi_\delta^*)^{-k}P_\delta^*$ of X_δ^* is just the partition $\bigvee_{k=0}^{n-1}T_*^{-k}P^*\times[0,\delta[$. Therefore we get, with $H_\mu(Q)=\sum_{C\in Q}-\mu(C)\log\mu(C)$:

$$\begin{aligned} h\left(\nu_\mu^\delta,\phi_\delta^*,P_\delta^*\right) &= \lim_n\frac{1}{n}H_{\nu_\mu^\delta}\left(\bigvee_{k=0}^{n-1}T_*^{-k}P^*\times[0,\delta[\right),\\ &= \lim_n\frac{1}{n}\sum_{C\in\bigvee_{k=0}^{n-1}T_*^{-k}P^*\times[0,\delta[}-(\nu_\mu^\delta(C)\log(\nu_\mu^\delta(C)),\end{aligned}$$

where ν_μ^δ denotes the probability measure induced by ν_μ on X_δ^*. For any $B \in \bigvee_{k=0}^{n-1} T_*^{-k} P^*$ and $C = B \times [0, \delta[$ we have $\nu_\mu^\delta(C) = \frac{\nu_\mu(C)}{\nu_\mu(X_\delta^*)} = \mu(B)$. Therefore we obtain finally :

$$h\left(\nu_\mu^\delta, \phi_\delta^*, P_\delta^*\right) = h(\mu_*, T_*, P^*),$$

which together (4·3) implies the required inequality (4·2).

□

A subset of X is said to have a *small boundary* when its boundary has zero μ-measure for any $\mu \in \mathcal{M}_T(X)$. The partition P has a small boundary, when its atoms have small boundary. We assume now (X, T) has the *small boundary property*, i.e. X has a topological base in which every element has a small boundary. Therefore there exists a sequence $(P_k)_k$ of small boundary partitions of X with $\mathrm{diam}(P_k) \xrightarrow{k} 0$. Then the sequence of entropy functions with respect to $(P_k)_k$ is an entropy structure (see [Dow05]). Let $(\delta_k)_k = (1/p_k)_k$ be the associated sequence of parameters $(\delta(P_k))_k$. Then $\frac{h(\nu_\mu, \phi_\delta, (P_k)_\delta)}{\delta} = h(\nu_\mu, \phi_1, Q_k)$ with $Q_k := \bigvee_{l=0}^{p_k - 1} \phi_\delta^{-l}(P_k)_{\delta_k}$. The partitions $Q = Q_k$ have also small boundary (w.r.t. the suspension flow) and for any $\mu \in \mathcal{M}_T^*(X)$ we get with $H(t) = -t\log t - (1-t)\log(1-t)$ for all $t \in [0, 1]$:

$$\begin{aligned} h(\nu_\mu, \phi_1, Q) &\geq \mu(X^*)\frac{h(\mu_*, T_*, P^*)}{\int f\, d\mu}, \\ &\geq \frac{h(\mu, T, P) - h(\mu, T, \{X^*, A_*\})}{\int f\, d\mu}, \\ &\geq \frac{h(\mu, T, P) - H(\mu(A_*))}{\int f\, d\mu}, \\ &\xrightarrow{Q=Q_k,\ k\to+\infty} \frac{h(\mu, T)}{\int f\, d\mu} = h(\nu_\mu, \phi_1). \end{aligned} \tag{4·4}$$

From Corollary 3.2 in [Bur19] it follows that $(h(\cdot, Q_k))_k$ defines an entropy structure of the flow (X^f, T^f). Following [Dow05] we say for two sequences $\mathcal{G} = (g_k)_k$ and $\mathcal{H} = (h_k)_k$ of real functions defined on the same metric space $\mathfrak{X}$ that $\mathcal{H}$ *yields* $\mathcal{G}$, written $\mathcal{H} \succ \mathcal{G}$, when for all $\epsilon > 0$ and $\mu \in \mathfrak{X}$ there exists a neighborhood U of μ such that

$$\limsup_{k\to\infty} \limsup_{j\to\infty} \sup_{\mu\in U} (g_j - h_k)(\mu) \leq \epsilon.$$

When $\mathfrak{X}$ is compact this is equivalent to

$$\limsup_{k\to\infty} \limsup_{j\to\infty} \sup_{\mu\in \mathfrak{X}} (g_j - h_k)(\mu) \leq 0.$$

Two entropy structures $\mathcal{H}$ and $\mathcal{G}$ of a topological system or flow satisfy $\mathcal{H} \succ \mathcal{G}$ and $\mathcal{G} \succ \mathcal{H}$ (see [Dow05, Bur19]).

Lemma 4.3. *Let $\mathcal{H}_T = (h_k)_k$ be an entropy structure of T and $\mathcal{G} = (g_k)_k$ be the sequence of real functions g_k defined on $\mathcal{M}_{T^f}^*(X^f)$ by $g_k = h_k \circ \Theta^{-1}$. Then the restriction to $\mathcal{M}_{T^f}^*(X^f)$ of any entropy structure of (X^f, T^f) yields $\mathcal{G}$.*

Proof. Without loss of generality we may assume $\mathcal{H}$ is the sequence $(h(\cdot, P_k))_k$ with $(P_k)_k$ as above and consider the associated entropy structure $h(\cdot, \phi_1, Q_k)$ of the suspension flow. Fix $\nu = \Theta(\mu) \in \mathcal{M}_{T^f}^*(X^f)$. For j large enough, $\mu(A_{P_j}^*)$ is so small that $\frac{H(\xi(A_{P_j}^*))}{\int f\, d\xi} \leq \epsilon$ for all ξ in a neighborhood U of μ in $\mathcal{M}_T^*(X)$. Then it follows from (4·4) with $\nu_\mu = \Theta(\mu)$ that

$$\begin{aligned} \limsup_{k\to\infty} \limsup_{j\to\infty} \sup_{\nu_\xi \in \Theta(U)} \left(h(\nu_\xi, \phi_1, Q_j) - \frac{h(\xi, T, P_k)}{\int f\, d\xi} \right) &\geq \limsup_{j\to\infty} \sup_{\xi\in U} \left(h(\nu_\xi, \phi_1, Q_j) - \frac{h(\xi, T, P_j)}{\int f\, d\xi} \right), \\ &\geq -\limsup_{j} \sup_{\xi\in U} \frac{H(\xi(A_*))}{\int f\, d\xi} \geq -\epsilon. \end{aligned}$$

□

4.2. **Singular suspension flows with small entropy at the singularity.** Under some criterion on the entropy function at the singularity, we manage to build a symbolic extension of the suspension flow (X^f, T^f) from a symbolic extension of (X, T).

Proposition 4.4. *Let (X^f, T^f) be a singular suspension flow. Assume the associated discrete dynamical system (X, T) admits a symbolic extension π with $\lim_{\mu\to\delta_*} \frac{h^\pi(\mu)}{\int f\, d\mu} = 0$.*

Then the suspension flow admits a symbolic extension π' with $h^{\pi'} \leq g^{har}$ where g^{har} denotes the harmonic extension of the function $g : \mathcal{M}_{T^f}(X^f) \to \mathbb{R}_{\geq 0}$ defined by $g(\nu) = \frac{h^\pi(\mu)}{\int f\, d\mu}$ for $\nu = \Theta(\mu) \in \mathcal{M}^*_{T^f}(X^f)$ and $g(\nu) = 0$ for others ν.

Proof. The function h^π is upper semi-continuous on $\mathcal{M}_T(X)$ and $\mu \mapsto \int f\, d\mu$ is continuous and positive on $\mathcal{M}^*_T(X)$. Therefore $G : \mu \mapsto \frac{h^\pi(\mu)}{\int f\, d\mu}$ is upper semi-continuous on $\mathcal{M}^*_T(X)$ and thus so is $g = G \circ \Theta^{-1}$ on $\mathcal{M}^*_{T^f}(X^f)$. When ν belongs to $\mathcal{M}^\dagger_{T^f}(X^f)$, we have also $\lim_{\xi\to\nu} g(\xi) \leq g(\nu) = 0$. Indeed we can assume ξ lies in $\mathcal{M}^*_{T^f}(X^f)$ because g is zero on $\mathcal{M}^\dagger_{T^f}(X^f)$. Then

$$\begin{aligned}\lim_{\xi\to\nu} g(\xi) &\leq \lim_{\xi\to\nu} G \circ \Theta^{-1}(\xi),\\ &\leq \lim_{\mu\to\delta_*} G(\mu) = 0.\end{aligned}$$

Arguing as in Lemma 3.13 in [Bur19] the function g is also affine on $\mathcal{M}^*_{T^f}(X^f)$. It is also affine on $\mathcal{M}^\dagger_{T^f}(X^f)$ as g is identically zero on this set. Then if $\nu \in \mathcal{M}^\dagger_{T^f}(X^f)$ and $\xi \in \mathcal{M}^*_{T^f}(X^f)$, the measure $\lambda\nu + (1-\lambda)\xi$ belongs to $\mathcal{M}^\dagger_{T^f}(X^f)$ for all $\lambda \in]0, 1]$, therefore

$$0 = g(\lambda\nu + (1-\lambda)\xi) \leq \lambda g(\nu) + (1-\lambda) g(\xi).$$

In conclusion, the function g is upper semi-continuous and convex. Its harmonic extension g^{har} is also upper semi-continuous by Fact A2.20 in [Dow11]. Observe also $g^{har} \geq h^{T^f}$. We show now g^{har} is a (affine) superenvelope, which will conclude the proof of the proposition by Theorem 3.1. By Lemma 8.2.14 in [Dow11], it is enough to check for some entropy structure $\mathcal{H}^{T^f} = (h^{T^f}_k)_k$ with harmonic functions $h^{T^f}_k$

$$\lim_k (g^{har} - h^{T^f}_k)\tilde{}^e = g^{har} - h^{T^f}$$

with $\tilde{f}^e(\nu) = \limsup_{\substack{\xi\to\nu \\ \xi\in\mathcal{M}^e_{T^f}(X^f)}} f(\xi)$ for any real function defined on $\mathcal{M}_{T^f}(X^f)$. When $\nu = \Theta(\mu)$ belongs to $\mathcal{M}^*_{T^f}(X^f)$ any ergodic measure ξ going to ν also lies in $\mathcal{M}^*_{T^f}(X^f)$, so that by Lemma 4.3 for any entropy structure $\mathcal{H}^T = (h^T_k)_k$ we get

$$\begin{aligned}\lim_k (g^{har} - h^{T^f}_k)\tilde{}^e(\nu) &\leq \lim_k \frac{(h^\pi - h^T_k)\tilde{}^e(\mu)}{\int f\, d\mu},\\ &\leq \frac{h^\pi - h^T(\mu)}{\int f\, d\mu} = (g^{har} - h^{T^f})(\nu).\end{aligned}$$

Finally let $\nu \notin \mathcal{M}^*_{T^f}(X^f)$ and ξ_n be a sequence of ergodic measures going to ν. Then we may write $\xi_n = \Theta(\mu_n)$ with $\mu_n \xrightarrow{n} \delta_*$ so that

$$\begin{aligned}\lim_k (g^{har} - h^{T^f}_k)\tilde{}^e(\nu) &\leq (g^{har})\tilde{}^e(\nu),\\ &\leq \limsup_{\mu\to\delta_*} \frac{h^\pi(\mu)}{\int f\, d\mu},\\ &= 0,\\ &\leq (g^{har} - h^{T^f})(\nu).\end{aligned}$$

□

Corollary 4.5. *Assume (X,T) admits a principal symbolic extension and $\lim_{\mu\to\delta_*}\frac{h(\mu)}{\int f\,d\mu}=0$, then the suspension flow (X^f,T^f) also admits a principal symbolic extension. In particular, if (X,T) has topological entropy zero, then (X^f,T^f) admits a symbolic extension with zero topological entropy.*

4.3. **Noninvariance under orbit equivalence.** Contrarily to regular flows, two singular flows with finite topological entropy may be orbit equivalent but one admitting a symbolic extension and not the other.

Proposition 4.6. *There are two orbit equivalent (singular) flows (X,Φ) and (X,Ψ) with $h_{top}(\Phi)=h_{top}(\Psi)<+\infty$ such that (X,Φ) admits a symbolic extension but not (X,Ψ).*

Proof. For any $n\in\mathbb{N}$ there exists a topological system (X_n,T_n) with $h_{top}(T_n)=4^{-n}$ and $h_{sex}(T_n)=3^{-n}$ (See Theorem D.1 in [BFF02]). We may let (X_0,T_0) be a subshift. Let $X=\coprod_{n\in\mathbb{N}}X_n\cup\{*\}$ be the one point compactification of the X_n's. We let $T:X\circlearrowleft$ defined by $T|_{X_n}=T_n$ and $T*=*$. The symbolic extension entropy $h_{sex}(T)$ of (X,T) satisfies $h_{sex}(T)=\sup_n h_{sex}(T_n)=1$. Then we consider the roof functions f and f' with $f(*)=f'(*)=*$ and $f|_{X_n}=\frac{1}{2^n}$, $f'|_{X_n}=\frac{1}{4^n}$ for all integers n. The hypothesis (4·1) is easily checked in this case. The topological entropies of the associated singular suspended flow Φ_f and $\Phi_{f'}$ satisfy $h_{top}(T^f)=h_{top}(\Phi_{f'})=h_{top}(T_0)=1$. Moreover we have

$$h_{sex}(T^{f'})\geq\frac{h_{sex}(T_n)}{f'|_{X_n}}=(4/3)^n$$

for all $n\in\mathbb{N}$ so that $\Phi_{f'}$ does not admit any symbolic extension. We check now $h_{sex}(T^f)=1$.

For any n there is a symbolic extension of the time-1 map of $(T^f)|_{X_n^f}$ with topological entropy less than $(3/4)^n$. We let $\underline{E_n}:\mathcal{M}(X_n^f,(T^f)_1)\to\mathbb{R}$ be the associated superenvelope given by the entropy function of this extension. Each $\underline{E_n}$ may be extended to an affine upper semi-continuous function E_n on $\mathcal{M}(X^f,(T^f)_1)$ with $E_n(\mu)=0$ for $\mu(X_n^f)=0$. As $\underline{E_n}$ and thus E_n is bounded from above by $(3/4)^n$, the function $E:=\sum_n E_n$ defines an affine upper semi-continuous function. Let $\mathcal{H}=(h_k)$ be an entropy strucutre of $\left(X^f,(T^f)_1\right)$. By using again Lemma 8.2.14 in [Dow11], to show E is superenvelope it is enough to check for all μ in the closure of ergodic measures and for all $\epsilon>0$, there exists k such that

$$\limsup_{\nu\to\mu,\ \nu\text{ ergodic}}(E-h_k)(\nu)\leq(E-h)(\mu)+\epsilon.$$

Either μ is supported on some X_n, then so does any ergodic measure ν close enough to μ and we may conclude in this case since $E(\mu)=E_n(\mu)$, $E(\nu)=E_n(\nu)$ and E_n is a superenvelope of the time-1 map of the flow on $(X_n)_f$. Or μ is the Dirac mass at $*$ and $\limsup_{\nu\to\mu,\ \nu\text{ ergodic}}(E-h_k)(\nu)\leq\limsup_n\sup_\xi E_n(\xi)=0=E(\delta_*)$. Therefore E is an affine superenvelope of the time-1 map of T^f, so that by Theorem 2.1 the suspension flow T^f admits a symbolic extension with topological entropy equal to $\sup_\mu E(\mu)=1$. □

Remark 4.7. *In the above example, the discrete system (X,T) admits a symbolic extension, however the singular suspension flow $(X^{f'},T^{f'})$ has finite topological entropy, but no symbolic extensions.*

4.4. **Universal symbolic suspension flow.** In the following we consider the two full shift with the singularity given by the infinite zero sequence, i.e. $(X,T)=\left(\{0,1\}^{\mathbb{Z}},\sigma\right)$ and $*=0^\infty$. We let d be the usual distance on X given by

$$d((u_n)_n,(v_n)_n)=2^{-\min\{|n|,\ u_n\neq v_n\}}.$$

In particular we have $d((u_n)_n,*)=2^{-k_u}$ with $k_u=\min\{|n|,\ u_n=1\}$.

We investigate the entropy and symbolic extension of a singular symbolic suspension flow over (X,T) associated to a roof function of the form $f(x)=R(d(x,*))$ for some continuous function $R:[0,1]\to\mathbb{R}_{\geq0}$ with $R(*)=0$ and $R(x)>0$ for $x\neq*$. We may also write f as $f(u)=g(k_u)$ for $u\neq0$ and for a function $g:\mathbb{N}\to\mathbb{R}_{\geq0}$ with $\lim_{k\to+\infty}g(k)=0$ by letting $g=R(2^{-\cdot})$. With these notations the singular symbolic flow X^f is well defined (i.e. (4·1) holds true) if and only if $\sum_{k\in\mathbb{N}} g(k) = +\infty$.

Theorem 4.8. *Assume the sequence $(kg(k))_{k\in\mathbb{N}}$ is converging to $l \in \mathbb{R}^+ \cup \{+\infty\}$ when k goes to infinity. Then*

(1) *if $l = 0$, the topological entropy of (X^f, T^f) is infinite,*
(2) *if $0 < l < +\infty$, then the symbolic extension entropy of (X^f, T^f) satisfies*

$$\forall \nu \in \mathcal{M}_{T^f}(X^f),\ h_{sex}(\nu) = \tilde{h}(\nu) = h(\nu) + \frac{1}{2l}\nu(*^f),$$

(3) *if $l = +\infty$ the flow (X^f, T^f) admits a principal symbolic extension.*

Question 4.9. *Does any singular symbolic suspension flow with finite topological entropy admit a symbolic extension?*

As a consequence of Theorem 4.8, we present the following examples. Let $g(k) = 1/k$, $g'(k) = 1/\sqrt{k}$ and $g''(k) = 1/k\log k$ for all k. The associated suspension flows (X^f, T^f), $(X^{f'}, T^{f'})$ and $(X^{f''}, T^{f''})$ are orbit equivalent. Since $kg(k) \to 1$ and $kg'(k) \to +\infty$ as $k \to +\infty$, it follows from Theorem 4.8 that $(X^{f'}, T^{f'})$ admits a principal symbolic extension contrarily to (X^f, T^f). Moreover they have both finite topological entropy, but $h_{top}(T^{f''}) = +\infty$. There exist also orbit equivalent singular topological flows (X, Φ) and (X, Ψ) with $h_{top}(\Phi) = 0$ and $h_{top}(\Psi) = +\infty$ [SZ11].

Corollary 4.10. *There are two orbit equivalent singular symbolic flows (X, Φ) and (X, Ψ) with $h_{top}(\Phi) = h_{top}(\Psi) < +\infty$ such that (X, Φ) admits a principal symbolic extension but not (X, Ψ).*

Corollary 4.10 and Proposition 4.6 raise the following question :

Question 4.11. *Do there exist orbit equivalent singular topological flows with equal topological entropy one without any symbolic extension and the other with a principal one?*

Now we present the proof of Theorem 4.8.

Proof of Theorem 4.8. (1) We first show that

$$\limsup_{\nu \to \delta_{*^f}} h(\nu) \geq \frac{1}{2l}. \tag{4·5}$$

This clearly implies the first item. Moreover the symbolic extension entropy function h_{sex} is upper semi-continuous and concave, therefore writing $\nu \in \mathcal{M}_{T^f}(X^f)$ as $\nu = \nu(*^f)\delta_{*^f} + (1-\nu(*^f))\xi$ with $\xi \in \mathcal{M}^*_{T^f}(X^f)$ we get a first inequality in the second item

$$\begin{aligned} h_{sex}(\nu) &\geq (1-\nu(*^f))h_{sex}(\xi) + \nu(*^f)h_{sex}(\delta_{*^f}), \\ &\geq (1-\nu(*^f))h(\xi) + \nu(*^f)\limsup_{\eta \to \delta_{*^f}} h(\eta), \\ &\geq h(\nu) + \frac{1}{2l}\nu(*^f). \end{aligned}$$

To prove (4·5) we consider the Bernoulli measure $\mu_\lambda \in \mathcal{M}_\sigma(\{0,1\}^{\mathbb{Z}})$ with parameter $\lambda = \mu([1])$ where $[1]$ denotes the cylinder $[1] := \{(u_n)_n \in \{0,1\}^{\mathbb{Z}} :\ u_0 = 1\}$. We also denote by $[0^{2k+1}]$ for $k \in \mathbb{N}$ the cylinder $[0^{2k+1}] := \{(u_n)_n \in \{0,1\}^{\mathbb{Z}} :\ u_m = 0 \text{ for } |m| \leq k\}$. We compute for λ close to 0 :

$$\begin{aligned} \int f\, d\mu_\lambda &= \mu_\lambda([1]) + \sum_{k\in\mathbb{N}} g(k)\mu_\lambda\left([0^{2k+1}] \setminus [0^{2(k+1)+1}]\right), \\ &= \lambda + l\sum_{k\in\mathbb{N}} \frac{\lambda(2-\lambda)(1-\lambda)^{2k+1}}{k} + O(\lambda), \\ &= \lambda - l\lambda(1-\lambda)(2-\lambda)\log(\lambda(2-\lambda)) + O(\lambda), \\ &= -2l\lambda\log\lambda + O(\lambda), \end{aligned}$$

where $\limsup_{\lambda\to 0}\left|\frac{O(\lambda)}{\lambda}\right| < +\infty$. As the entropy of μ_λ is equal to $-\lambda\log\lambda-(1-\lambda)\log(1-\lambda) = -\lambda\log\lambda+O(\lambda)$, we get

$$h(\nu_{\mu_\lambda}) = \frac{h(\mu_\lambda)}{\int f\,d\mu_\lambda} = \frac{-\lambda\log\lambda+O(\lambda)}{-2l\lambda\log\lambda+O(\lambda)},$$
$$\xrightarrow{\lambda\to 0} \frac{1}{2l}.$$

Moreover the T^f-invariant measure ν_{μ_λ} is going to the Dirac measure at the singularity when λ goes to zero. Indeed for all fixed k, we have for some polynomial P_k with $P_k(0)\neq 0$:

$$\nu_{\mu_\lambda}([0^{2k+1}]\times\mathbb{R}/\sim) = 1-\frac{\int_{X\setminus[0^{2k+1}]} f\,d\mu_\lambda}{\int_X f\,d\mu_\lambda},$$
$$\geq 1-\frac{\lambda P_k(\lambda)+O(\lambda)}{\int_X f\,d\mu_\lambda},$$
$$\xrightarrow{\lambda\to 0} 1.$$

(3) Let us show now the last item assuming the second one. Assume $kg(k)$ goes to infinity as k tends to infinity. For $a>0$, let $g_a(k)=\min(g(k),\frac{a}{k})$ for all k. The associated roof function f_a satisfies the hypothesis of the second item, that is, $\lim_{k\to\infty} kg_a(k)=a$. The symbolic extension entropy function is upper semi-continuous, therefore $\tilde{h}\leq h_{sex}$ and we get for all $a>0$:

$$\limsup_{\mu\to\delta_*}\frac{h(\mu)}{\int f\,d\mu} \leq \limsup_{\mu\to\delta_*}\frac{h(\mu)}{\int f_a\,d\mu},$$
$$\leq \tilde{h}(\delta_{*^{f_a}}),$$
$$\leq h_{sex}(\delta_{*^{f_a}}) = \frac{1}{2a}.$$

By letting a go to $+\infty$, we get $\limsup_{\mu\to\delta_*}\frac{h(\mu)}{\int f\,d\mu}=0$. By applying Corollary 4.5, the singular suspension flow (X^f,T^f) admits a principal symbolic extension.

(2) We prove now the second item. For g with $\lim_k kg(k)=l$ we build a symbolic extension of (X^f,T^f) with $h^\pi(\nu)\leq h(\nu)+\frac{\nu(*^f)}{2l}$ for all $\nu\in\mathcal{M}(X^f,T^f)$. This will conclude the proof of Theorem 4.8.

Step 1 : Construction of a principal extension by a regular suspension flow $\pi:(Y^{f'},S^{f'})\to(X^f,T^f)$.
We first build a principal extension of (X^f,T^f) by a regular suspension flow $(Y^{f'},S^{f'})$. For $x\in\{0,1\}^{\mathbb{Z}}$ we let $k_x^+=\min\{n\in\mathbb{N}^*,\ x_n=1\}$ et $k_x^-=\min\{n\in\mathbb{N},\ x_{-n}=1\}$ and we denote by $\mathbf{k}_x$ the pair (k_x^-,k_x^+). We consider the partition of $\mathbb{E}:=(\mathbb{N}\cup\{+\infty\})\times(\mathbb{N}\cup\{+\infty\})\setminus\{(+\infty,+\infty)\}$ into the following subsets of points $\mathbf{k}=(k^-,k^+)\in\mathbb{E}$:

- $\mathcal{R}_1:=\{k^-=0\}$,
- $\mathcal{R}_2:=\{0<k^-\leq\frac{1}{3}k^+\}$,
- $\mathcal{R}_3:=\{k^+>k^->\frac{1}{3}k^+>0\}$,
- $\mathcal{R}_4:=\{0<k^+\leq k^-\}$.

We let $L:\mathbb{E}\to\mathbb{N}$ be the function satisfying for all $\mathbf{k}=(k^-,k^+)\in\mathbb{E}$:

(1) $L(\mathbf{k})=1$ for $\mathbf{k}\in\mathcal{R}_1$,
(2) $L(\mathbf{k})=k^-$ for $\mathbf{k}\in\mathcal{R}_2$,
(3) $L(\mathbf{k})=k^+-\lfloor\frac{(k^-+k^+)^2}{8k^-}\rfloor$ for $\mathbf{k}\in\mathcal{R}_3$,
(4) $L(\mathbf{k})=\lceil k^+/2\rceil$ for $\mathbf{k}\in\mathcal{R}_4$.

Lemma 4.12. *For all $\mathbf{k}=(k^-,k^+)\in\mathcal{R}_3$, we have with $R(p,q)=\sum_{j=p}^{q}\frac{1}{j}$ for positive integers $q>p$:*

(1) $L(\mathbf{k})>\frac{k^+-k^-}{2}\geq 0$,

(2) $L(\mathbf{k}) \leq \lceil \frac{k^+}{2} \rceil$,
(3) $\left| k^+ + k^- - \lceil \sqrt{8k^-(k^+ - L(\mathbf{k}))} \rceil \right| \leq 4$,
(4) $R\left(k^-, \lfloor \frac{k^+ + k^-}{2} \rfloor\right) + R\left(k^+ - L(\mathbf{k}), \lceil \frac{k^+ + k^-}{2} \rceil\right) \xrightarrow{k^- \to +\infty} \log 2$.

Proof. One checks easily the two first items. Let us just show the two last ones. From the definition of L on $\mathcal{R}_3$ we have by using $\sqrt{1-x} \geq 1 - x$ for all $x \in [0,1]$:

$$\sqrt{8k^- \lfloor \frac{(k^- + k^+)^2}{8k^-} \rfloor} = \sqrt{8k^-(k^+ - L(\mathbf{k}))} \leq k^+ + k^-,$$
$$\sqrt{(k^- + k^+)^2 - 8k^-} \leq \sqrt{8k^-(k^+ - L(\mathbf{k}))} \leq k^+ + k^-,$$
$$(k^+ + k^-)\left(1 - \frac{8k^-}{(k^- + k^+)^2}\right) \leq \sqrt{8k^-(k^+ - L(\mathbf{k}))} \leq k^+ + k^-,$$
$$k^+ + k^- - 4 \leq \sqrt{8k^-(k^+ - L(\mathbf{k}))} \leq k^+ + k^-.$$

The second line makes sense only if $8k^- \leq (k^+ + k^-)^2$. But $8k^- > (k^+ + k^-)^2$ implies $k^- \leq 1$ and $k^+ \leq 2$. In this remaining case we have therefore again $\left| k^+ + k^- - \lceil \sqrt{8k^-(k^+ - L(\mathbf{k}))} \rceil \right| \leq 4$.

Finally we prove item (4). Observe that k^+ and k^- are going simultaneously to infinity for $\mathbf{k} \in \mathcal{R}_3$. Then as $R(p,q) = \log \frac{q}{p} + \frac{o(p)}{p}$ for large p where $\lim_{p \to \infty} \frac{o(p)}{p} = 0$, we get

$$R\left(k^-, \lfloor \frac{k^+ + k^-}{2} \rfloor\right) + R\left(k^+ - L(\mathbf{k}), \lceil \frac{k^+ + k^-}{2} \rceil\right) = \log\left(\frac{(k^+ + k^-)^2}{4k^-(k^+ - L(\mathbf{k}))}\right) + \frac{o(k^-)}{k^-},$$
$$\xrightarrow{k^- \to +\infty} \log 2.$$

□

Let $S : \{0,1\}^{\mathbb{Z}} \circlearrowleft$ be the topological system defined as $Sx = \sigma^{L(\mathbf{k}_x)}x$ for $x \neq *$ and $S* = *$. The map S is continuous at any $x \neq *$ because in this case $L(\mathbf{k}_y)$ is constant for y in some neighborhood of x. Now if x is close to the zero sequence then k_x^+ and k_x^- are large. Moreover we have always $0 \leq L(\mathbf{k}_x) \leq \lceil \frac{k_x^+}{2} \rceil$ and therefore $y = \sigma^{l(\mathbf{k}_x)}x$, which satisfies $y_n = 0$ for $n = -k_x^-, \cdots, \lfloor \frac{k_x^+}{2} \rfloor$, is also close to the zero sequence. For $x \in X$ with $x_0 = 1$, we let $p_x = \min\{n \in \mathbb{N}^*,\ (S^n x)_0 = 1\}$.

Lemma 4.13. *For all $x \in X$ with $x_0 = 1$ and $3 \leq k_x^+ < +\infty$ there exists a unique $0 < r < p_x$ such that*

- $\mathbf{k}_{S^q x} \in \mathcal{R}_2$ *for* $1 \leq q < r$,
- $\mathbf{k}_{S^r x} \in \mathcal{R}_3$,
- $\mathbf{k}_{S^q x} \in \mathcal{R}_4$ *for* $r < q \leq p_x - 1$.

Moreover

(1) *for* $1 \leq q \leq r$ *we have*
$$k^-_{S^q x} = 2^{q-1},$$
(2) *there exists a sequence* $\epsilon_{r+1}, \cdots, \epsilon_{p_x - 2} \in \{0,1\}$ *depending only on* k_x^+ *such that for* $r < q < p_x$ *we have*
$$k^+_{S^q x} = 2^{p_x - 1 - q} + \sum_{0 \leq i < p_x - q - 1} \epsilon_{q+i} 2^i.$$
(3) $p_x - 2 - r \leq \lceil \frac{p_x}{2} \rceil - 1$.

Proof. We only prove the two last items (2) and (3), as the other conclusion is easily checked. For $p_x - 1 > q > r$ we have $k^+_{S^{q+1}x} = k^+_{S^q x} - \lceil \frac{k^+_{S^q x}}{2} \rceil$, therefore $k^+_{T^q x} = 2k^+_{S^{q+1}x} + \epsilon_q$ with $\epsilon_q \in \{0,1\}$ depending on the parity of $k^+_{S^q x}$. Also $k^+_{S^{p_x - 1}x} = 1$. Then we get by a direct induction the desired formula for $k^+_{T^q x}$.

Concerning the last item, we have $\mathbf{k}_{S^r x} \in \mathcal{R}_3$, so that $k^+_{S^r y} > k^-_{S^r y} = 2^{r-1} > \frac{k^+_{S^r y}}{3}$. Then we have $2^{p_x - r - 2} \leq k^+_{S^{r+1}x} \leq k^+_{S^r y} \leq 3 \cdot 2^{r-1}$, thus $2^{p_x} \leq 2^{2r+3}$, i.e. $r \geq \frac{p_x - 3}{2}$.

□

Remark 4.14. *The integers p_x, r and the sequence $\epsilon_{r+1}, \cdots, \epsilon_{p_x-2} \in \{0,1\}$ only depend on k_x^+ for $x \in X$ satisfying $3 \leq k_x^+ < +\infty$.*

Now we consider the roof function f' on X given by

$$\forall x \in X \setminus \{*\},\ f'(x) = \sum_{k=0}^{L(\mathbf{k}_x)-1} f(\sigma^k x)$$

$$f'(*) = l \log 2.$$

Lemma 4.15. *f' is continuous on X.*

Proof. Clearly it is enough to check the continuity at $*$, i.e. when k_x^- and k_x^+ both go to infinity. We have

- $f'(x) \sim lR(k_x^-, 2k_x^-)$ when k_x lies in $\mathcal{R}_2$,
- $f'(x) \sim lR\left(k_x^-, \lfloor \frac{k_x^+ + k_x^-}{2} \rfloor\right) + lR\left(k_x^+ - L(\mathbf{k}_x), \lceil \frac{k_x^+ + k_x^-}{2} \rceil\right)$, when k_x lies in $\mathcal{R}_3$,
- $f'(x) \sim lR(\lfloor \frac{k_x^+}{2} \rfloor, k_x^+)$, when k_x lies in $\mathcal{R}_4$.

In all cases (see Lemma 4.12 (4) for the second case) we get $f'(x) \xrightarrow{x \to *} l \log 2$.

□

Let $Y = \bigcap_{n \in \mathbb{N}} S^n X$. Any sequence $x = (x_n)_n \in X = \{0,1\}^{\mathbb{Z}}$ with $x_n = 0$ for $n \leq 0$ or with $x_0 = 1$ belongs to Y. In particular for any $x \in X$ there is $k > 0$ with $\sigma^{-k} x \in Y$. We let $(Y^{f'}, S^{f'})$ be the suspension flow over $(Y, S|_Y)$ with roof function f'.

We also denote by Z the subset of X given by sequences with infinitely many 1's in the future and in the past. The set of recurrent points of $(Y, S|_Y)$ is given by $(Y \cap Z) \cup \{*\}$.

Lemma 4.16. *The map $\pi : (Y^{f'}, S^{f'}) \to (X^f, T^f)$, $(x,t) \mapsto T_t^f(x,0)$ is a principal topological extension.*

Proof. To prove the extension property, it is enough to see $\pi(x, f'(x)) = \pi(Sx, 0)$, i.e. $T^f_{f'(x)}(x,0) = (Sx, 0)$ for all $x \in Y$ which follows from

$$\begin{aligned}
T^f_{f'(x)}(x,0) &= T^f_{\sum_{l=0}^{L(\mathbf{k}_x)-1} f(\sigma^k x)}(x,0), \\
&= T^f_{f(\sigma^{L(\mathbf{k}_x)-1}x)} \circ \cdots \circ T^f_{f(x)}(x,0), \\
&= (\underbrace{\sigma \circ \cdots \circ \sigma}_{L(\mathbf{k}_x) \text{ times}}(x), 0), \\
&= (Sx, 0).
\end{aligned}$$

The factor map π is surjective. Indeed for any $x \in \{0,1\}^{\mathbb{Z}}$ there is $k > 0$ with $\sigma^{-k} x = y \in Y$. Therefore for $(x,t) \in X^f$ there is $s \geq 0$ with $\pi(y,s) = T_s^f(y,0) = (x,t)$. The recurrent points different from $*^f$ in X^f are contained in the subset $Z \times \mathbb{R}/\sim$ of X^f. For $x \in Z$ we let $y = \sigma^{-k} x \in Y$ where k is the smallest nonnegative integer with $\sigma^{-k} x \in Y$. Then $\pi^{-1}(x,t) = \{(y,s)\}$ for some $0 \leq s \leq f'(y)$. Moreover $\pi^{-1} *^f$ is contained in the subset $\{0\} \times \mathbb{R}/\sim$ of $Y^{f'}$ and the suspension flow $T^{f'}$ restricted to this set is topologically conjugated to the translation flow on the circle. In any case we have $h_{top}(\pi^{-1}(x,t)) = 0$ for all recurrent points (x,t) of (X^f, T^f). By Ledrappier-Walters formula, π is a principal extension. □

Step 2 : A symbolic extension χ of (Y,S). To conclude the proof of Theorem 4.8 we investigate the symbolic extensions of (Y,S). We will build a symbolic extension χ of (Y,S) (with an embedding) with entropy function h^χ equal to $\mu \mapsto h(\mu) + \frac{\mu(*) \log 2}{2}$.

Let us call a *block* any finite word of the form $10^l := 1\underbrace{0 \cdots 0}_{l \text{ times}}$ with $l \in \mathbb{N}$. Any sequence in $Y \cap Z$ is an infinite concatenation of such blocks. A map $\psi : Y \cap Z \to \mathcal{A}^{\mathbb{Z}}$ for some alphabet $\mathcal{A}$ is said a *block code map* if for any $x \in Z \cap Y$ whose orbit is given by the concatenation of blocks $(B_n)_{n \in \mathbb{Z}}$

the orbit of $\psi(x)$ under the shift on $\mathcal{A}^{\mathbb{Z}}$ is the concatenation of $(\Psi(B_n))_n$ for some map Ψ from the set of all blocks to $\bigcup_{n\in\mathbb{N}}\mathcal{A}^n$. For Ψ given, the map ψ will be completely defined by letting the zero coordinate of $\psi(x)$ be equal to the zero-coordinate of $\Psi(B_0)$ for any $x\in Y\cap Z$ with $x_0=1$ and by ensuring $\psi\circ S=\sigma\circ\psi$ where σ denotes here the shift on $\mathcal{A}^{\mathbb{Z}}$.

We define now the map Ψ. The alphabet $\mathcal{A}$ is given by

$$\mathcal{A}=\{y^z,\ y\in\{1,2,3,4\} \text{ and } z\in\{1,2,3,4,\times\}\}.$$

For $y^z\in\mathcal{A}$ we will refer to y as the y-coordinate of y^z. For any $l\in\mathbb{N}$, we let $\Psi(10^l)=y_1^{z_1}\cdots y_p^{z_p}$ be the word of length $p=p_x$ for any $x\in Y$ with $x_0=x_{l+1}=1$ and $x_1=\cdots=x_l=0$ such that we have $y_q=i$ whenever $\mathbf{k}_{S^{q-1}x}\in\mathcal{R}_i$ for any $1\leq q\leq p$. Finally we put with the notations of Lemma 4.13:

- $z_1=\left|k^++k^--\lceil\sqrt{8k^-(k^+-L(\mathbf{k}))}\rceil\right|$ with $(k^-,k^+)=\mathbf{k}_{T^rx}$, then $0\leq z_1\leq 4$ by Lemma 4.12 (3),
- $z_{p-2i}=\epsilon_{p-2-i}$ for $0\leq i\leq p-3-r$ (note that $p-2(p-3-r)>1$ by Lemma 4.13 (3)).
- $z_i=\times$ for others i.

Lemma 4.17. *The block code map ψ defines a uniform generator of (Y,S), i.e. ψ is a Borel embedding of $(Y\cap Z,S)$ to $(\mathcal{A}^{\mathbb{Z}},\sigma)$ such that $\chi:=\psi^{-1}$ extends in a symbolic extension of (Y,S) on $\overline{\psi(Y\cap Z)}\subset\mathcal{A}^{\mathbb{Z}}$.*

Proof. We first check Ψ is injective on the set of blocks. Assume $\Psi(10^l)=y_1^{z_1}\cdots y_p^{z_p}$. Then we may recover l from the y_i and z_i as follows. For $x\in Y$ with $x_0=x_{l+1}=1$ and $x_1=\cdots=x_l=0$ we have

- $r=\sharp\{i,\,y_i=2\}+1$,
- $z_{p-2i}=\epsilon_{p-2-i}$ for $0\leq i\leq p-3-r$.
- $k^-_{S^rx}=2^{r-1}$,
- $k^+_{S^{r+1}x}=2^{p-r-2}+\sum_{0\leq i<p-r-2}\epsilon_{r+1+i}2^i$,
- $l=k^-_{S^rx}+k^+_{S^rx}=z_1+\lceil\sqrt{8k^-_{S^rx}k^+_{S^{r+1}x}}\rceil$.

Then we may rebuild the decomposition of $x\in Y\cap Z$ into blocks which are delimited by the y-coordinates equal to 1 in the sequence $\psi(x)$. Therefore, to prove the injectivity of ψ on $Y\cap Z$, it only remains to show how to repair the position of x inside the blocks, that is k^-_x or k^+_x. With the above notations, assume $(\psi(x))_0=y_q^{z_q}$. We have :

- if $y_q=1$ then $k^-_x=0$
- if $y_q=2$ or 3, we get $k^-_x=2^{q-2}$,
- if $y_q=4$, then $k^+_x=2^{p-1-l}+\sum_{0\leq i<p-q-1}\epsilon_{q+i}2^i$.

Let us check now that ψ^{-1} extends to a symbolic extension of (Y,S). Any $u\in\overline{\psi(Y\cap Z)}\setminus\psi(Y\cap Z)$ may be written as the concatenation of words of the form $\Psi(B)$ for blocks B and semi-infinite or bi-infinite words in the alphabet $\mathcal{A}$ without any y-coordinate equal to 1. In the last case, when there is no y-coordinate equal to 1 in u, we just let $\chi(u)$ be the zero sequence, i.e. $\chi(u)=*$. Moreover a semi-infinite block without y-coordinates equal to 1 is sent to a semi-infinite sequence of 0's. Then the position of $x=\chi(u)$ inside such a semi-infinite sequence is defined as below :

- if the semi-infinite sequence lies in the future then $k^-_x=2^{l-2}$,
- if it lies in the past $k^+_x=2^{p-1-l}+\sum_{0\leq i<p-l-1}\epsilon_{l+i}2^i$ where l is the position of u inside the semi-infinite block.

Defined in this way, the extension χ of ψ^{-1} on $\overline{\psi(Y\cap Z)}$ is continuous. Observe finally that χ is surjective on Y because the image of χ contains $Y\cap Z$ which is dense in Y.

□

Remark 4.18. *By letting $\psi(*)$ be the $1^\times$ sequence $(1^\times)^\infty$, we get a Borel embedding of the set of recurrent points of (Y,S) to $\mathcal{A}^{\mathbb{Z}}$. By Remark 1.4 in* [BD19] *we may then extend ψ to a Borel embedding of the whole system (Y,S) satisfying $\chi\circ\psi=\mathrm{Id}_Y$.*

We finish now the proof of the second item of Theorem 4.8. The only recurrent points in $\overline{\psi(Z \cap Y)} \setminus \psi(Z \cap Y)$ belong to the two subshifts of finite type generated [5] respectively by the 2-words $2^0 2^\times$, $2^1 2^\times$ and $4^0 4^\times$, $4^1 4^\times$, which are contained in $\chi^{-1}*$. Therefore $h^\chi(\mu) = h(\mu)$ for any $\mu \in \mathcal{M}_S(Y)$ with $\mu(Y \cap Z) = 1$. Moreover $h^\chi(\delta_*) = h_{top}(\chi^{-1}*) = \frac{\log 2}{2}$. The function h^χ being affine we get $h^\chi(\mu) = h(\mu) + \mu(*)\frac{\log 2}{2}$ for all $\mu \in \mathcal{M}_S(Y)$.

Step 3 : Conclusion. Finally the symbolic extension π' induced by χ of $(Y^{f'}, S^{f'})$ by the suspension flow over $(\overline{\psi(Y \cap Z)}, \sigma)$ under the roof function $f' \circ \chi$ satisfies $h^{\pi'}(\nu) = h(\nu)$ for all $\nu \in \mathcal{M}^*_{S^{f'}}(Y^{f'})$ and $h^{\pi'}(\delta_{*f'}) = \frac{h^\chi(\delta_*)}{f'(*)} = \frac{1}{2l}$. As the extension $\pi : (Y^{f'}, S^{f'}) \to (X^f, T^f)$ is principal and satisfies $\pi^{-1}\mathcal{M}^*_{X^f}(X^f) = \mathcal{M}^*_{S^{f'}}(Y^{f'})$, the extension $\Pi = \pi \circ \pi'$ also satisfies $h^\Pi(\nu) = h(\nu)$ for all $\nu \in \mathcal{M}^*_{T^f}(X^f)$ and $h^\Pi(\delta_{*f}) = \frac{1}{2l}$. This concludes the proof of the second item of Theorem 4.8, because the function h^Π is affine. □

References

[Abr59] Leonid M Abramov. On the entropy of a flow. In *Dokl. Akad. Nauk SSSR*, volume 128, pages 873–875, 1959.

[BD04] Mike Boyle and Tomasz Downarowicz. The entropy theory of symbolic extensions. *Invent. Math.*, 156(1):119–161, 2004.

[BD19] David Burguet and Tomasz Downarowicz. Uniform generators, symbolic extensions with an embedding, and structure of periodic orbits. *J. Dynam. Differential Equations*, 31(2):815–852, 2019.

[BFF02] Mike Boyle, Doris Fiebig, and Ulf Fiebig. Residual entropy, conditional entropy and subshift covers. *Forum Math.*, 14(5):713–757, 2002.

[BS40] M Beboutoff and W Stepanoff. Sur la mesure invariante dans les systemes dynamiques qui ne different que par le temps. *Matematicheskii Sbornik*, 7(1):143–166, 1940.

[Bur19] David Burguet. Symbolic extensions and uniform generators for topological regular flows. *Journal of Differential Equations*, 267(7):4320–4372, 2019.

[BW72] Rufus Bowen and Peter Walters. Expansive one-parameter flows. *Journal of differential Equations*, 12(1):180–193, 1972.

[DH13] Tomasz Downarowicz and Dawid Huczek. Zero-dimensional principal extensions. *Acta Applicandae Mathematicae*, 126(1):117–129, 2013.

[Dow05] Tomasz Downarowicz. Entropy structure. *J. Anal. Math.*, 96:57–116, 2005.

[Dow11] Tomasz Downarowicz. *Entropy in dynamical systems*, volume 18. Cambridge University Press, 2011.

[Lin99] Elon Lindenstrauss. Mean dimension, small entropy factors and an embedding theorem. *Inst. Hautes Études Sci. Publ. Math.*, 89(1):227–262, 1999.

[LW77] François Ledrappier and Peter Walters. A relativised variational principle for continuous transformations. *J. London Math. Soc. (2)*, 16(3):568–576, 1977.

[Ohn80] Taijiro Ohno. A weak equivalence and topological entropy. *Publications of the Research Institute for Mathematical Sciences*, 16(1):289–298, 1980.

[SZ11] Wenxiang Sun and Cheng Zhang. Zero entropy versus infinite entropy. *Discrete Contin. Dyn. Syst.*, 30(4):1237–1242, 2011.

Sorbonne Universite, LPSM, 75005 Paris, France
E-mail address: david.burguet@upmc.fr

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland
E-mail address: rshi@impan.pl

[5] In other terms any element of the subshift of finite type is a concatenation of these two words.